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Introduction

بسم الله الرحمن الرحيم

We are pleased to introduce this book to show the philosophy on which the academic content has been prepared. This philosophy aims at:

1 Developing and integrating the knowledgeable unit in Math, combining the concepts and relating all the school mathematical curricula to each other.

2 Providing learners with the data, concepts, and plans to solve problems.

3 Consolidate the national criteria and the educational levels in Egypt through:

A) Determining what the learner should learn and why.

B) Determining the learning outcomes accurately. Outcomes have seriously focused on the following: learning Math remains an endless objective that the learners do their best to learn it all their lifetime. Learners should like to learn Math. Learners are to be able to work individually or in teamwork. Learners should be active, patient, assiduous and innovative. Learners should finally be able to communicate mathematically.

4 Suggesting new methodologies for teaching through (teacher guide).

5 Suggesting various activities that suit the content to help the learner choose the most proper activities for him/her.

6 Considering Math and the human contributions internationally and nationally and identifying the contributions of the achievements of Arab, Muslim and foreign scientists.

In the light of what previously mentioned, the following details have been considered:

★ This book contains three domains: algebra, relations and functions, calculus and trigonometry. The book has been divided into related and integrated units. Each unit has an introduction illustrating the learning outcomes, the unit planning guide, and the related key terms. In addition, the unit is divided into lessons where each lesson shows the objective of learning it through the title You will learn. Each lesson starts with the main idea of the lesson content. It is taken into consideration to introduce the content gradually from easy to hard. The lesson includes some activities, which relate Math to other school subjects and the practical life. These activities suit the students' different abilities, consider the individual differences throughout Discover the error to correct some common mistakes of the students, confirm the principle of working together and integrate with the topic. Furthermore, this book contains some issues related to the surrounding environment and how to deal with.

★ Each lesson contains examples starting gradually from easy to hard and containing various levels of thoughts accompanied with some exercises titled Try to solve. Each lesson ends in Exercises that contain various problems related to the concepts and skills that the students learned through the lesson.

★ Each unit ends in Unit summary containing the concepts and the instructions mentioned and General exams containing various problems related to the concepts and skills, which the student learned through the unit.

★ Each unit ends in an Accumulative test to measure some necessary skills to be gained to fulfill the learning outcome of the unit.

★ The book ends in General exams including some concepts and skills, which the student learned throughout the term.

Last but not least. We wish we had done our best to accomplish this work for the benefits of our dear youngsters and our dearest Egypt.

Contents

Contents

Unit One

Sequences and series

Unit introduction

Fibonacci (1170 - 1250)

Fibonacci was born in pisa in Italy. He originally delivered his education in Maghreb Countries.He had brought the Arab numbers and algebraic exponents used up today. Furthermore, he had introduced and identified the accounting systems to Europeans. These systems had greately exceeded the Roman systems which had been widely spreaded on that time in Europe. He had been best known for a problem that leads us to the following number sequence which has been named after him up to date. This sequence is : (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,.......)

It is noticed that each number starting with the third number is the sum of the two numbers preceding directly to it. The rule of this sequence is identified as follows:

$T_{n+2} = T_n + T_{n+1}$, for each $n \in Z^+$

Fibonacci's numbers had been used for analyzing the financial markets and the computer algorithms such as Fibonacci technology for searching and structuring the overstock of Fibonacci's data. It clearly appears in the biological orders such as branching of trees, ordering of leaves on the stem and the order of the pine cone.

Unit objectives

By the end of this unit and doing all the activities included, the student should be able to :

- Identify the concept of the sequence.
- Distinguish between the sequence and the series.
- Identify the arithmetic sequence.
- Deduce the general term of an arithmetic sequence in different forms .
- Find the arithmetic mean of an arithmetic sequence.
- Identify the finite and infinite series.
- Find the sum of a finite number of the terms of an arithmetic sequence in different forms.
- Identify the geometric sequence.
- Deduce the general term of the geometric sequence.
- Find the geometric mean of a geometric sequence.
- Deduce the relation between the arithmetic mean and the geometric mean of two different positive numbers .
- Find the sum of a finite number of the terms of a geometric sequence in different forms.
- Find the sum of an infinite number of the terms of a geometric sequence.
- Find the sum of an infinite geometric series.
- Convert the recurring decimal into a common fraction.
- Function the arithmetic and geometric sequences to interpret some life problems such as overpopulation.
- Solve life applications on series.
- Use the calculators to solve math and life problems using the sequences and series.

Key terms

- Function
- Term
- Finite Sequence
- Infinite Sequence
- Increasing Sequence
- Decreasing Sequence
- Series
- Summation Notation
- Arithmetic Sequence
- Common Difference
- Arithmetic Means
- Arithmetic Series
- Geometric Sequence
- Common Ratio
- Geometric Means
- Geometric Series
- Infinite Geometric Series
- Infinity

Lessons of the unit

Lesson (1 - 1): Sequences.

Lesson (1 - 2): Series and summation notation.

Lesson (1 - 3): Arthmetic sequences.

Lesson (1 - 4): Arthmetic series.

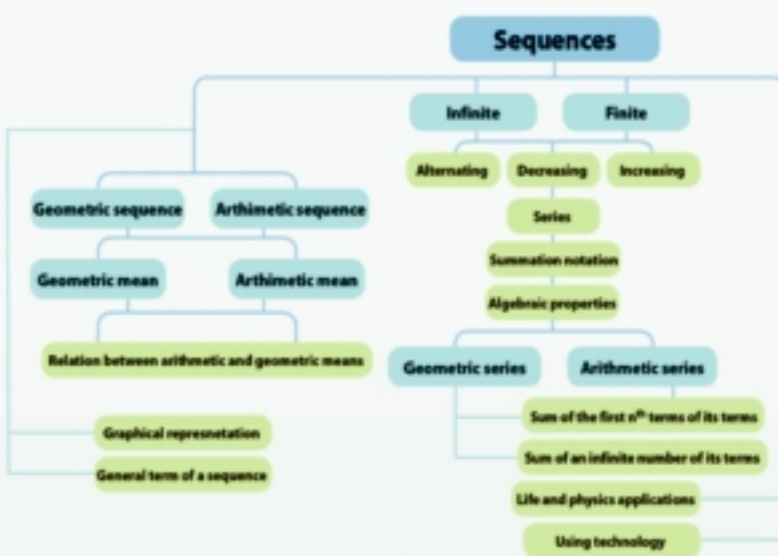
Lesson (1 - 5): Geometric sequences.

Lesson (1 - 6): Geometric series.

Materials

Scientific calculator - Graphics.

Unit planning guide



Unit One

1 - 1 Sequences

You will learn

- Definition of sequence.
- Finite and infinite sequence.
- n^{th} term of a sequence.
- Increasing and decreasing sequence.

Key - terms

- Sequence
- Finite Sequence
- Infinite Sequence
- Term
- Increasing Sequence
- Decreasing Sequence
- Constant Sequence

Matrials

- Scientific Calculator
- Graphic programs

Introduction:

You have previously learned the patterns such as (1, 3, 5, 7, ...) and learned that the pattern is an order of a set of real numbers . In this unit, we are going to shed light on some of these patterns and learn them more deeply.

Think and discuss

Learn the following pattern, then answer the questions: (1, 4, 7, 10, ...)

1) What is the relation between each term and the term next to it?
2) Can you find the next two terms of the pattern?
3) Can you find the ninth term of this pattern . (don't try to find the previous terms)?

Remember

The function is a relation between the two sets X and Y so that each element of X appears as a first projection only one time in a limited ordered pairs of the relation.

Learn

Definition 1

➤ The sequence is a function whose domain is the set of the positive integers Z^+ or a subset of it and its range is a set of the real numbers R where the first term is denoted by T_1, the second term is denoted by T_2, and the third term is denoted by T_3 and son on.... and the n^{th} term is denoted by T_n the sequence can be expressed by writing down its terms between two brackets as follows:
$(T_1 , T_2 , T_3 , ... , T_n)$
or denoted by the symbol (T_n).

Notice

(1) Terms of a sequence are the elements of the sequence domain.
(2) The symbol (T_n) expresses the sequence while the symbol T_n expresses its n^{th} term.
(3) The sequence is subjected to the order of its elements while the set is not subjected to the order of its elements.
(4) The elements of the set don not recur while the elements of the sequence may recur.

Definition:

Finite and Infinite Sequences

The sequence is finite if the number of its terms is finite (**i.e. can be counted**) the sequence is infinite if the number of its terms is infinite (**an infinite number of elements**)

Example

1) Write down each of the sequences whose n^{th} term is given by the relation:

a) $T_n = n^2 - 1$ (to five terms starting with first term)

b) $T_n = \frac{(-1)^n}{2n+1}$ (to an infinite number of terms starting with first term)

Solution

a) let n = 1 then $T_1 = (1)^2 - 1 = 0$, let n = 2 then $T_2 = (2)^2 - 1 = 3$,

let n = 3 then $T_3 = (3)^2 - 1 = 8$, let n = 4 then $T_4 = (4)^2 - 1 = 15$,

let n = 5 then $T_5 = (5)^2 - 1 = 24$

the sequence is: (0 , 3 , 8 , 15 , 24)

b) let n = 1 then $T_1 = \frac{(-1)^1}{2 \times 1 + 1} = \frac{-1}{3}$, let n = 2 then $T_2 = \frac{(-1)^2}{2 \times 2 + 1} = \frac{1}{5}$,

let n = 3 then $T_3 = \frac{(-1)^3}{2 \times 3 + 1} = \frac{-1}{7}$, let n = 4 then $T_4 = \frac{(-1)^4}{2 \times 4 + 1} = \frac{1}{9}$,

let n = 5 then $T_5 = \frac{(-1)^5}{2 \times 5 + 1} = \frac{-1}{11}$

The sequence is: $(\frac{-1}{3}, \frac{1}{5}, \frac{-1}{7}, \frac{1}{9}, \frac{-1}{11}, ...)$ the sequence in this case is called an alternating sequence (i.e. a term of its terms is positive and the other is negative or vice versa).

Try to solve

1) Write down each of the sequences whose n^{th} term is given by the relation:

a) $T_n = 1 + \frac{2}{\sqrt{n}}$ (to an infinite number of terms starting with first term)

b) $T_n = \sin \frac{\pi}{n}$ (to five terms starting with first term)

General Term of a Sequence

The general term of a sequence (it is sometimes called n^{th} term) is written as T_n where T_n is the image of the element whose order is n and can be found sometimes through the given terms of the sequence by realizing the relation between the value and order of the term.

For example:

- The general term of the sequence of even numbers: (2 , 4 , 6 , 8 , ...) is $T_n = 2n$
- The general term of the sequence of odd numbers:(1 , 3 , 5 , 7 , ...) is $T_n = 2n - 1$
- The general term of the sequence:$(\frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \frac{1}{6}, ...)$ is $T_n = \frac{(-1)^n}{n+2}$

Example

2) Write down the first five terms and the general term of the sequence (T_n) which is defined as follows:

$T_1 = 2$ and $T_{n+1} = 2T_n$ where $n \geqslant 1$

Solution

by substituting the value of n = 1, 2, 3, 4, 5 in the relation $T_{n+1} = 2T_n$

let n = 1 $T_2 = 2T_1$ **i.e.:** $T_2 = 2 \times 2 = 4 = 2^2$ (by substituting $T_1 = 2$)

let n = 2 $T_3 = 2T_2$ **i.e.:** $T_3 = 2 \times 4 = 8 = 2^3$ (by substituting $T_2 = 4$)

let n = 3 $T_4 = 2T_3$ **i.e.:** $T_4 = 2 \times 8 = 16 = 2^4$ (by substituting $T_3 = 8$)

let n = 4 $T_5 = 2T_4$ **i.e.:** $T_5 = 2 \times 16 = 32 = 2^5$ (by substituting $T_4 = 16$)

The first five terms of the sequence are: (2 , 4 , 8, 16 , 32)

The general term of the sequence (T_n) is: $T_n = 2^n$

Tip

Some sequences donot have a defined rule till now such as the sequence of the prime numbers (2, 3, 5, 7,...)

Think:

1- How can you check the solution above?

Try to solve

(2) Write down the first six terms of the sequence (T_n) where: $T_{n+2} = T_{n+1} + T_n$ where $n \geqslant 1$, $T_1 = 2$ and $T_2 = 3$

Increasing and Decreasing Sequences

Check the following sequences:

1) (-5 , -1 , 3 , 7 , 11 , 15 , ...) **(What do you notice?)**

2) (4 , 2 , 1 , $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ...) **(What do you notice?)**

- **In the first sequence:** -1 > -5 i.e. $T_2 > T_1$, 3 > -1 i.e. $T_3 > T_2$ and so on for the rest of the terms. I.e. each term of the sequence is greater than the directly previous term.
- **In the second sequence:** 2 < 4 i.e. $T_2 < T_1$, 1 < 2 i.e. $T_3 < T_2$ and so on i.e. each term of the sequence is lesser than the directly previous term.

Definition:

- The sequence (T_n) is called **increasing** if $T_{n+1} > T_n$
- The sequence (T_n) is called **decreasing** if $T_{n+1} < T_n$

Tip

Constant sequence :
The whole terms of the sequence are equal. i.e.: $T_n = a$. It may be finite or infinite.

Example

(3) Show which of the sequences (T_n) is in increasing , decreasing or otherwise.

a $T_n = 2n + 3$ b $T_n = \frac{1}{3n - 1}$ c $T_n = \frac{(-1)^n}{2n} + 4$

Solution

a Find T_{n+1} as follows: $T_{n+1} = 2(n+1) + 3 = 2n + 5$

Find the result of: $T_{n+1} - T_n$: $T_{n+1} - T_n = (2n+5) - (2n+3) = 2 > 0$

$T_{n+1} > T_n$ i.e. the sequence is **increasing** for all **the values of n**

b **Find** T_{n+1} as follows: $T_{n+1} = \frac{1}{3(n+1)-1} = \frac{1}{3n+2}$

Find the result of: $T_{n+1} - T_n$: $T_{n+1} - T_n = \frac{1}{3n+2} - \frac{1}{3n-1} = \frac{3n-1-3n-2}{(3n+2)(3n-1)}$

$= \frac{-3}{(3n+2)(3n-1)} < 0$

$T_{n+1} < T_n$ i.e. the sequence is **decreasing** for all **the values of n**

c $T_{n+1} - T_n = \frac{(-1)^{n+1}}{2(n+1)} - \frac{(-1)^n}{2n} = \frac{(-1)^{n+1}}{2(n+1)} + \frac{(-1)^{n+1}}{2n}$

$= (-1)^{n+1}[\frac{1}{2(n+1)} + \frac{1}{2n}] = (-1)^{n+1}[\frac{n+n+1}{2n(n+1)}]$

$= (-1)^{n+1}[\frac{2n+1}{2n(n+1)}]$

This expression is positive when n is an odd number and negative when n is an even number i.e. the sequence is neither increasing nor decreasing.

Try to solve

(3) Show which of the sequences (T_n) is increasing , decreasing or otherwise.

a $T_n = \frac{2}{n} - 3$ **b** $T_n = (\frac{1}{2})^n$ **c** $T_n = (-2)^n$

Exercises (1 - 1)

Complete:

(1) The sequence is a function whose domain is .... or ....

(2) The seventh term of the sequence (T_n) where $T_n = 2n^2 + 3$ is ....

(3) The fourth term of the sequence (T_n) where $T_n = \frac{\sqrt{n}}{n+1}$ is ....

(4) In the sequence (T_n) where $T_{n+1} = n\,T_n$, $n \geqslant 1$ if $T_1 = 1$ then $T_2 =$ ....

(5) In the sequence (T_n) where $T_n = 3n^2 - 1$ if $T_n = 74$, then n = ...

(6) The n^{th} term of the sequence (1 , 8 , 27 , 64 , ...) is ....

(7) The n^{th} of the sequence (- 1 , 4 , - 9 , 16 , ...) is ....

Complete the following using: > , < or =

(8) The sequence is decreasing if: T_{n+1} .... T_n for each $n \geqslant 1$

(9) The sequence is constant if: T_{n+1} .... T_n for each $n \geqslant 1$

(10) The sequence is increasing if: T_{n+1} .... T_n for each $n \geqslant 1$

Choose the correct answer:

(11) The sequence whose n^{th} term is $T_n = \frac{2}{n} - 1$ where $n \in Z^+$ represents:

- a Increasing sequence
- b Decreasing sequence
- c Constant sequence
- d Alternating sequence

(12) The general term of the sequence $((2 \times 3), (3 \times 4), (4 \times 5), (5 \times 6), ...)$ is

- a $(n-1)(n+1)$
- b $n(n+1)$
- c $2n(n+1)$
- d $(n+1)(n+2)$

Answer the following questions:

(13) Show whether each of the following sequences is finite or infinte:

- a $(1, 4, 7, 11, ...)$
- b $(3, 5, 7, 9, ..., 21)$
- c The sequence (T_n) where $T_n = n^2 - 1$, $n \in Z^+$
- d The sequence (T_n) where $T_n = \frac{2}{n} + 3$, $n \in \{1, 2, 3, 4, 5\}$

(14) Write the first five terms for each of the following sequences whose general term is given by the following rules:

- a $T_n = n + n^2$
- b $T_n = \frac{1}{2n-5}$
- c $T_n = (\frac{1}{3})^n$
- d $T_n = \sin(\frac{n}{4}\pi)$
- d $T_n = (-1)^n (n-2)^2$
- e $T_n = \frac{(-1)^n}{n^2}$

(15) Write down the general term for each of the following sequences:

- a $(2, 5, 8, 11, ...)$
- b $(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...)$
- c $(1, 8, 27, 64, ..)$
- d $(1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, ...)$
- e $(-1, 2, -4, 8, -16, ...)$
- f $(\cos\frac{\pi}{3}, \cos\frac{2\pi}{3}, \cos\pi, \cos\frac{4\pi}{3}, ...)$

(16) Show whether each of the sequences (T_n) is increasing , decreasing or otherwise in each of the following:

- a $T_n = 3n + 5$
- b $T_n = \frac{1}{n} + 2$
- c $T_n = (\frac{1}{2})^{n+1}$
- d $T_n = (-1)^n (n^2 + 1)$

(17) **Sports:** Kareem does physical fitness exercises for 8 minutes in the first day and increases the training period for two minutes daily.

- **a** Write the first five terms of this sequence.
- **b** Find the general term of this sequence.
- **c** How long does kareem take in the seventh day?
- **d** In which day does kareem take half an hour? Explain.

(18) **Discover the error:**

- **a** Each function whose domain is Z is a sequence.
- **b** Each function whose domain is {0 , 1 , 2 , 3 , 4 , ...} is an infinite sequence.
- **c** In the sequence (T_n) where $T_n = n^2$ then $T_n > T_{n+1}$

(19) **Creative thinking:** In the sequence (T_n) if $T_1 = 9$, $T_3 = 36$ and $T_{n+1} = T_n + n\,x$, find the value of x.

Activity

1- If (T_n) is a sequence whose terms are (1, 1 , 2 , 3 , 5 , 8 , 13 , ...).

- **a** Study the pattern of the sequence , then find the eighth and ninth terms.
- **b** Represent the first nine terms of the sequence graphically.
- **c** Do we consider such a sequence increasing , decreasing or otherwise? Explain.
- **d** Write down the relation between the terms of this sequence.
- **d** For more information about fibonacci's sequence, log in (www.go.hrw.com Fibonacci) .

Unit One

Series amd summation notation

1 - 2

In our daily language, we use the two expressions; sequence and series as two equivalent words. Although those two words are correlated meaning fully, there is a mathematical difference between them. The sequence is an ordered list of numbers whereas the series is the sum of terms of the sequence.

For example: (2 , 5 , 8 , 11 , ..) is a sequence while 2 + 5 + 8 + 11 + .. is the series correlated with the previous sequence. The summation notation "Σ" can be used for writing the series in short forms and can be read as "sigma" .

You will learn

- The concept of series
- Finite series
- Infinite series
- Algebraic properties of summation.

Key - terms

- Series
- Finite Series
- Infinite Series
- Summation Notation

Matrials

- Scientific Calculator

Learn

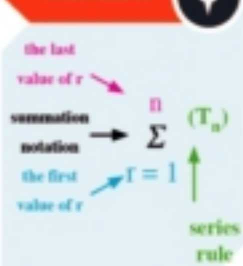
Definition 4

Finite Series

It is written in the form: $T_1 + T_2 + T_3 + ... + T_r + ... + T_n$ where n is a positive integer, T_n is the terms whose order is n in the series and the numerical value of the finite series is called the sum of the terms of the corresponding sequence.

The finite series: $a_1 + a_2 + a_3 + ... + a_r \; ... + a_n$ can be written in the form $\sum_{r=1}^{n} (a_r)$ and read as the sum of a_r from r = 1 to r = n

Remember



Example

1. Expand each of the following series, then find the expansion sum.

a) $\sum_{r=1}^{4} (r^2)$ b) $\sum_{r=1}^{7} (2r - 1)$ c) $\sum_{r=1}^{n} (\frac{1}{r+1} - \frac{1}{r})$

Solution

a) $\sum_{r=1}^{4} (r^2) = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$

b) $\sum_{r=1}^{7} (2r - 1) = (2 \times 1 - 1) + (2 \times 2 - 1) + (2 \times 3 - 1) + (2 \times 4 - 1) + (2 \times 5 - 1) + (2 \times 6 - 1) + (2 \times 7 - 1)$

$= 1 + 3 + 5 + 7 + 9 + 11 + 13 = 49$

c) $\sum_{r=1}^{n} (\frac{1}{r+1} - \frac{1}{r}) = (\frac{1}{2} - \frac{1}{1}) + (\frac{1}{3} - \frac{1}{2}) + (\frac{1}{4} - \frac{1}{3}) + ... + (\frac{1}{n+1} - \frac{1}{n})$

$= \frac{1}{2} - 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + ... + \frac{1}{n+1} - \frac{1}{n}$

$= \frac{1}{n+1} - 1 = \frac{1 - n - 1}{n+1} = \frac{-n}{n+1}$

Try to solve

(1) Write down the series in the form of an expansion , then find its sum:

a $\sum_{r=1}^{5} (1+r^2)$ b $\sum_{r=1}^{9} (3r+2)$ c $\sum_{r=1}^{n} (\frac{1}{r+2} - \frac{1}{r+1})$

Infinite Series

The terms of the infinite series cannot be counted . For example the series:
$-3+9-27+81-243+ \dots$ can be written in the form of $\sum_{r=1}^{\infty} (-3)^r$. The symbol ∞ is used to express infinity.

Example

(2) Use the summation notation Σ to write down the series: $2\times3+3\times4+4\times5+\dots$

Solution

$\because$ The general term of the sequence is: $T_r = (r+1)(r+2)$ where $r \in Z^+$

$\therefore 2\times3+3\times4+4\times5+\dots = \sum_{r=1}^{\infty} (r+1)(r+2)$

Try to solve

(2) Use the summation notation Σ to write down the series:

$1\times2\times3+3\times4\times5+5\times6\times7+\dots$

Algebraic Properties of Summation

1- If (T_r) and (E_r) are two sequences, $n \in Z^+$ and $C \in R$, then:

a $\sum_{r=1}^{n} C = Cn$ b $\sum_{r=1}^{n} cT_r = c\sum_{r=1}^{n} T_r$

c $\sum_{r=1}^{n} (T_r \pm E_r) = \sum_{r=1}^{n} T_r \pm \sum_{r=1}^{n} E_r$

2- $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$, $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

Example

(3) Find in two different methods $\sum_{r=1}^{4} (3-2r+r^2)$

Solution

1- Method 1 (direct substitution)

$$\sum_{r=1}^{4} (3-2r+r^2) = (3-2\times1+1^2)+(3-2\times2+2^2)+(3-2\times3+3^2)+(3-2\times4+4^2)$$
$$= 2+3+6+11 = 22$$

2- Method 2 (using algebraic properties of summation)

$$\sum_{r=1}^{4}(3-2r+r^2)=\sum_{r=1}^{4}3-2\sum_{r=1}^{4}r+\sum_{r=1}^{4}r^2$$

$$=3\times 4-2\times\frac{4(4+1)}{2}+\frac{4(4+1)(2\times 4+1)}{6}$$

$$=12-2\times\frac{4\times 5}{2}+\frac{4\times 5\times 9}{6}$$

$$=12-20+30=22$$

Using the Scientific Calculator to Find the Sum of a Series

- Press the summation notation button Σ according to the specific color .
- Write the rule of the sequence $(3-2r+r^2)$ as follows:

3 − 2 ALPHA ((×) + ALPHA ((×) x^2

- Use the button (REPLAY) to move as follows:
- Upward, write 4 and downward, write 1
- Press the button = to give the sum 22 on the screen

Try to solve

(3) Find in two methods: $\sum_{r=1}^{5}(2r^2-3r+5)$ known that:

$\sum_{r=1}^{n} r=\frac{n(n+1)}{2}$ and $\sum_{r=1}^{n} r^2=\frac{n(n+1)(2n+1)}{6}$

Exercises (1 - 2)

(1) Complete:

a) The series: $5+10+15+20+\ldots+50$ is written using the summation notation in the form of ....

b) The series: $7\times 1+7\times 2+7\times 3+\ldots+7\times 20$ is written using the summation notation in the form of ....

c) The series: $-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}\ldots$ is written using the summation notation in the form of ....

d) The series : $9+99+999+9999+\ldots$ to n terms is written using the summation notation in the form of ....

(2) Write down the following series using the summation notation:

a) $1+2+3+4+5+\ldots\ldots+20$

b) $2+4+6+8+10+\ldots\ldots+60$

c) $3+6+9+12+15+18+21$

d) $2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots$

e) $1+4+9+16+\ldots\ldots+64$

f) $\frac{1}{5}+\frac{1}{10}+\frac{1}{15}+\frac{1}{20}+\ldots\ldots+\frac{1}{50}$

3. Expand each of the following series:

a) $\sum_{r=1}^{5} (3r-2)$ b) $\sum_{r=1}^{8} ((-1)^r + 4r)$

c) $\sum_{r=1}^{\infty} \left((\frac{1}{2})^r - 1\right)$ d) $\sum_{r=1}^{\infty} (\frac{1}{r} - \frac{1}{r+1})$

4. If you know that $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ and $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, use the properties of the summation notation Σ to find the value for each of the following:

a) $\sum_{r=8}^{12} 2(r+5)$ b) $\sum_{r=5}^{8} (2r^2 - 3r)$

5. **Discover the error:**

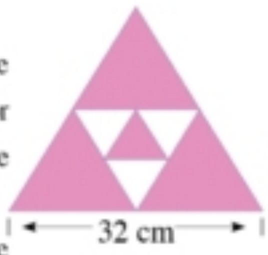
a) The series is a function whose domain is the set of the positive integers or a subset of it.

b) $\sum_{r=0}^{5} (2r+1) = 1+3+5+7+9$

6. **Mining:** A gold mine produced 4200 kg of gold in the first year and the production is decreased at a rate of 10 % yearly of the directly previous year.

➢ Use the summation notation to write the sum of production of the gold mine during the first five years, then find this sum.

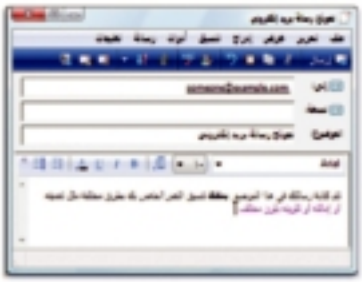
7. **Geometry:** The figure opposite represents an equilateral triangle whose side length is 32cm. Its sides are bisected and the interior triangle is drawn and this pattern lasts interiorly till we get three triangles including the first triangle



a) Write down the series of the triangle perimeters using the summation notation .

b) Find the sum of all the triangle perimeters in centimetres.

8. **Technology:** Kareem has electronically mailed a message for three friends, then each friend has e-mailed the same message to other three friends and so on .. E-mailing the messages lasts the same pattern (known that each person has received the message once)



a) Write down the series using the summation notation.

b) How many persons do they receive the message in the fifth stage?

c) Find the sum of all persons who will share this message till the fifth stage.

Arithmetic Sequence

1 - 3

You will learn

- Definition the arithmetic sequence.
- The graphical representation of the arithmetic sequence.
- n^{th} term of the arithmetic sequence.
- Identify the arithmetic sequence
- Definition of the arithmetic mean.
- Insert a finite number of arithmetic means between two numbers.

Key - term

- Arithematic Sequence
- n^{th} Term
- Commen Difference
- Order of a Term
- Arithmetic Mean

Matrials

- Scientific Calculator
- Graphic programs

Doaa has started to read a novel. On the first day, she read 10 pages, on second day , she read 15 pages, on the third day, she read 20 pages and she kept reading in this pattern. The sequence of the number of the pages read on each day is : **(10 , 15 , 20 , 25 , ...)**

What do you notice in this sequence?

Definition 1

Arithematic Sequence

The arithmetic sequence is the sequence in which the difference between a term and the directly previous term to it equals a constant and it is called the common difference of the sequence. it is denoted by the symbol (d)

i.e.: $d = T_{n+1} - T_n$ for each $n \in Z^+$

Tip

Harmonic Sequence) The sequence is called a harmonic sequence if the reciprocals of its terms form an arithmetic sequence such as the sequence $(\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, ...)$

Example

(1) Which of the following is an arithmetic sequence? why ?

a (7 , 10 , 13 , 16 , 19)

b (27 , 23 , 19 , 15 , 11, ...)

c $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6})$

Solution

a $\because T_2 - T_1 = 10 - 7 = 3$, $T_3 - T_2 = 13 - 10 = 3$, ...

$\therefore T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = T_5 - T_4 = 3$

$\therefore$ The sequence is arithmetic and its common difference $= 3$

b $\because T_2 - T_1 = 23 - 27 = -4$, $T_3 - T_2 = 19 - 23 = -4$, ...

$\therefore T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = T_5 - T_4 = -4$

$\therefore$ The sequence is arithmetic and its common difference $= -4$

c $\because T_2 - T_1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$, $T_3 - T_2 = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$

$\therefore T_2 - T_1 \neq T_3 - T_2$

$\therefore$ **The sequence is not arithmetic**

Try to solve

(1) Which of the following is an arithmetic sequence ? why

a (38 , 33 , 28 , 23 , 18, ...) b (- 14 , - 8 , - 2 , 4, 10) c (23 , 28 , 33 , 38 , 42)

Example

(2) Show which of the following sequences whose n^{th} term is given by the following relations is arithmetic, then find its common difference in case it is arithmetic.

a $T_n = 2n + 3$ b $T_n = \frac{3}{n} + 2$

Solution

a $\because T_{n+1} - T_n = (2(n+1)+3) - (2n+3) = 2n + 2 + 3 - 2n - 3 = 2$

$\therefore$ Th sequence is arithmetic and its common difference is 2

b $\because T_{n+1} - T_n = \left(\frac{3}{n+1} + 2\right) - \left(\frac{3}{n} + 2\right)$

$= \frac{3}{n+1} - \frac{3}{n} = \frac{3n - 3n - 3}{n(n+1)} = \frac{-3}{n(n+1)}$ doesnot equal to a constant.

$\therefore$ **The sequence is not arithemtic.**

Try to solve

(2) Show which of the following sequences whose n^{th} term is given by the following relations is arithmetic, then find its common difference in case it is arithmetic.

a $T_n = 5 - 3n$ b $T_n = (n+1)^2$

Graphical representation of an arithematic sequence

Example

(3) Find the next four terms of the arithmetic sequence (10, 7, 4, ...), then represent the seven terms graphically.

Solution

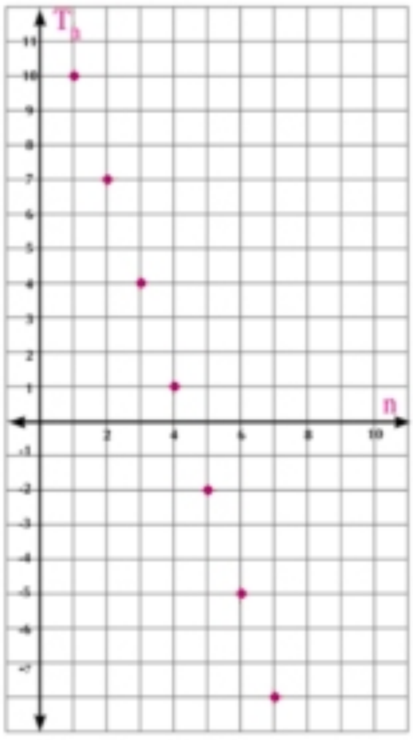
$\because d = T_2 - T_1 = 7 - 10 = -3$

$\therefore$ the next four terms are: 1, - 2, - 5, - 8

the domain of the sequence is {1 , 2 , 3 , 4 , 5 , 6 , 7, ...}

The range of the sequence is {10 , 7 , 4 , 1 , - 2 , - 5 , - 8, ...}

The opposite figure shows the graphical representation of the sequence.



From the figure opposite , we notice that:

- The points that represent the terms of the arithmetic sequence are collinear .

This means that the arithmetic sequence is a function of the first degree in n where $n \in Z^+$ and the coefficient of n is the common difference of the sequence.

From the previous, we deduce that:

- The relation between the two variables n and T_n is $T_n = d\,n + b$ where b and d are two constants and d is the common difference of the sequence .
- The sequence is increasing if $d > 0$, and is decreasing if $d < 0$

Using the scientific calulator to write the arithmetic sequence:

To write the arithmetic sequence in which ($a = 10$ and $d = -3$) we do as follows:

- We write the value of a (10) then press [=] and put the value of d by pressing [−] then (3) and press [=] to give the second term of the sequence. By repeating the process of pressing [=] the next terms are given and so on ...

Try to solve

(3) In the sequence (T_n) where $T_n = 3n - 5$:

- **a** Prove that (T_n) is an arithmetic sequence and find its common difference.
- **b** Show that this sequence is increasing.
- **c** Find the fifteenth term of the sequence.
- **d** What's the value of n if $T_n = 85$?

The n^{th} Term of the arithmetic sequence

From difinition (1) the n^{th} term of the arithmetic sequence (T_n) whose first term is a and common difference is d can be deduced as follows :

$T_1 = a$, $T_2 = a + d$ and $T_3 = a + 2\,d$ and by keeping this pattern, we find that the n^{th} term of this sequence is :

$T_n = a + (n - 1)\,d$ and if $T_n = \ell$ (where ℓ is the last term, then $\ell = a + (n-1)\,d$

Example

(4) In the arithmetic sequence (13 , 16 , 19 , ... , 100)

- **a** Find the tenth term.
- **b** Find the number of the terms of the sequence.

Solution

$\because$ The sequence is arithmetic $\qquad \therefore a = 13,\ d = 16 - 13 = 3$

a $\because T_n = a + (n - 1)\,d \qquad \therefore T_{10} = 13 + (10 - 1) \times 3$

$= 13 + 9 \times 3 = 13 + 27 = 40$

b The required is to find the value of n when $\ell = 100$

$\because T_n = a + (n - 1)\,d \qquad \therefore 100 = 13 + (n - 1) \times 3$

$\therefore\ 100 = 13 + 3n - 3$ **i.e.:** $3n = 100 - 10 = 90$ $\qquad \therefore\ n = 30$

Try to solve

(4) Find the number of the terms of the arithmetic sequence (7 , 9 , 11 , ... , 65) then find the value of the tenth term from the end.

Identifying the Arithmatic Sequence

The arithmetic sequence can be identified when its first and common difference are known.

Example

(5) If the seventh and fifteenth terms of an arithmetic sequence are 18 and 34 respectively, find the common difference and the first term then find the n^{th} term of this sequence.

Solution

$T_7 = 18, T_{15} = 34$ **from the given data**

$\because T_n = a + (n-1)\,d$

$\therefore a + 6d = 18 \ , \ a + 14d = 34$

$\therefore 8d = 16$ **by subtracting the two equations**

$d = 2$ **By dividing by 8**

$\therefore a + 6 \times 2 = 18$ **by substituting in equation 1**

$\therefore a = 18 - 12 = 6$

To find the n^{th} term , substitute in the rule: $T_n = a + (n-1)\,d$ about the two values of a and d

$T_n = 6 + (n-1) \times 2 = 6 + 2n - 2 = 2n + 4$

The first term equals 6 the common difference equals 2 and the n^{th} term is $2n + 4$

Try to solve

(5) Find the arithmetic sequence whose sixth term = 17 and the sum of its third and tenth terms = 37.

Arithmtic means

When there are two non-consecutive terms in an arithmetic sequence, then all the terms lying between those two terms are called arithmetic means. This concept can be used to find the missing terms between those two terms in the arithmetic sequence.

Definition 2

If a , b and c are three consecutive terms in an arithmetic sequence, then b is called the arithmetic mean between the two terms a and c where b- a = c-b,

i.e.: $2b = a + c$ then $b = \frac{a+c}{2}$ So: $\left(a , \frac{a+c}{2} , c\right)$ **is an arithmetic sequence.**

several arithmetic means: $x_1 , x_2 , x_3 , ... , x_n$ can be inserted between the two numbers a and b in a way that the numbers : $(a , x_1 , x_2 , x_3 , ... , x_n , b)$ **form an arithmetic sequence.**

Verbal expression: What is the relation between the number of the arithmetic means and the number of the terms of the sequence involving such means?

Inserting a Finite Number of Arithmatic Means Between Two Numbers

Example

(6) Insert 5 arithmetic means between 6 and 48

Solution

1- Find the number of the sequence terms.

Thera are five means between the first and the last terms of the sequencer. Thus, the number of the terms of the arithmetic sequence is $\mathbf{n = 2 + 5 = 7}$

2- Find the value of d

The n^{th} term of the arithmetic sequence is: $T_n = a + (n - 1)\, d$

by substituting: $a = 6$, $T_n = 48$ and $n = 7$

$48 = 6 + (7 - 1)\, d$ i.e.: $6d = 42$

by dividing the two sides by 6 $d = 7$

3- We use the value of d to find the required arithmetic means

(6) , 13 , 20 , 27 , 34 , 41 , (48)

+7 , +7 , +7 , +7 , +7 , +7

The required means are: 13 , 20 , 27 , 34 , 41

Try to solve

(6) Insert seven arithmetic means between the two numbers – 24 and 16

Exercises (1 - 3)

Determine which of the following sequences are arithmetic and which are non-arithmetic, then find the common difference in case the sequence is arithmetic:

(1) (34 , 30 , 26 , 22 , 18)

(2) (7 , 12 , 17 , 22 , 27)

(3) (-12 , -18 , - 24 , - 30 , - 36)

(4) (7 , 7 , 7 , 7 , 7)

Complete:

(5) The seventh term of the arithmetic sequence (2 , 5 , 8 , ...) is ....

(6) The eleventh term of the arithmetic sequence (T_n) where $T_n = 3n - 5$ is ....

(7) The n^{th} term of the arithmetic sequence (81 , 77 , 73 , ...) is ....

(8) The arithmetic mean of the two numbers 8 and 12 is ....

(9) If the arithmetic mean of the two numbers x and 26 is 21, then x equals ....

Choose the correct answer:

(10) All the following sequences are arithmetic except :

a $(3, 7, 11, 15, ...)$ b $(-11, -15, -19, -23, ...)$

c $(\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} ...)$ d $(\frac{21}{5}, \frac{16}{5}, \frac{11}{5}, \frac{6}{5}, ...)$

(11) Of all the following sequences, the arithmetic sequence is:

a $(T_n) = (\frac{n+1}{n})$ b $(T_n) = (n+1)^2$

c $(T_n) = \left(\frac{3}{n}(n+2)\right)$ d $(T_n) = (\frac{n^3-1}{n^2+n+1})$

(12) If (T_n) is an arithmetic sequence where $T_n = 3n + 2$, then the arithmetic mean between T_5 and T_{11} equals:

a 8 b 16 c 22 d 26

(13) If $2a + 1$, $5a - 1$ and $6a + 3$ are three consecutive terms of an arithmetic sequence , then a equals.

a 1 b 2 c 3 d 5

(14) If a and b are two arithmetic means between x and y, then: $\frac{y - x}{b - a}$ equals:

a 2 b 3 c 4 d 6

Discover the error:

(15) The common difference of the arithmetic sequence is defined as the difference between a term and the term directly previous to it i.e.:

$d = T_n - T_{n-1}$ for each $n \in Z^+$

(16) The relation between n and T_n of an arithmetic sequence is given as follows: $T_n = an + b$ where a and b are two constants and b is the common difference of the sequence in this relation.

Answer the following questions:

(17) Find the number of the terms of the arithmetic sequence (2 , 5 , 8 , ... ,80).

(18) In the arithmetic sequence (63 , 59 , 55 , ... , - 133) find :

a The value of the seventh term b the number of the sequence terms.

(19) Find the order and value of the first negative term in the arithmetic sequence (67, 64 , 61 , ...)

(20) Find the order and value of the first term whose value is greater than 180 in the arithmetic sequence:

(21) Write down the first three terms of the sequence $(T_n) = (2 + 5n)$, then find the order of the term whose value is 72 of the sequence and find the order of the first term whose value is more than 100

(22) What is the value of n in the arithmetic sequence whose first term = 3, $T_n = 39$ and $T_{2n} = 79$? find the sequence.

(23) Find the arithmetic sequence whose fifth term = 21, and the tenth term = 3 times the second term.

(24) (T_n) is an arithmetic sequence in which $T_1 + T_2 = 9$ and $T_5 = 22$, find this sequence.

(25) Find the arithmetic sequence whose sixth term = 20 and the ratio between the fourth and tenth terms is 4: 7.

(26) Find the arithmetic sequence whose fourth term = 11 and the sum of the fifth and ninth terms = 40, then find the order of the term whose value is 152 in this sequence.

(27) Find the arithmetic sequence in which the arithmetic mean between its third and seventh terms is 19 and its tenth term is greater than twice of its fourth term by 2.

(28) (T_n) is an arithmetic sequence in which: $T_2 + T_4 = 42$ and $T_3 \times T_5 = 315$, find this sequence.

(29) If (8, a, ..., b, 68) form an arithmetic sequence of sixteen terms, find the values of a and b.

(30) If 36, a, 24, b are consecutive terms of an arithmetic sequence, find the value of a and b.

(31) If the arithmetic mean between a and b is 8 and the arithmetic mean between 4 a and 2 b is 20, find the value of a and b

(32) Insert 16 arithmetic means between 27 and - 24

(33) Find the arithmetic sequence whose ninth term = 25 and the arithmetic mean between the third and fifth terms is 10

(34) Find the number of the arithmetic means are inserted between 1 and 17 and the seventh mean equals three times the second mean

(35) **Physics:** Kareem has started to ride his motorcycle from the highest point of a downhill. He covered a distance of 100 cm in the first second. In each next second, he was covering a distance that increases in a magnitude of 120 cm from the directly previous distance. Find the distance which he covers in the tenth second.

(36) **Trade:** A man has bought a motorcycle and he has agreed with the seller to pay its price in monthly installment forming an arithmetic sequence whose n^{th} term is 120 n + 80. If the last installment was 1400L.E., find the number of the installments.

(37) **Creative thinking:** If ℓ and m are two arithmetic means between x and y where $\ell > m$, prove that: $\ell - m = \frac{1}{3}(x - y)$.

Arithmetic Series

1 - 4

Sum of Arithmetic Series

German scientist-
Karl Gauss
1777 - 1855

The students has made loud noise in the class and their teacher wanted to punish them indirectly. He had them sum the numbers from 1 up to 100 to involve them at work for longer time and to get quiet. After a few minutes, a seven - year old child named (Karl Gauss) had told the teacher that the sum is (5050) . In turn , the teacher asked him how he reached the correct answer.

The child had answered that the addition sum of 101 is repeated from adding the following numbers:

$1 + 100$, $2 + 99$, $3 + 98$, ..., $50 + 51$ for fifty times

Thus, the addition sum is $50 \times 101 = 5050$

You will learn

- Arithmetic series.
- Find the sum of n terms of an arithmetic series in terms of its first and last terms
- Find the sum of n terms of an arithmetic series in terms of its first term and the common difference.

Key - terms

- arithmetic series
- Summation notation

Matrials

- Scientific calculator

Learn

The Sum of the First n Terms of an Arithmetic Sequence

First: the sum of n^{th} terms of an arithmetic sequence in terms of its first term(a) and the last term (ℓ)

its common difference d, the number of its terms is denoted by the symbol S_n and is given by the following series:

$$S_n = a + (a + d) + (a + 2d) + ... + (\ell - d) + \ell \quad (1)$$

The series can be also written in the form:

$$S_n = \ell + (\ell - d) + (\ell - 2d) + ... + (a + d) + a \quad (2)$$

By adding the two equations (1) and (2), we deduce that:

$2S_n = (a + \ell) + (a + \ell) + (a + \ell) + ... + (a + \ell)$ to n times

i.e. $2S = n(a + \ell)$ by dividing the two sides by 2 $\boxed{S_n = \frac{n}{2}(a + \ell)}$

Example Use the summation notation Σ

(1) Find $\sum_{r=5}^{24} (4r - 3)$

Remember

Solution

$n = 24 - 5 + 1 = 20$ **Find the number of the sequence terms**

$T_n = 4n - 3$ **n^{th} term of the sequence**

$T_5 = 4 \times 5 - 3 = 17$, $T_{24} = 4 \times 24 - 3 = 93$

$S_n = \frac{n}{2}(a + \ell)$ **summation formula**

$S_{20} = \frac{20}{2}(17 + 93) = 1100$ **by substituting $a = 17, \ell = 93, n = 20$**

Try to solve

(1) Find:

a) $\sum_{K=1}^{20} (6K + 5)$

b) $\sum_{m=7}^{32} (12 - 5m)$

Example

(2) Find the sum of the arithmetic series $2 + 5 + 8 + ... + 62$

Solution

$\ell = a + (n - 1)d$ **n^{th} term of the sequence**

$62 = 2 + (n - 1) \times 3$ **by substituting $a = 2, d = 3$ and $\ell_n = 62$**

i.e.: $3n - 3 + 2 = 62$

$3n - 1 = 62$ **then $n = 21$**

$S_n = \frac{n}{2}(a + \ell)$ **summation formula**

$S_{21} = \frac{21}{2}(2 + 62) = 672$ **by substituting $a = 2, n = 21, T_n = 62$**

Try to solve

(2) Find:

a) The sum of the terms of the arithmetic series $89 + 85 + 81 + ... + 33$

b) The number of the terms of the arithmetic sequence whose first term equals 3, last term equals 39 and the sum of the first n terms equals 210

Second: Finding the sum of n^{th} terms of an arithmetic sequence in terms of its first term and its common difference.

You know that $\ell = a + (n - 1)d$ and $S_n = \frac{n}{2}(a + \ell)$

by substituting from the first relation in the second relation, then:

$S_n = \frac{n}{2}[a + a + (n - 1)d]$ **i.e.:** $\boxed{S_n = \frac{n}{2}[2a + (n - 1)d]}$

Example

(3) In the arithmetic series 5 + 8 + 11 + ... find:

a) The sum of its first twenty terms of the series .

b) The sum of ten terms starting from the seventh term .

c) The sum of the sequence terms starting from T_{10} up to T_{20}

Solution

$a = 5$, $d = 8 - 5 = 3$

a) $S_n = \frac{n}{2}[2a + (n-1)d]$ **summation formula**

$S_{20} = \frac{20}{2} \times [2 \times 5 + (20-1) \times 3]$ **by substituting $a = 5$ and $d = 8 - 5 = 3$**

$S_n = 10(10 + 19 \times 3)$

$= 10 \times 67 = 670$ **by simplifying**

b) $T_n = a + (n-1)d$ **n^{th} term of the sequence**

$T_7 = a + 6d$

$= 5 + 6 \times 3 = 23$ **by substituting $a = 5$, $d = 3$ and $n = 7$**

$S_{10} = \frac{10}{2} \times [2T_7 + (10-1) \times 3]$ **by substituting in summation formula**

$S_{10} = 5 \times [2 \times 23 + 27]$

$= 5 \times 73 = 365$ **by simplifying**

c) The sum of the sequence terms starting from T_{10} up to T_{20}

$T_n = a + (n-1)d$ **n^{th} term of the sequence**

$T_{10} = a + 9d$

$= 5 + 9 \times 3 = 32$ **by substituting $a = 5$, $d = 3$**

$T_{20} = a + 19d = 5 + 19 \times 3 = 62$

$S_n = \frac{n}{2}(a + \ell)$ **summation formula**

$S_{11} = \frac{11}{2}(T_{10} + T_{20})$

$= \frac{11}{2}(32 + 62) = 517$ **by substituting $T_{10} = 32$, $T_{20} = 62$, $n = 11$**

Think:

Are there other solutions to find the sum of the sequence terms starting from T_{10} up to T_{20}

Try to solve

(3) In the arithmetic sequence (9 , 12 , 15 , ...), find :

a) The sum of its first fifteen terms .

b) The sum of the sequence terms starting from the fifth term up to the fifteenth term.

c) The number of terms whose sum equals 750 starting from the first term .

Example

(4) Find the arithmetic sequence in which : $T_1 = 11$, $T_n = 87$ and $S_n = 980$

Solution

a Finding the value of n

$S_n = \frac{n}{2}(a + \ell)$ — summation formula

$980 = \frac{n}{2}(11 + 87)$ — by substituting $T_1 = 11, T_n = 87$ and $S_n = 980$

$98 \times \frac{n}{2} = 980$ then: $n = 20$ terms — by simplifying

b Finding the value of d

$T_n = a + (n - 1)d$ — n^{th} term

$87 = 11 + 19d$ — by substituting $T_1 = 11, n = 20$ and $T_n = 87$

$19d = 87 - 11 = 76$ — by simplifying

$d = 4$ — and by dividing by 19

c Forming the sequence: $T_2 = 11 + 4 = 15$, $T_3 = 15 + 4 = 19$

The arithmetic sequence is (11 , 15 , 19 , ... , 87)

Try to solve

(4) Find the arithmetic sequence in which:

a $T_1 = 23$, $T_n = 86$, $S_n = 545$ **b** $T_1 = 17$, $T_n = -95$, $S_n = -585$

Example

Physics:

(5) An object fell from a height of 490m under the effect of the gravity. Disregarding the air resistance, it covers a distance of 4.9m in the first second, 14.7 in the second second and 24.5 in the third second and so on.. , **Find:**

a The distance covered by the object in the sixth second.

b The sum of distances covered in the first eight seconds.

c When does the object hit the ground?

Solution

The distances covered in the first three seconds respectively are: 4.9 , 14.7 , 24.5 , ...

The distances represent an arithmetic sequence in which : $a = 4.9$ and $d = 14.7 - 4.9 = 9.8$

a By substituting: $a = 4.9$ and $d = 9.8$ in the rule: $T_n = a + (n - 1)d$

$T_6 = a + 5d$ then $T_6 = 4.9 + 5 \times 9.8 = 53.9$ meters

b By substituting: $a = 4.9$ and $d = 9.8, n = 8$ in the rule: $S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_8 = \frac{8}{2}(2 \times 4.9 + 7 \times 9.8) = 313.6$ meters.

c by substituting a = 4.9 , d = 9.8, S = 490 in the rule: $S_n = \frac{n}{2}[2a + (n - 1) d]$

$490 = \frac{n}{2}[2 \times 4.9 + (n - 1) \times 9.8]$ **by expanding the brackets and simplifying**

$490 = n(4.9 + 4.9n - 4.9)$ **i.e.:** $4.9n^2 = 490$

by dividing the two sides by 4.9 **then** $n^2 = 100$

by taking the positive square root of two sides: n = 10 (notice that n > 0)

Try to solve

5. **Sports:** Kareem is preparing himself for a long-distance race. He has decided to run 4 km on the first day, then increased the distance for half a kilometer daily.
   - **a** Find the distance , which kreem covers on the seventh day.
   - **b** Find the sum of the distances, which kareem covers in the first week (A week is seven days).
   - **c** How many days does kareem need to cover 81 km, if he continues to exercise according to this pattern? .

Exercises (1 - 4)

Complete

1. The sum of the first n terms of an arithmetic sequence whose first term is a and last term is ℓ is ....
2. The sum of the first n^{th} terms of an arithmetic sequence whose first term is a and its common difference is d is ....
3. $\sum_{k=1}^{5} (2k + 1) =$ ....
4. The sum of the first ten even number in the set of the natural numbers equals ....
5. The sum of the odd natural numbers which is greater than 10 and less than 30 equals ....
6. The sum of the natural numbers which is divisible by three and included between 30 and 50 equals ....

Choose the correct answer

7. The value of the arithmetic series $\sum_{r=1}^{5} (2r + 1)$ equals:
   - **a** 25 **b** 30 **c** 35 **d** 40
8. The value of the series: 4 + 9 + 14 + ... + 5n – 1 by using the summation notation equals :
   - **a** $\sum_{r=4}^{n} (5r - 1)$ **b** $\sum_{r=1}^{n} (5r + 1)$ **c** $\sum_{r=1}^{n} (5r - 1)$ **d** $\sum_{r=1}^{5n-1} (3r + 1)$
9. The value of the series: 7 + 12 + 17 + 22 by using the summation notation equals:
   - **a** $\sum_{r=1}^{4} (5r + 2)$ **b** $\sum_{r=1}^{4} (4r + 3)$ **c** $\sum_{r=1}^{4} (7r + 1)$ **d** $\sum_{r=1}^{4} (3r + 4)$

(10) The sum of the terms of the arithmetic sequence $(3, 5, 7, \ldots, (2n+1))$ starting from its first term equals :

a $n(n+1)$ b $n(n+2)$ c $n(n+5)$ d $n(n+2)\,n(n+3)$

Discover the error

(11) To find the greatest sum of an arithmetic sequence, we find the number of its positive terms by putting $S_n > 0$ to find the value of n , and then we can find the greatest sum.

(12) To find the least sum of an arithmetic sequence, we find the number of its negative terms by putting $S_n < 0$ to find the value of n and then we can find the least sum.

(13) To find the number of the terms of an arithmetic sequence at which the sum is vanished, we put $S_n = 0$, then $a + (n-1)d = 0$
[where $n \neq 0$] , then we find the number of terms.

(14) If the sum of the first n terms of the terms of an arithmetic sequence is given by the relation $S_n = \frac{n}{2}(3n+5)$ then $T_n = S_{n+1} - S_n$.

Answer the following questions

(15) Find the sum of the first ten terms of the arithmetic sequence $(14, 18, 22, \ldots)$.

(16) Find the sum of the first thirty terms of the sequence (T_n) where $T_n = 2n + 3$

(17) Find the sum of the terms of the arithmetic sequence $(2, 5, 8, \ldots, 80)$.

(18) Find the number of the terms necessary to be taken from the sequence $(16, 20, 24, \ldots)$ starting from the first term to get a sum equal to 456

(19) Find the number of the terms necessary to be taken from the sequence $(27, 24, 21, \ldots)$ starting from the first term to vanish the sum.

(20) If the sum of n-terms from an arithmetic sequence is identified by the rule: $S_n = 2n(7-n)$, find:

a T_7

b The number of terms necessary to be taken from the sequence starting from the first term to make the sum equal to -240

(21) Find the least number of terms that can be taken from the sequence $(89, 81, 73, \ldots)$ starting from the first term to make the negative sum.

(22) Find the greatest number of terms that can be taken from the sequence $(25, 21, 17, \ldots)$ starting from the first term to make the positive sum

23) In the arithmetic sequence (5 , 8 , 11 , ...) ,find:

a) The sum of its first twenty terms.

b) The sum of ten terms starting from the seventh term.

c) The sum of the sequence terms starting from T_{10} up to T_{20}

24) In the sequence $(T_n) = (32 , 28 , 24 , ...)$.

a) Find the order and value of its first negative term.

b) Find the number of terms which makes the sum greater than zero.

25) In the arithmetic sequence (25 , 23 , 21 , ...), find:

a) The greatest sum of the sequence.

b) The number of terms whose sum = 120 starting from the first term " Explain the existance of two answers".

26) Find the arithmetic sequence whose first term = 12 , last term = - 26 , and the sum of its terms = - 140.

27) Find the arithmetic sequence (T_n) whose second term = 13 , and the sum of its first ten terms = 235 .

28) An arithmetic sequence in which $T_{36} = 0$. If the sum of the first n terms = twice the sum of its first five terms, find the value of n.

29) If n arithmetic means are inserted between 1 and 31 and the ratio of the seventh mean to the last mean is $\frac{15}{29}$ find n and sum of the sequence?

30) Find the arithmetic sequence in which $T_4 = 24$ and the ratio between the sum of the first five terms to the sum of the next five terms directly is 1: 2 .

31) What is the arithmetic sequence whose first term is greater than twice its fifth term by 2 and the arithmetic mean of the third and sixth terms is 16? How many terms should be taken starting from the first term to make the sum equal to zero?

32) **Saving:** Zyad saves 15 L.E from his daily work. If he saves an amount of money increasing for 2 L.E every day more than the day before directly. Find the sum of what he can save within 15 days.

33) **Arts:** A school stage accommodates for 16 rows, if the first row contains 16 seats and each row next to it contains a number of seats more than the directly previous row for 4 seats, find the number of seats in this stage.

(34) **Income:** Kareem has started his working life with a yearly salary of 19200 L.E. If he gets a yearly raise of 480 L.E, how much he will get by the end of the tenth year

(35) **Trade:** A man has borrowed an amount of money, he has handled to pay it back over 10 installments. How much is the loan if the first installment is 500 L.E and each next installment is more than the directly previous installment for 200 L.E?

(36) **Maintenance:** A company has assigned one of its buildings to be completely maintained and it has stated a date to receive the building. There was a condition in the maintenance contract stating that ; in case of delaying the date of receiving the building, the contractor will pay 100 L.E as a fine for the first day and this fine increases 100 L.E more than the day before . How much will the contractor pay if he delays to receive the building for five days?

Activities

Kareem has a store for food commodities. He arranges the tuna cans in rows so that he puts 12 cans in the (first) lowest row , 11 in the next row and 10 in the next and so on.

1- Find the number of the tuna cans in the seventh row.

2- In which row are the tuna cans three cans?

3- Find the number of the tuna cans starting from the first row up to the last row containing one tuna can.

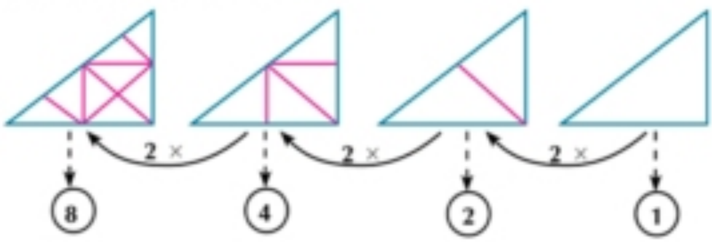
4- Tell why Kreem arranges the tuna can in rows in this way.

5- Log in internet and search for the places where tuna widly live. What are the healthy benefits of tuna?

Geometric Sequences

1 - 5

Check the small triangles in each of the following figures, then find their numbers. What do you notice?



Definition 1

The sequence (T_n) where $T_n \neq 0$ is called a geometric sequence if $\frac{T_{n+1}}{T_n}$ = a constant for each $n \in Z^+$

The constant is called the common ratio of the sequence and is denoted by the symbol (r)

Example

1. Show which of the following sequences (T_n) is geometric , then find the common ratio of each :

   a $(T_n) = (2 \times 3^n)$ **b** $(T_n) = (4\,n^2)$

   c The sequence (T_n) where: $T_1 = 12$, $T_n = \frac{1}{4} \times T_{n-1}$ (where $n > 1$)

Solution

a $\because \frac{T_{n+1}}{T_n} = \frac{2 \times 3^{n+1}}{2 \times 3^n} = 3^{n+1-n} = 3$ **(constant)**

$\therefore$ the sequence is geometric and its common ratio $r = 3$

b $\because \frac{T_{n+1}}{T_n} = \frac{4\,(n+1)^2}{4\,n^2}$ **(is not a constant)**

$\therefore$ The sequence is not geometric

c $\because T_n = \frac{1}{4} \times T_{n-1}$ (where $n > 1$)

$\therefore \frac{T_n}{T_{n-1}} = \frac{1}{4}$ **(constant)**

$\therefore$ The sequence is geometric and its common ratio $r = \frac{1}{4}$

You will learn

- Definition of the geometric sequence
- Graphical representation of the geometric sequence
- n^{th} term of the geometric sequence
- Identifying the geometric sequence
- Geometric means
- The relation between an arithmetic mean and a geometric mean of two numbers

Key - term

- Geometric Seqence
- n^{th} Term
- Increasing Seqence
- decreasing Sequence
- Alternating sign Sequence
- Geometric Mean

Matrials

- Scientific Calculator
- Graphic programs

Try to solve

(1) Show which of the following sequences is geometric , then find its common ratio in case it is a geometric sequence:

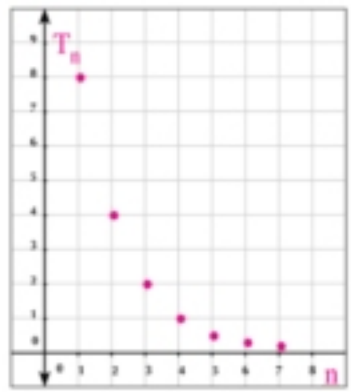
a $(T_n) = (96 , 48 , 24 , 12 , 6 , 3)$ b $(T_n) = (\frac{1}{243} , -\frac{1}{81} , -\frac{1}{27} , \frac{1}{9} , \frac{1}{3})$

c $(T_n) = (5 \times 2^n)$ d $(T_n) = (3 (n + 1)^2)$

Graphical Representation of The Geometric Sequence

Example

(2) Find the next four terms of the geometric sequence (8, 4, 2, ...) then represent the first seven terms graphically.



Solution

The sequence common ratio $= \frac{1}{2}$ thus, the next four terms are:

$1 , \frac{1}{2} , \frac{1}{4} , \frac{1}{8}$

the domain is $\{ 1 , 2 , 3 , 4 , 5 , 6 , 7\}$

the range is $\{ 8 , 4 , 2 , 1 , \frac{1}{2} , \frac{1}{4} , \frac{1}{8}\}$

From the graph , we notice that:

- The terms of the sequence are decreasing where $a > 0$, $0 < r < 1$
- The graphical representation of the geometric sequence follows the exponential function and not a first - degree function as in the arithmetic sequence .

Verbal expression: Write down a description to the geometric sequence so that it is (increasing , decreasing or alternating sign) in each of the following cases.

First term	Positive				negative			
common ratio	$r > 1$	$0 < r < 1$	$-1 < r < 0$	$r < -1$	$r > 1$	$0 < r < 1$	$-1 < r < 0$	$r < -1$
description	increasing	....	....	....	....	....	alternating sign	....

Try to solve

(2) Find the next four terms of the geometric sequence $(\frac{1}{27} , \frac{1}{9} , \frac{1}{3} ...)$, then represent the first seven terms graphically.

The n th Term of the geometric sequence

From definition (1) the n^{th} term of the geometric sequence (T_n) whose first term is a and common ratio is r can be deduced as follows:

$T_1 = a$, $T_2 = ar$ and $T_3 = ar^2$, by continuing this pattern, we find that the n^{th} term of this sequence is: $T_n = a\, r^{n-1}$

Example

(3) In the geometric sequence (2 , 4 , 8 , ...), find:

(a) The fifth term (b) the order of the term whose value is 512

Solution

$\because a = 2,\ r = \frac{4}{2} = 2,\ T_n = a \times r^{n-1}$

$\therefore T_5 = a\,r^4 = 2 \times 2^4 = 2 \times 16 = 32$ **i.e. the value of the fifth term is 32**

$\because T_n = a \times r^{n-1}$ $\therefore 2 \times 2^{n-1} = 512$ **By dividing the two sides by 2**

$\therefore 2^{n-1} = 2^8$ $\therefore n - 1 = 8$ $\therefore n = 9$

i.e. the term whose value is 512 is the ninth term.

Try to solve

(3) Prove that the sequence (T_n) where $T_n = 2 \times 3^{n-5}$ is a geometric sequence, then find its seventh term.

Identifying The Geometric Sequence

The geometric sequence can be identified whenever its first and common ratio are known (given).

Example

(4) (T_n) is a geometric sequence and all of its terms are positive. If $T_3 + T_4 = 6T_2$, $T_7 = 320$, find this sequence.

Solution

$\because T_3 + T_4 = 6T_2$ $\therefore ar^2 + ar^3 = 6 \times ar$

$\therefore ar\,(r + r^2) = 6 \times ar$ **by dividing the two sides by** ar (where a and r donot equal zero)

$\therefore r^2 + r = 6$ **i.e.:** $r^2 + r - 6 = 0$

$\therefore (r - 2)(r + 3) = 0$ $\therefore r = 2$ or $r = -3$ **refused (because the terms are positive)**

$\because T_7 = 320$ $\therefore ar^6 = 320$ **by substituting $r = 2$**

$\therefore a \times 2^6 = 320$ $\therefore 64\,a = 320$ **by dividing the two sides by 64**

$\therefore a = 5$ the sequence is (5, 10, 20, ...)

Using the scientific calculator to write the geometric sequence

To write down the geometric sequence in which a = 5 and r = 2 we do the following:

We write the value of a (number 5) then press (=) then press(×) and put the value of r (number 2) then press (=) then we get the second term of the sequence. By pressing (=) repeatedly, we get the next terms and so on ....

Try to solve

(4) (T_n) is a geometric sequence in which $T_5 = 8\,T_2$, $T_4 + T_6 = 240$. find this sequence.

(5) If the sum of the second and third terms of a geometric sequence is 12 and the product of its first and fourth terms equals 27, find this sequence.

Example

(5) **Education:** If the number of the students accepted in the high school stage in an educational administration increases at a rate of 4% yearly. How many students will be there after 6 years if the number of the students is 2400 students right now?

Solution

$\because$ Number of students now = 2400

$\therefore$ Number of students in the second year $= 2400\,(1 + 0.04)$

$= 2400\,(1.04)$

$\therefore$ Number of students in the third year $= 2400(1.04) + 2400(1.04) \times 0.04$

$= 2400(1.04)\,(1 + 0.04) = 2400(1.04)^2$ and so on ...

i.e. the numbers of students form the geometric sequence

$(2400\ ,\ 2400(1.04)\ ,\ 2400(1.04)^2\ ,\ ...)$

$a = 2400, r = 1.04\ ,\ n = 6$ by substituting the rule of n^{th} term of the geometric sequence

$T_n = a \times r^{n-1}$ $\qquad T_n = (2400) \times (1.04)^5 = 2919.966966$

i.e. the number of students after 6 years equals 2920 students approximately.

Try to solve

(6) **Physics:** A rubber ball falls from a height of 240 above the ground surface. How high is the ball after the seventh bounce, If the ball bounces back $\frac{3}{4}$ of its directly previous height?

Geometric Means

The geometric means are similar to the arithmetic means. They are the terms located between two non- successive terms in the geometric sequence . the common ratio of the geometric sequence is used to find these means .

Definition 2

If a, b and c are three successive terms of a geometric sequence, then b is known as the geometric mean between the two numbers a and c where: $\frac{b}{a} = \frac{c}{b}$ i.e. $b^2 = a\,c$ then $b = \pm\sqrt{a\,c}$

Tip

the geometric mean of a set of positive real values $T_1, T_2, T_3, \dots, T_n$ is known as the n^{th} root of the product of these values. I.e. the geometric mean = $\sqrt[n]{T_1 \times T_2 \times \dots \times T_n}$

Verbal expression:

The geometric means which can be inserted between two numbers depend on the sign of those two numbers. Explain.

Inserting a number of geometric means between two known quantities:

Example

(6) Find the geometric means of the sequence: (4 , ... , ... , ... , ... , ... , 2916)

Solution

1- Find the number of the sequence terms

There are five means between the first and last terms in the geometric sequence

Thus , the number of the terms of the sequence n = 2 + 5 = 7

2- Find the value of r

By using the rule: $\ell = a\, r^{n-1}$

$2916 = 4 \times r^{7-1}$ by substituting: a = 4, ℓ = 2916, n = 7

i.e.: $4 \times r^6 = 2916$ by dividing the two sides by 4 $r^6 = 729$

i.e.: $r^6 = (\pm 3)^6$ then $r = \pm 3$

3- Use the value of r to find the geometric means required

The terms are:

4 , 12 , 36 , 108 , 324 , 972 , 2916 (each term × 3) **or**

4 , -12 , 36 , -108 , 324 , -972 , 2916 (each term × −3)

the required means are 12, 36, 108, 324, 972 **or** - 12, 36, - 108, 324, - 972

Try to solve

(7) Insert six geometric means between $\frac{1}{4}$ and 32

Tip

If: $T_1, T_2, T_3, \ldots T_n$ are positive real numbers , then : $\frac{T_1 + T_2 + T_3 + \ldots T_n}{n} \geqslant \sqrt[n]{T_1 T_2 T_3 \ldots T_n}$ and the equality is satisfied only when: $T_1 = T_2 = T_3 = \ldots = T_n$

The relation between the arithmetic and geometric means of two numbers:

If $x, y \in R^+ , x \neq y$

then: the arithmetic mean (A) = $\frac{x+y}{2}$ and the positive geometric mean (G) = $\sqrt{xy}$

$\therefore A - G = \frac{x+y}{2} - \sqrt{xy} = \frac{x - 2\sqrt{xy} + y}{2}$

$\therefore = \frac{(\sqrt{x} - \sqrt{y})^2}{2} > 0$

(by putting the expression in a form of a complete square)

$\therefore A > G$. So that the positive geometric mean is greater than the negative geometric mean

Then the arithmetic mean of two different positive real numbers is greater than their geometric mean.

Critical thinking: What do you expect about the relation between the arithmetic and geometric means of two equal positive real numbers?

Example

(7) If 6a , 3b , 2c , 2 d are positive quantities in an arithmetic sequence, prove that $b\,c > 2\,a\,d$

Solution

$\because$ 3b is an arithmetic mean between 6a and 2c

where the arithmetic mean > the positive geometric mean

$\therefore 3\,b > \sqrt{6\,a \times 2\,c}$ by squaring the two sides

$\therefore 9\,b^2 > 12\,a\,c$ (1)

Similarly 2c is an arithmetic mean between 3b and 2d

$\therefore 2c > \sqrt{3\,b \times 2\,d}$ by squaring the two sides

$4c^2 > 6b\,d$ (2)

from (1) and (2)

$9b^2 \times 4c^2 > 12\,ac \times 6b\,d$

By dividing the two sides by 36b c (b , c $\in R_+$)

$\therefore b\,c > 2\,a\,d$

Exercises (1 - 5)

Complete

(1) The fifth term of the sequence (T_n) where $T_n = 2 \times (3)^{n-1}$ equals ....

(2) The n^{th} term of the geometric sequence (3 , - 6 , 12 , ..) is ....

(3) The sixth term of geometric sequence $(\frac{1}{243}, \frac{1}{81}, \frac{1}{27}, ...)$ is ....

(4) The geometric mean of the two numbers 4 and 16 is ....

(5) If the geometric mean of the two numbers 9 and y is 15, then y = ...

(6) If a , b and c are three consecutive positive terms of a geometric sequence , then b < ...

choose the correct answer

(7) The next term of the geometric sequence $(8, 6, \frac{9}{2}, \frac{27}{8}, ...)$ is:

a $\frac{11}{8}$ b $\frac{27}{16}$ c $\frac{9}{4}$ d $\frac{81}{32}$

(8) All the following sequence are geometric except:

a $(3\ ,\ -6\ ,\ 12\ ,\ -24\ ,\ ...)$

b $(\log a\ ,\ \log a^2\ ,\ \log a^3\ ,\ \log a^4\ ,\ ...)$

c $(\frac{3}{2}\ ,\ 1\ ,\ \frac{2}{3}\ ,\ \frac{4}{9})$

d $(\frac{3b}{2a}\ ,\ 3\ ,\ \frac{6a}{b}\ ,\ \frac{12a^2}{b^2})$

(9) Of all the following sequence , the geometric sequence is:

a $(T_n) = (4n^2)$ for each $n \geqslant 1$

b $(T_n) = (\frac{1}{4} \times T_{n-1})$ for each $n \geqslant 2$

c $(T_n) = (2^n - 1)$ for each $n \geqslant 1$

d $(T_n) = (\log(3 \times 2^n))$ for each $n \geqslant 1$

(10) If a , b and c three consecutive positive terms of a geometric sequence, then:

a $\frac{a+c}{2} > b$

b $\frac{a+c}{2} < b$

c $\frac{a+c}{2} = b$

d $b^2 = a + c$

Discover the error:

(11) The terms of the geometric sequence are represented by a set of separated points which are collinear .

(12) The sequence (T_n) is called geometric if $\frac{T_n}{T_{n+1}}$ equals a constant known as the common ratio of the sequence (for each $n \geqslant 1$).

(13) The geometric sequence is decreasing if its common ratio $r \in]-1\ ,\ 0[$

(14) The geometric means are known that they are the terms lying between two non-consecutive terms of a geometric sequence and can be found whenever the value of those two terms are known.

(15) The arithmetic mean of two different real numbers is greater than their geometric mean.

Answer the following questions:

(16) If (T_n) is a sequence where $T_n = 5 \times 2^n$, prove that it is a geometric sequence, then find its first three terms.

(17) In the geometric sequence $(\frac{1}{8}\ ,\ \frac{1}{4}\ ,\ \frac{1}{2}\ ,\ -1\ ,\ ...)$, find:

a Its tenth term.

b The order of the term whose value = 1024

(18) Find the geometric sequence whose common ratio $= \frac{1}{2}$ and its third term = 24 .

(19) Find the geometric sequence in which $T_3 = 12$ and $T_8 = 384$.

20. Show that the sequence (T_n) where: $T_n = \frac{3}{8}(2)^n$ is a geometric sequence, then find its eighth term and the order of the term whose value is 768

21. Find the geometric mean between 16 and 49.

22. Find the two numbers whose arithmetic mean is 5 and their geometric mean is 3.

23. Find two positive numbers whose positive geometric mean is greater than one of those two numbers by 2 and is less than the other number by 3.

24. Insert five positive geometric means between $\frac{8}{27}$, $\frac{27}{8}$

25. If several geometric means are inserted between 2 and 1458 and the ratio between the sum of the first two means to the sum of the last two means is 1:27, find the number of these means.

26. Find the geometric sequence whose all terms are positive, the first term is four times the third term and the sum of the second and fifth terms = 36.

27. If x, y, z and ℓ are positive quantities of a geometric sequence, prove that $x + \ell > y + z$.

28. **Environment:** Water is poured in a water tank at a rate of twice the directly previous day. How many days does the water tank hold 1536 liters if 12 liters has been poured in the tank on the first day?

29. **Population:** The population of a city increases at a constant rate of 3% per year. How many populations are there in this city after 5 years if the populations are 600000 at present?

30. **Percentage:** The price of a new brand car is 120000 L.E. How much is that car after 5 years if its price decreases at a rate of 12% yearly?

31. **Income:** The salary of an employee is 1200 L.E and he gets a yearly constant raise at a rate of 6% more than his salary in the directly previous year. How much is his salary after 6 years?

32. **Creative thinking:**

  a If $a + b + c = 1$ where: a, b, c are different and positive quantities.
  prove that: $(1 - a)(1 - b)(1 - c) > 8abc$

  b If : $x \in R^+$, $x \neq 1$
  prove that: $x + \frac{1}{x} > 2$

Geometric Series

1 - 6

Sum of the Geometric Series

We know that the geometric series is the sum of the terms of the geometric sequence and the sum of n terms is denoted by the symbol S_n.

Sum of the First n Terms of a Geometric Series

First: To find the sum of n terms of a geometric series in terms of its first term and common ratio:

If $a + ar + ar^2 + \dots + ar^{n-1}$ is a geometric series whose fist term is a and its common ratio is r , then the sum of S_n of this series can be found as follows:

$$S_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} \qquad (1)$$

By multiplying the two sides by r then:

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n \qquad (2)$$

By subtracting the two equations, then:

$$S_n - rS_n = a - ar^n$$

i.e.: $S_n(1 - r) = a(1 - r^n)$

By dividing the two sides by $(1 - r)$ in a condition $1 - r \neq 0$

$$S_n = \frac{a(1 - r^n)}{1 - r}, r \neq 1$$

You will learn

- Sum of the geometric series.
- Forming the geometric sequence
- Infinite geometric series
- Sum of infinite geometric series.
- Converting the recurring decimal into a common fraction .

Key - term

- Geometric Series
- Infinite Geometric Series

Matrials

- Scientific Calculator
- Graphical Programs

Example

1) Find the sum of the geometric sequence in which : $a = 3$, $r = 2$, $n = 8$

Solution

$S_n = \frac{a(1 - r^n)}{1 - r}$ **the sum formula of the geometric sequence**

$S_8 = \frac{3(1 - 2^8)}{1 - 2}$ **by substituting:** $a = 3$, $r = 2$ **and** $n = 8$

$S_8 = 3 \times 255 = 765$ **by simplifying**

Try to solve

1) Find the sum of the following two geometric sequences in which:

a) $a = 4$, $r = 3$, $n = 6$ b) $a = 1000$, $r = \frac{1}{2}$, $n = 10$

Second: to find the n terms of a geometric series in terms of its first and last terms

We know that: $S_n = \frac{a - a r^n}{1 - r}$ **(1)**

and: $\ell = a r^{n-1}$ by multiplying the two sides by r

then $\ell r = a r^n$ **(2)**

by substituting from **(2)** in **(1)** then: $S_n = \frac{a - \ell r}{1 - r}, r \neq 1$

Example

(2) Find the sum of the geometric series : $1 + 3 + 9 + ... + 6561$

Solution

$S_n = \frac{a - \ell r}{1 - r}$ **the sum formula of the geometric sequence**

$S = \frac{1 - 6561 \times 3}{1 - 3}$ **by substituting :** a **= 1, r = 3 and ℓ = 6561**

$S = \frac{19682}{2} = 9841$ **by simplifying**

Try to solve

(2) Find the sum of the following two geometric sequences:

a $a = 9$, $r = 3$, $\ell = 6561$

b $a = 2048$, $r = \frac{1}{2}$, $\ell = 128$

Using the Summation Notation

Example

(3) Find $\sum_{r=5}^{12} 3(2)^{r-1}$

Solution

$T_5 = a = 3(2)^{5-1} = 48$, $r = 2$, $n = 12 - 5 + 1 = 8$

$S_n = \frac{a(1 - r^n)}{1 - r}$ **the sum formula of the geometric sequence**

$S_8 = \frac{48(1 - 2^8)}{1 - 2}$ **by substituting:** a **= 48, r = 2, n = 8**

$S_8 = 48 \times 255 = 12240$ **by simplifying**

Think: Can you find the sum in the previous example in terms of a , ℓ and r ? How?

Try to solve

(3) Find the value for each of the following two geometric series:

a $\sum_{r=7}^{16} \frac{1}{8}(2)^{r-1}$

b $\sum_{r=3}^{11} 16\left(\frac{1}{2}\right)^{r-1}$

Example Forming the geometric sequence

(4) If the sum of the first n terms of a geometric sequence is given by the rule: $S_n = 128 - 2^{7-n}$, find the sequence and its seventh term.

Solution

let: n = 1 $\therefore S_1 = 128 - 2^{7-1} = 128 - 64 = 64$ **i.e.** $T_1 = 64$

let: n = 2 $\therefore S_2 = 128 - 2^{7-2} = 128 - 32 = 96$

$\because T_1 + T_2 = S_2 \quad \therefore 64 + T_2 = 96 \quad \therefore T_2 = 96 - 64 = 32$

let: n = 3 $c_3 = 128 - 2^{7-3}$ **i.e.** $S_3 = 128 - 16 = 112$

$\because T_1 + T_2 + T_3 = S_3$ **i.e.** $T_3 = S_3 - S_2$

$\therefore T_3 = 112 - 96 = 16$

$\therefore$ the sequence is: (64, 32, 16, ...), $T_7 = ar^6 = 64\left(\frac{1}{2}\right)^6 = 1$

Notice: from the previous, we can deduce that $T_n = S_n - S_{n-1}$ and find the two values of a and r then $T_7 = S_7 - S_6 = (128 - 2^{7-7}) - (128 - 2^{7-6}) = -1 + 2 = 1$

Try to solve

(4) If the sum of the first n terms of a geometric sequence is given by the rule $S_n = 1024 - 2^{10-n}$ find the sequence.

(5) Find the geometric sequence whose first term = 243, last term = 1, and the sum of its terms = 364

Example

(5) **Geometry** The opposite figure shows an equilateral triangle whose side length is 40cm. Another triangle is drawn interiorly through connecting the points representing the mid-points of the greatest triangle. The interior triangles are repeatedly drawn the same way, find the sum of the perimeters of the first 10 triangles in this pattern to the nearest integer.

Solution

The perimeter of the greatest triangle = 3 × 40 = 120

The perimeter of the next smaller triangle = 3 × 20 = 60

The perimeter of the next smaller triangle = 3 × 10 = 30

i.e. the pattern is : 120, 60, 30, ... to 10 terms

sum of perimeters = 120 + 60 + 30 + ... it is the sum of a geometric series

the sum formula of the geometric sequence: $S_n = \frac{a(1 - r^n)}{1 - r}$

by substituting: a = 120, $r = \frac{1}{2}$, n = 10

By simplifying and using the calculator $S_{10} = \frac{120\left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}} \simeq 240$

Remember

Perimeter of equilateral triangle = 3 × its side length

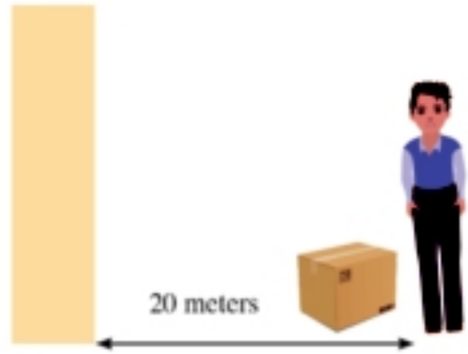
Try to solve

(6) **Biology:** Culturing bacteria is replicated every day (in a nutritional medium) . How many bacteria are there after ten days if their number is 800 on the first day ?

Infinite geometric series

Think and discuss

Zyad has wanted to move a box in the direction of a wall distant 20 m from him over some stages so that the distance traveled by the box equals half the remaining distance after each stage. Can Zyad reach the wall?

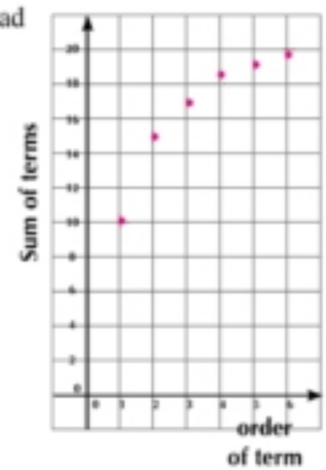


You can answer this question through learning infinite geometric series .

Definition 2

The infinite geometric series has an infinite number of terms. If their sum is a real number, the series is convergent because its sum gets near to a real number . If the series doesnot have a sum , it is divergent

In Think and discuss, the sum of the distances traveled by Zyad is given by the series 10 + 5 + 2.5 +1.25 + ... when ever its terms increase, the sum gets nearer to 20 m. It is the real sum of the series. As a result , we can assume that Zyad will reach the wall when the terms of the sequence increase infinitely. The figure opposite illustrates the graphical representation of the sum of S_n , so the sum of the convergent series gets near to a real number where $|r| < 1$ and the series is divergent if the sum does not get near to a real number where $|r| \geqslant 1$



Example

(6) Which of the following series can you sum an infinite number of its terms ? Explain

a 75 + 45 + 27 + ... **b** 24 + 36 + 54 + ...

Solution

a Find the common ratio of the geometric series $r = \frac{45}{75} = \frac{3}{5}$, then the series is covergent because $-1 < \frac{3}{5} < 1$

b Find the common ratio of the geometric series $r = \frac{36}{24} = \frac{3}{2}$ then the series is divergent because: $\frac{3}{2} > 1$

Remember

If $|r| < 1$ then:
$-1 < r < 1$
If $|r| \geqslant 1$ then:
$r \geqslant 1$ or $r \leqslant -1$

Try to solve

(7) Which of the following geometric series can you sum an infinite number of its terms? Explain

(a) $7 + 21 + 63 + ...$ (b) $\frac{27}{4} + \frac{9}{2} + 3 + 2 + ...$

Sum of Infinite Geometric Series

We knew that the sum of n terms of the geometric series is given by the relation $S_n = \frac{a(1 - r^n)}{1 - r}$

and when we sum an infinite number of its terms, then r^n gets near to zero when $-1 < r < 1$

and the sum is: $S_\infty = \frac{a}{1 - r}$

Critical thinking: Can the sum of an infinite geometric series be found when $|r| \geqslant 1$? Explain .

Verbal expression: Can you find the sum of the geometric series in **think and discuss**? Find the two values of a and b and use the rule of the sum of an infinite number of terms of a geometric series.

Example

(7) Find the sum for each of the following two geometric series if found:

(a) $\frac{81}{8} + \frac{27}{4} + \frac{9}{2} + ...$ (b) $\frac{2}{3} + \frac{5}{6} + \frac{25}{24} + ...$

Solution

(a) Find the common ratio of the geometric sequence: $r = \frac{27}{4} \div \frac{81}{8} = \frac{27}{4} \times \frac{8}{81} = \frac{2}{3}$

$\because -1 < \frac{2}{3} < 1$ $\therefore$ The series has a sum

$\because a = \frac{81}{8}, \ r = \frac{2}{3}$ **by substituting in the sum formula** $S_\infty = \frac{a}{1 - r}$

$\therefore S_\infty = \frac{\frac{81}{8}}{1 - \frac{2}{3}} = \frac{\frac{81}{8}}{\frac{1}{3}} = \frac{243}{8}$

(b) Find the common ratio of the geometric sequence: $r = \frac{5}{6} \div \frac{2}{3} = \frac{5}{6} \times \frac{3}{2} = \frac{5}{4}$

$\because \frac{5}{4} > 1$ $\therefore$ The series is divergent and has no sum

Try to solve

(8) Find the sum for each of the following two geometric series if found:

(a) $96 - 48 + 24 - 12 + ...$ (b) $\frac{7}{5} + \frac{21}{10} + \frac{63}{20} + ...$

Example **Use the summation notation**

(8) Find $\sum_{r=1}^{\infty} 42 \left(\frac{6}{7}\right)^{r-1}$

Solution

Sum formula of the geometric sequence: $S_\infty = \frac{a}{1 - r}$

by substituting: $a = 42$ and $r = \frac{6}{7}$: then $S_\infty = \frac{42}{1 - \frac{6}{7}} = 294$

Try to solve

(9) Find: $\sum_{r=1}^{\infty} 56\left(\frac{3}{4}\right)^{r-1}$

Converting a Recurring Decimal Into a Common Fraction

Example

(9) Put $0.\overline{432}$ in the form of a common fraction

Solution

first: using the sum of an infinite geometric series

$0.\overline{432} = 0.432 + 0.000432 + 0.000000432 + ...$

sum formula of the geometric sequence: $S_\infty = \frac{a}{1 - r}$

let $a = \frac{432}{1000}$, $r = \frac{1}{1000}$ then $S_\infty = \frac{\frac{432}{1000}}{1 - \frac{1}{1000}}$

$S_\infty = \frac{432}{1000} \times \frac{1000}{999} = \frac{16}{37}$ **by simplifying**

Try to solve

(10) Put each of the following decimals in the form of a common fraction $0.\overline{6}, 0.\overline{63}, 0.4\overline{6}, 0.6\overline{54}$

Applications on the Sequences

Example

(10) The following table represents the salary of an employee in five consecutive years in L.E known that this income is subjected to a geometric sequence.

Year	first	second	third	fourth	fifth
Salary in L.E	288	432	648	....	....

a) Find the common ratio of the geometric sequence.

b) Find the salary of that employee within the fourth and fifth years.

c) Find the sum of what this employee deserves during this period in L.E.

d) Find the monthly average income of this employee during this period.

Solution

a) $\because$ The sequence is geometric $\therefore$ The common ratio (r) $= \frac{\text{second term}}{\text{first term}} = \frac{432}{288} = \frac{3}{2}$

b The monthly income of the fourth year = $T_3 \times r = 648 \times \frac{3}{2} = 962$ L.E,

The monthly income of the fifth year = $T_4 \times r = 962 \times \frac{3}{2} = 1458$ L.E

c The sum of what the employee deserves = 12 (288 + 432 + 648 + ... to five terms)

$\because S_n = \frac{a(1 - r^n)}{1 - r}$

$\therefore S_n = \frac{288 \times 12\ [(\frac{3}{2})^5 - 1]}{1 - \frac{3}{2}} = 45576$ L.E

Use the arrangement of the calculator buttons as follows:

(3 ÷ 2) $x^{■}$ 5 ▶ - 1 = × 1 2 ×

2 8 8 = ÷ (3 ÷ 2 - 1) =

d The monthly average of the employee's income = $\frac{\text{sum of income}}{\text{number of months}} = \frac{45576}{5 \times 12} = 759.6$ L.E

Critical thinking: Can you find the monthly average of the employee's income by dividing (288 + 432 + 648 + .. to 5 terms) by 5 years ? Explain.

Try to solve

(11) **Physics:** A small ball of iron has been rolled over on a horizontal plane. If the ball covered 25m in the first minute , then it started to cover 60% only in each next minute from the distance it covered in the directly previous minute. Find the total distance which the ball covered till it stopped completely.

Exercises (1 - 6)

Choose the correct answer

(1) The sum of an infinite number of terms of the sequence (8 , 4 , 2 , ...) is:

a 16 **b** 20 **c** 24 **d** 30

(2) If the sum of an infinite number of terms of a geometric sequence whose common ratio is $\frac{1}{3}$ is $13\frac{1}{2}$ then its first term equals :

a 6 **b** 8 **c** 9 **d** 12

(3) If the sum of an infinite number of terms of a geometric sequence whose first term is 12 is 96 , then its common ratio equals:

a $\frac{1}{3}$ **b** $\frac{1}{2}$ **c** $\frac{7}{8}$ **d** $\frac{3}{4}$

(4) A geometric sequence in which the sum of the first n terms is given by the relation $S_n = 3^{n+1} - 4$, then the third term equals:

a 18 **b** 23 **c** 54 **d** 77

(5) A geometric sequence whose first term equals the sum of the next infinite terms, then its common ratio equals:

a 0.5 b 0.333 c 0.25 d 0.666

Discover the error:

(6) The sum of an infinite geometric series can be found when $|r| \leqslant 1$

(7) The sum of an infinite number of terms of the sequence (16 , 8 , 4 , ...) is greater than twice its first term.

Answer the following questions:

(8) Find the sum for each of the following geometric sequences:

a (6 , 12 , 24 ,.. to 6 terms) b (125 , 25 , 5 , ... to 6 terms)

c (3 , - 6 , 12 , ... , 768)

(9) Which of the following geometric sequences can be summed up to ∞ , then find the sum if possible:

a (24 , 12 , 6 , ...) b (3 , - 6 , 12 , ...)

c $(\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, ...)$ d $(2 \times 5^{1-n})$

(10) Find the sum of an infinite number of terms for each of the following geometric sequence:

a $(3, \sqrt{3}, 1, ..)$ b $(T_n) = (3^{3-n})$

(11) Put each of the following recurring decimals in the form of common fractions using the infinite geometric sequence: $0.\overline{7}$, $0.2\overline{4}$, $0.8\overline{63}$

(12) Find the geometric sequence whose first term = 243 , last term = 1 , and the sum of its terms = 364

(13) Find the geometric sequence whose sum is 1093 , last term is 729 and its common ratio is 3

(14) How many terms are to be taken from the geometric sequence (3 , 6 , 12 , ..) starting from its first term to make the sum of these terms = 381?

(15) Prove that the sequence $(T_n) = (10 \times 2^{n-2})$ is a geometric sequance, then find the number of its terms whose sum is 2555 starting from the first term.

(16) Find the number of the terms of the geometric sequence whose sum of its terms is $121\frac{4}{9}$ and its first term equals 18 and the last term equals $\frac{1}{9}$.

(17) (T_n) is a geometric sequence whose terms are positive in which $T_2 = 6$ and $T_3 - T_1 = 9$. Find the sequence and sum of the first twelve terms.

(18) Find the geometric sequence whose sum of its first five terms = 7.75 and the sum of the consecutive five terms = 248 .

(19) The first term of a geometric sequence of an infinite number of terms = 18 and the fourth term $= \frac{16}{3}$. What is the sum of this sequence?

(20) If the sum of an infinite geometric sequence is $\frac{375}{4}$, and the sum of its first and second terms is 90, prove that there are two sequences and find them .

21. Find the geometric sequence whose sum of its first and second terms = 16 , and the sum of an infinite number of its terms = 25.

22. Find the infinite geometric sequence whose terms are positive, its first term is greater than its second term by 30 and the sum of an infinite number of its terms equals $\frac{135}{2}$.

23. (T_n) is a geometric sequence in which $T_1 + T_4 = 70$, $T_2 + T_3 = 60$, prove that there are two sequences and an infinite number of the terms of one of them can be summed and find this sum starting from its first term.

24. $(T_n)$is a geometric sequence in which $T_2 - T_6 = 45$ and the first $S_4 = 180$ find the sequence and show that there are an infinite number of its terms can be summed and find this sum .

25. Find the infinite geometric sequence whose first term = the sum of the next infinite terms of it and the sum of its first and second terms = 9.

26. Find the infinite geometric sequence in which any of its terms equals twice the sum of the next infinite terms of it, and its fourth term =3.

27. **Mining:** A gold mine produces 4200 kg in the first year. If the production of that mine decreases at a rate of 10% yearly of the production of the directly previous year, find the production of that mine in the eighth year, then calculate the production of the mine within the first eight years.

28. **Income:** A worker has started to work at a factory for a yearly income of 7200 L.E so that he got a yearly raise of 0.6 % of the directly previous year. Calculate his income in the seventh year and the sum of what he got within the first seven years.

Applications on overpopulation:

Activity

If the population of a city by the end of year 2000 was $\frac{1}{2}$ a milion populations and it is supposed to be 8 millions in year 2016. If the increasing of populations is constant and is subjected to an arithmetic sequence.

Calculate:

1- The constant ratio of the increasing populations.

2- The linear equation of estimating the number of populations.

3- Estimate the number of populations in year 2020.

4- Estimate the number of populations in year 2025.

5- Log in the Internet and do a report about the steady increase in the number of population in Egypt, how much it badly effects the national income and the suggested solutions of such a problem.

Unit summary

sequence: It is a function whose domain is the set of positive integers Z_+ or a subset of it and its range is a set of real numbers R with regarding the follows:

- Terms of the sequence are the image of its domain elements.
- The symbol (T_n) expresses the sequence while the symbol T_n express its n^{th} term.
- The sequence is subjected to the order of its elements and these elements can be repeated.
- The sequence is finite if the number of its terms is finite . The sequence is infinite if the number of its terms is infinite.
- The sequence is called **increasing** if $T_{n+1} > T_n$
- The sequence is called **decreasing** if $T_{n+1} < T_n$
- The sequence is called **constant** if **all of its terms are equal**
- The sequence is represented graphically as a function throughout representing the ordered pairs by separated points on the coordinate plane.

Series: it is the sum of the sequence terms. The summation notation " Σ " is used to write the series in a short form .

- **Finite series:** it contains a finite number of elements and it is written in the form $\sum_{r=1}^{n} (T_r)$
- **Infinite series:** its elements are countless and it's written in the form $\sum_{r=1}^{\infty} (T_r)$

Arithmetic sequence : it is the sequence in which the difference between each term and the directly previous term equals a constant called the common difference of the sequence. It is usually denoted by the symbol (d) where $d = T_{n+1} - T_n$ for each $n \in Z^+$

- The arithmetic sequence can be formed in terms of its first term(a) and its common difference (d).
- The relation between n and T_n is $T_n = dn + b$ where d and b are two constants and d is the sequence common difference. This is a liner relation and it can be graphically represented by a set of points.
- The sequence is increasing if $d > 0$ and is decreasing if $d < 0$
- The n^{th} term of the arithmetic sequence (T_n) whose first term a and common difference d is given by the relation : $T_n = a + (n - 1) d$
- If a , b and c are three consecutive terms of an arithmetic sequence, then b is known as the arithmetic mean between the two numbers a and c where $b = \frac{a + c}{2}$

- Several arithmetic means can be inserted between two number to form an arithmetic sequence and the number of the sequence terms = the number of the means +2
- The sum of the first n terms of an arithmetic series:

  First: in terms of its first and last terms: $S_n = \frac{n}{2}(a + \ell)$

  second: In terms of its first term and the common difference : $S_n = \frac{n}{2}[2a + (n - 1)d]$

Geometric sequence: The sequence (T_n) where $T_n \neq 0$ is called an arithmetic sequence if $\frac{T_{n+1}}{T_n}$ = a constant for $n \in Z_+$ and the constant is called the sequence common ratio and is denoted by the symbol (r)

- The n^{th} term of the geometric sequence (T_n) whose first term a and common ratio r is given by the relation $T_n = a \times r^{n-1}$
- If a , b and c are three consecutive terms of a geometric sequence, then b is known as the geometric mean between the two numbers a and c where $b = \pm\sqrt{ac}$

The relation between the arithmetic and geometric means:

- The arithmetic mean of two different positive real numbers is greater than their geometric mean.
- The arithmetic mean of two equal positive real numbers is equal to their geometric mean.
- The sum of the first n terms of a geometric series:

  First: in terms of its first term and common ratio : $S_n = \frac{a(1 - r^n)}{1 - r}, r \neq 1$

  Second: in terms of its first and last terms: $S_n = \frac{a - \ell r}{1 - r}, r \neq 1$

The infinite geometric sequence: has an infinite number of terms and it has a sum if $|r| < 1$

- Th sum of the infinite geometric sequence $S_\infty = \frac{a}{1 - r}, r \neq 1$

@ Enrichment Information

Please visit the following links.

Accumulative test

(1) Find the solution set for each of the equations in R:

a $x^2 - 3x - 4 = 0$.

b $x + \frac{2}{x} = 3$.

(2) Find the solution set for each pair of the following equations in R :

a $x + 2y = 5$, $2x - y = 5$

b $a - b = 1$, $a + 2b = 10$

c $x - y = 2$, $xy = 3$

d $3a + 5f = 21$, $2a - 7f = -17$

(3) Factorize each of the following thoroughly:

a $ar^3 - ar$

b $x^2y + xy^2 - 2xy$

c $a + ar + ar^2$

d $1 + r + r^2 + r^3$

(4) Find the solution set for each of the following equations and inequalities in R:

a $2^{x-1} = \frac{1}{8}$

b $(-\frac{1}{2})^{n+2} = \frac{1}{64}$

c $2^{1-5n} > 7$

d $(\frac{1}{2})^{n+2} < 0.01$

Comparisons: Show the aspects of similarity and difference betwee each of the following:

(5) Sequences and series.

(6) The n^{th} term of the arithmetic sequence and the n^{th} term of the geometric sequence.

(7) The sum of the arithmetic series and the sum of the geometric series.

Short answerd questions:

Find the value of the first term (a) in each of the following sequences and series:

(8) The arithmetic sequence in which : $d = 3$, $T_5 = 32$

(9) The geometric sequence in which : $r = \frac{1}{2}$, $T_4 = \frac{1}{16}$

(10) The arithmetic series in which: $d = 6$, the first $S_{16} = 128$

(11) The geometric series in which : $r = 2$, the first $S_8 = 255$

(12) The geometric series in which: $r = \frac{1}{2}$ and $S_\infty = 128$

Multi answered question:

(13) Write down three sequences so that one of them is arithmetic, the second is geometric and the third is neither arithmetic nor geometric and each one of them starts as follows : 2 , 6 , ...

(14) Write down three geometric sequences such that the first term of each is 16 and the first sequence is increasing , the second is decreasing and the third is constant, then find one of these infinite sequences .

15. Write down the geometric series $64 + 32 + 16 + \ldots$ in two different ways using the summation notation.

16. **Justification:** when does the geometric series have a sum and when it doesn't have a sum? Justify.

Life application:

Long answered questions:

17. If $(15, 3k + 2, 4k - 5, \ldots)$ form an arithmetic sequence, then find the value of K.

18. Find the arithmetic sequence whose sum of its fifth and tenth terms equals 20 and its seventh term is 3 times of its fourth term .

19. Find the arithmetic sequence whose twentieth term equals 41 and the sum of its third and sixth terms is greater than its ninth term by a unit.

20. Find the geometric sequence (T_n) in which $T_2 = 5$, $\frac{T_7}{T_4} = \frac{1}{27}$.

21. If a , 2 b , 2c and 5 d are positive quantities in an arithmetic sequence, prove that : $2(b^2 + c^2) > ac + 5bd$.

22. An arithmetic sequence in which $T_{12} = 38$, $T_{35} = 245$, find the sum of the first fifteen terms.

23. An arithmetic sequence in which the sum of the nineteenth and twentieth terms equals 144, and the sum of the twentieth and twenty first terms equals 152, Find :

    a. The twentieth term .

    b. The number of terms necessary to be taken from the sequence starting from the first term to make the sum equal 160.

24. Find the arithmetic sequence whose terms are positive, the sum of its first four terms equals 50 and the product of its second and third terms equals 150, then find the sum of its first fifteen terms..

25. Find the geometric sequence whose terms are positive, the sum of the first twelve terms equals 273 times of the sum of its first four terms. Find its common ratio if its eight h term = 640.

26. If the sum of the first seven terms of a geometric sequence is ℓ, and the sum of the next seven terms is m, prove that the sequence common ratio = $\sqrt[7]{\frac{m}{\ell}}$

General Exercises

For more exercises, please visit the website of Ministry of Education.

Unit Two

Permutations, Combinations

Unit introduction

Counting is an important part of the basic skills in mathematics. We regularly face a lot of problems that need to be solved. We need to do counting operations in deferent ways to solve them. In this unit, we are going to identify different strategies for counting such as the fundamental counting principle and the most important applications of it:

Permutations. They are used to know the number of methods used to order the elements of a set with all possible methods.

Combinations. They mean to choose disregarding the order.

Scientists such as Omar Alkhaiam, Isac Newton and pascal had played a great role in this field which is still ongoing up to day.

Unit objectives

By the end of the unit and doing the activities involved , the student should be able to:

- Identify the counting principle and simple applications on it.
- Identify an introduction about the permutation and combination and the relation between them .
- Use the computer to calculate each of permutations and combinations.

Key terms

- Fundamental Counting Principle
- Conditional Counting Principle
- Operation
- Tree Diagram
- Factorial
- Permutations
- Combinations
- Elements
- Order
- Committee
- Subset

Lessons of the unit

Lesson (2 - 1): Counting Principle .

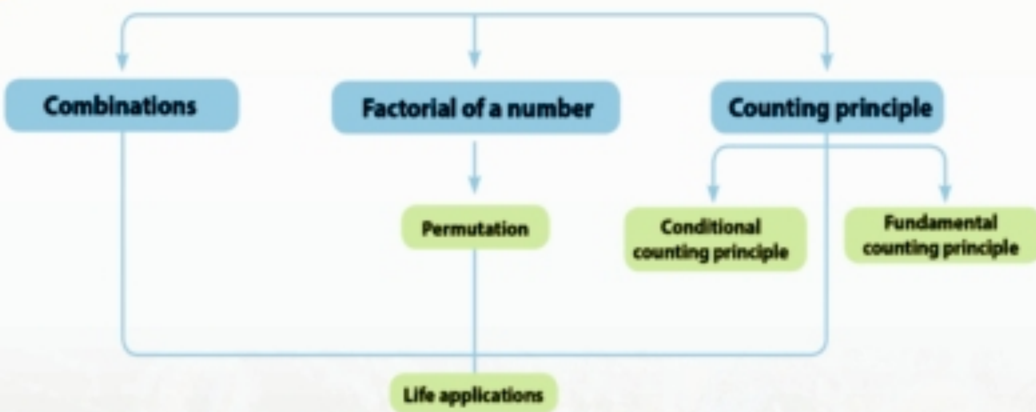
Lesson (2 - 2): Permutations.

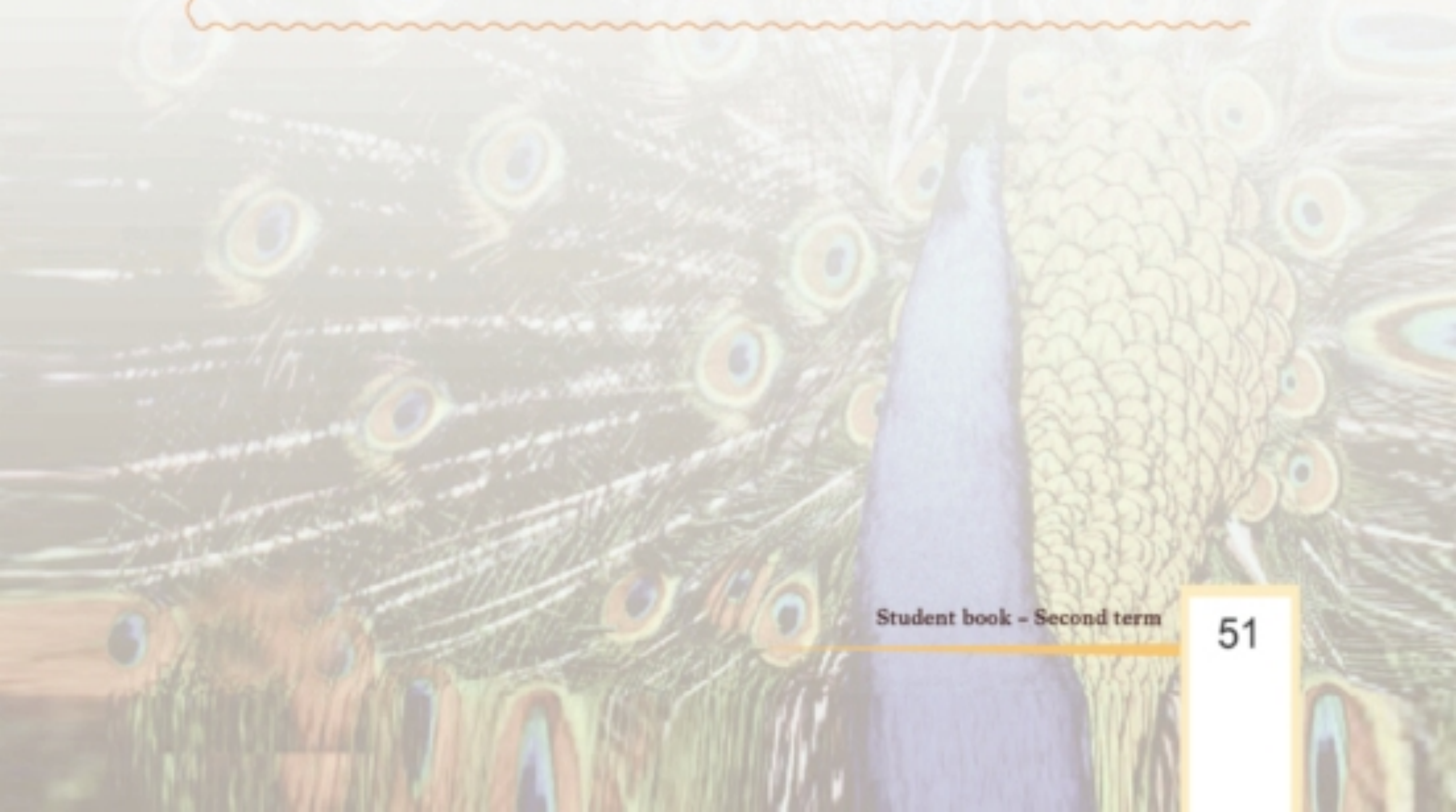
Lesson (2 - 3): Combinations

Materials

Scientific calculator - computer - graphic programs

Unit planning guide





Counting Principle

2 - 1

We will learn

- The concept of counting principle and simple applications on it.
- Fundamental counting principle.
- Conditional counting principle.

Key - term

- Fundamental Counting Principle
- Operation
- Tree Diagram

Matrials

- Scientific Calculator
- Graphic program

Think and discuss

How many ways are there to choose, if you are asked to wear a t-shirt and a pair of pants out of 2 t-shirts and 3 pairs of pants?

Example

1 How many ways are there to choose a male student out of three students (Ashraf - Mohamed - Hassan) and a female student out of two students (Samar - Mona)?

Solution

In this example, we find that it is easy to know the number of ways of choosing. For example, we can choose Ashraf, Samar or Ashraf, Mona or Mohamed, Mona or Hassan, Samar .. etc. We are going to express that by the following graphical diagram called tree diagram:

Male students	Female students	Choice
Ashraf	Samar	Ashraf, Samar
	Mona	Ashraf, Mona
Mohamed	Samar	Mohamed, Samar
	Mona	Mohamed, Mona
Hassan	Samar	Hassan, Samar
	Mona	Hassan, Mona

The number of ways to choose a male student out of three students = 3 ways.

The number of ways to choose a female student out of two students = 2 ways

$\therefore$ Number of ways of choosing = $3 \times 2 = 6$ ways

Try to solve

1 In **think and discuss** , How many possible ways of choosing are there?

Example

2 How many three- digit numbers so that the unit digit is from the elements {3, 7} the tens digit is from the elements {2, 4, 9} and the hundreds digit is from the elements is {1, 5} are there?

Solution

Unit digit	tens digit	hundreds digit	Number
3	2	1	123
		5	523
	4	1	143
		5	543
	9	1	193
		5	593
7	2	1	127
		5	527
	4	1	147
		5	547
	9	1	197
		5	597

From the tree diagram, we find that:

The number of ways to choose the unit digit × number of ways to choose tens digit × number of ways to choose hundreds digit $= 2 \times 3 \times 2 = 12$ ways

The previous examples support the following definition:

Learn

Fundamental Counting Principle

Definition: If the number of ways to perform a certain task is m_1 way , the number of ways to perform another certain task is m_2 way and the number of ways to perform a third certain task is m_3 way and so on ...., then the number of ways to perform these tasks together $= m_1 \times m_2 \times m_3 \times ... \times m_n$

Example

3) How many ways can khaled have a meal out of three meals (Liver - chicken - fish) and a drink out of three drinks (orange - lemon - Mango)?

Solution

Number of ways to choose a meal $= 3$ ways and the number of ways to choose a drink$= 3$ ways.

The total number of ways to choose $= 3 \times 3 = 9$ ways.

Try to solve

2) A restaurant presents 6 types of pies, 4 types of salads and 3 types of drinks. How many meals can the restaurant present daily so that a meal includes a type from each of pies, salads and drinks?

Example **Conditional Counting Principle**

4) How many different three -digit numbers can be formed from the numbers $\{0 , 1 , 2 , 3 , 4\}$?

Solution

Start with the conditional hundreds digit (Zero can't be used left side)

Number of ways of choosing the digit in hundreds digit = 4

place	hundred	tens	ones
No of ways	4	4	3

Number of ways of choosing the digit in tens digit = 4

Number of ways of choosing the digit in unit digit = 3

$\therefore$ The total number of ways of choosing $= 4 \times 4 \times 3 = 48$ ways

Try to solve

(3) How many ways can a different four -digit number be formed from the numbers {2, 3, 4, 7}, so that the tens digit is even.

Exercises (2 - 1)

Choose the correct answer:

(1) The number of ways of sitting 4 students on 4 seats in a row equals :

a) 1 b) $4+4$ c) 4×4 d) $4 \times 3 \times 2 \times 1$

(2) The number of the different two - digit numbers taken from the numbers { 5, 3, 0, 2} equals:

a) 3×2 b) 4×2 c) 3×3 d) 3×4

(3) The number of the different three-digit odd numbers taken from the numbers {2 , 3, 6, 8} equals:

a) $8 \times 6 \times 3$ b) $4 \times 3 \times 3$ c) $4 \times 3 \times 2$ d) $2 \times 3 \times 1$

(4) How many three -digit numbers can be formed from the elements {2, 3, 5}?

(5) How many different four - digit numbers can be formed from the elements {2, 3, 6, 8} so that the unit digit is 6?

(6) The licence plates of cars in a governorate start with three letters followed by three digits except Zero. How many plates can be got assuming that there is no repetition for any letter or digit in the licence plates?

(7) How many different three-digit numbers can be taken from the numbers {2, 5, 8, 9} so that these numbers are less than 900?

(8) If you know that the set of the numbers of mobile networks in a country is made up of an eleven-digit number. If the number (025) is constant on left side, find the greatest number of phone lines which the mobile network can stand .

Factorial of a Number - Permutations

2 - 2

We will learn
- Factorial of a number
- Permutations

Key - term
- Factorial of a Number
- Permutatuins
- Sub-Permutation

Matrials
- Scientific calculator
- Graphical programs

Think and discuss

Use what you learned in the previous lesson to answer the following questions:

1) How many ways can 4 students sit on three seats in a row?

2) How many ways can 5 racers stand on the edge of a swimming pool to jump?

Learn

Definition 1

Factorial: The factorial of a positive integer n is written as $\underline{|n}$ and equals the product of all the positive integers which are lesser than or equal n where:

$$\underline{|n} = n(n-1)(n-2)\ldots 3\times 2\times 1$$

Notice that

- When $n = 0$ **then** $\underline{|0} = 1$
- When $n = 1$ **then** $\underline{|1} = 1$
- $\underline{|4} = 4\times 3\times 2\times 1 = 4\,\underline{|3}$, $\underline{|6} = 6\times 5\times 4\times 3\times 2\times 1 = 6\,\underline{|5}$

In general: $\underline{|n} = n\,\underline{|n-1}$ where $n \in Z^+$

Example

1 a) Find $\dfrac{\underline{|10}}{\underline{|8}}$ b) If $\underline{|n} = 120$ find the value of n

Solution

a) $\dfrac{\underline{|10}}{\underline{|8}} = \dfrac{10\times 9\,\underline{|8}}{\underline{|8}} = 10\times 9 = 90$

b) $\underline{|n} = 5\times 4\times 3\times 2\times 1$ $\therefore \underline{|n} = \underline{|5}$ then $n = 5$

Try to solve

1 **Find:** a) $\dfrac{\underline{|15}}{\underline{|12}}$ b) $\dfrac{\underline{|7}}{\underline{|5}} + \dfrac{\underline{|9}}{\underline{|7}}$

Example

2 Find the solution set of the equation:- $\dfrac{\underline{|n}}{\underline{|n-2}} = 30$

Solution

$\because \dfrac{\underline{|n}}{\underline{|n-2}} = \dfrac{n(n-1)\,\underline{|n-2}}{\underline{|n-2}} = 30 \qquad \therefore n(n-1) = 6 \times 5 \qquad \therefore n = 6$

Try to solve

(2) If $\dfrac{1}{\underline{|n}} + \dfrac{2}{\underline{|n+1}} = \dfrac{56}{\underline{|n+2}}$ find the value of n

Critical thinking: if: $\underline{|a} = \underline{|0}$ find the value of a.

Permutations

Introductory example: How many ways can different three-digit numbers be formed from the set of numbers {2, 3, 5} ?

The numbers are: 532 , 352 , 523 , 253 , 325, 235. Each number of these numbers is called a permutation of the numbers

and its number $= 3 \times 2 \times 1$ **and is written as** $^3p_3$ **and is read as** (3 p 3).

The following table illustrates that:

Hundreds	Tens	Unit digit
3	2	1

So, the permutation for a number of objects is to put them in a certain arrangement

Definition 1

The number of permutations of n different objects taking r at a time is denoted by the symbol ^np_r where:

$^np_r = n(n-1)(n-2) \dots (n-r+1)$ where $r \leqslant n$, $r \in N$, $n \in Z^+$

$^np_0 = 1$

For example:

- $^6p_3 = 6 \times 5 \times 4 \times \dfrac{3 \times 2 \times 1}{3 \times 2 \times 1} = \dfrac{\underline{|6}}{\underline{|6-3}}$
- $^7p_5 = 7 \times 6 \times 5 \times 4 \times 3 \times \dfrac{2 \times 1}{2 \times 1} = \dfrac{\underline{|7}}{\underline{|7-5}}$

$^6p_3 = \dfrac{\underline{|6}}{\underline{|6-3}}$

from the previous , we deduce that:

$^np_r = \dfrac{\underline{|n}}{\underline{|n-r}}$ where $r \in N$, $n \in Z^+$, $r \leqslant n$

using calculato

Permutations are denoted by the symbol ^np_r in the calculator and we use the buttons [Shift] [×] **To calculate the value of $^5p_2$ by the calculator, press the following buttons consecutively:**

[5] [Shift] [×] [2] [=]

The answer = 20

Example

(3) Find the value for reach of the following:

a $^7p_4$ **b** $^4p_4$ **c** $^4p_3$

Solution

a $^7p_4 = 7 \times 6 \times 5 \times 4 = 840$

b $^4p_4 = 4 \times 3 \times 2 \times 1 = 24$

c $^4p_3 = 4 \times 3 \times 2 = 24$. What do you notice in the two phrases b and c?

Try to solve

3) Calculate the value of the following: **a** ${}^{5}p_{2} + {}^{6}p_{3}$ **b** $\frac{{}^{5}p_{5}}{{}^{5}p_{4}}$

Example

4) Find the number of the different ways, for 5 students to sit on 7 seats in one row.

Solution

We have 7 seats. Among the 7 seats, 5 should be selected each time

$\therefore$ Number of ways $= {}^{7}p_{5} = 7 \times 6 \times 5 \times 4 \times 3 = 2520$

Use the calculator: 7 SHIFT × (npr) 5 =

Try to solve

4) How many words can be formed from five different letters?

Example **(Arranging (ordering) in a circle)**

5) How many ways can 4 persons sit on 4 seats in the form of a circle ?

Solution

In this case , the first person sits in one way, second person sits in three ways, third person sits in 2 ways and the last person sits in an only way.

The number of ways of their arrangement $= 3 \times 2 \times 1 = \lfloor\underline{3} = 6$

Critical thinking: Can you deduce how many ways n of persons can be arranged in a circle?

Try to solve

5) How many ways can 9 persons in 9 seats be arranged in the form of a circle?

Example

6) If ${}^{7}p_{r} = 840$, find the value of $\lfloor\underline{r - 4}$

Solution

Start by dividing the number 840 by 7, then divide the quotient by 6, then divide the resulted quotient by 5 and so on till you reach number 1

840	7
120	6
20	5
4	4
1	

$\therefore$ Number $840 = 7 \times 6 \times 5 \times 4 = {}^{7}p_{4}$

$\therefore {}^{7}p_{r} = {}^{7}p_{4}$ $\therefore r = 4$ $\therefore \lfloor\underline{r - 4} = \lfloor\underline{0} = 1$

Try to solve

6) If ${}^{9}p_{r-1} = 504$, find the value of $\lfloor\underline{r + 1}$

Critical thinking: **1)** Find the value of: ${}^{7}p_{7}$, $\lfloor\underline{7}$. **What do you notice**?

Exercises (2 - 2)

Choose the correct answer:

1. How many ways can a president and vice president be selected from a 12 - member committee?
   a. 2 b. 23 c. 66 d. 132

2. If $^5p_r = 60$, then r equals:
   a. 4 b. 3 c. 2 d. 5

3. If $^np_3 = 120$, then the value of n equals:
   a. 6 b. 5 c. 4 d. 3

4. The number of ways to arrange the letters of the word HELP equals:
   a. 4 b. 9 c. 10 d. 24

5. The number of ways to select a different two - digit number of the set of numbers {3, 4, 5, 6} equals:
   a. 48 b. 30 c. 12 d. 4

6. The number of ways to arrange 7 children in a circle equals :
   a. 1 b. 7 c. 720 d. 5040

7. A phone number is made up of 8 places

9	c	.....	.....	.....	.....	.....	.....

   c should be a number of 3, 4, 5, 8 while the rest of the digits are made up of any unconditional digit. How many different telephone numbers are available?
   a. 499999 b. 4000000 c. 4999999 d. 10000000

8. How many ways can Hossam have a meal out of three meals (Kofta - chicken - fish) and a drink out of two drinks (Juice - soft drink)? Represent it using the tree diagram.

9. How many ways can a two -digit number be formed from the numbers 1 , 2 , 3 , 4?

10. How many ways can a different two -digit number be formed from the numbers 1 , 2 , 3 , 4?

11. How many ways can a different two -digit even number be formed from the numbers 1 , 2 , 3, 4?

12. How many ways can a committee of a man and a woman be formed from 3 men and 4 woman?

(13) An Ice-cream shop offers three sizes (small - medium - large) and five flavors (strawberry , mango, lemon , milk, chocolate)
How many ways are available to buy an ice -cream cone?

(14) From the set of the letters {a , b, c , d, e , f}, find
- **a** Number of ways to select a letter.
- **b** Number of ways to select two letters.

(15) **Find the value of the following:**

- **a** $\lfloor\underline{7} \div \lfloor\underline{5}$
- **b** $3\lfloor\underline{2} - \lfloor\underline{3}$
- **c** ${}^{5}p_{3} \times \lfloor\underline{2}$
- **d** ${}^{3}p_{3} \times {}^{2}p_{2}$
- **e** ${}^{8}p_{1} + {}^{8}p_{2}$
- **f** ${}^{7}p_{0} + {}^{7}p_{7}$

(16) **Find the value of n which satisfies the following:**

- **a** $\lfloor\underline{n} = 24$
- **b** $\frac{\lfloor\underline{n+1}}{\lfloor\underline{n-1}} = 42$
- **c** ${}^{15}p_{n} = 2730$
- **d** ${}^{n}p_{0} + {}^{n}p_{1} + {}^{n}p_{2} = 50$

(17) Find the value of n if:

- **a** ${}^{7}p_{n} = 210$
- **b** $n\lfloor\underline{2n-1} = 12$

(18) If ${}^{n}p_{4} = 14 \times {}^{n-2}p_{3}$ find the value of n.

(19) Find the number of ways to select a president, vice president and a secretary out of a ten-person committee.

(20) How many ways can the physical education instructor choose three students (one after another) to participate in the teams of soccer , basketball and volleyball respectively from eight students?

(21) Prove that: $\frac{\lfloor\underline{n+2}}{\lfloor\underline{n}} = {}^{n+2}p_{2}$

Activities

1- If x = {2 , 3 , 5 , 7 , 9}

- **a** How many ways can a different two-digit number be formed from these numbers?
- **b** How many ways can a two-digit number be formed from these numbers?
- **c** How many ways can a different three-digit number be formed from these numbers?
- **d** How many ways can a different three-digit number less than 500 be formed from these numbers?
- **e** How many ways can a different four-digit number whose unit digit is 2 be formed from these numbers?

2 - 3 Combinations

We will learn

- The concept of combinations and simple applications on them.
- Pascal's triangle.

Key - term

- Combinations
- Elements
- Order
- Committee
- Subset

Matrials

- Scientific calculator
- Computer

Tip

Combination can be written in the form of $^nC_r = \binom{n}{r}$

Introduction

Two clubs of a four - club set { a , b, c , d} are to be selected for a soccer match, then the possible permutations are:

(a, b), (a, c), (a, d), (b, a) , (b, c), (b, d), (c, a), (c, b), (c, d), (d, a), (d, b), (d, c).

From the previous data, we notice that selecting (a, b) is different from selecting (b, a) and so on...

If we want to select from the previous disregarding the order, then all the possible choices are: {a, b}, {a, c}, {a, c}, {b, c}, {b, d}, {c, d}. and each choice of these choices is called" Combination"

Combinations

Definition 1

The number of combinations formed from r of objects chosen from n elements at the same time is ^nC_r where, $r \leqslant n$, $r \in N$, $n \in Z^+$

In the previous introduction , we find that:

the number of combinations of two elements taken from four elements is denoted by the symbol $^4C_2$ and is read as (4 c 2) or by the symbol $\binom{4}{2}$.

In the introduction above, we notice that the number of ways of choosing = 6 ways

i.e.: $^4C_2 = \frac{^4P_2}{\underline{|2}} = \frac{4 \times 3}{2 \times 1} = 6$, $\boxed{^nC_r = \frac{^nP_r}{\underline{|r}}}$

Example

1) Find the value of each of the following

a) $^7C_5$ b) $^7C_2$ (what do you notice)?

Solution

a) $^7C_5 = \frac{^7P_5}{\underline{|5}} = \frac{7 \times 6 \times 5 \times 4 \times 3}{5 \times 4 \times 3 \times 2 \times 1} = 21$ b) $^7C_2 = \frac{7 \times 6}{2 \times 1} = 21$

We notice that: ${}^{7}C_5 = {}^{7}C_2$ $(5 + 2 = 7)$

Important corollaries: ${}^{n}c_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}$ ${}^{n}C_r = {}^{n}C_{n-r}$

Try to solve

(1) Find the value of ${}^{12}C_9$: ${}^{17}C_{14}$ without using the calculator.

Activity

Using the calculator

The buttons [SHIFT] [+] can be used from left to right to write the symbol of combinations(${}^{n}C_r$)

1) Use the calculator to find the value of ${}^{5}C_4 + {}^{7}C_2$

Solution

Press the following buttons consecutively

start ⟶ [5] [SHIFT] [+] [4] [+] [7] [SHIFT] [+] [2] [=]

The sum = 26

Example

(2) If : ${}^{28}C_r = {}^{28}C_{2r-47}$

Solution

$\because$ ${}^{28}C_r = {}^{28}C_{2r-47}$

either : $r = 2r - 47$ i.e.: $r = 47$

it is greater than the value of n, so it is refused.

or : $r + 2r - 47 = 28$ $\therefore 3r = 75$ $\therefore r = 25$

Try to solve

(2) If ${}^{28}C_r = {}^{28}C_{2r-5}$, then find the value of r.

Example

(3) How many ways can a four - person team be chosen from a 9 - person set?

Solution

So that the choice disregards the order, then each choice is called a combination.

Number of choices $= {}^{9}C_4 = \frac{{}^{9}P_4}{\lfloor 4} = 126$

Try to solve

(3) 7 people have participated in a chess game so that a game is held between each two players. How many matches are there?

Example Counting principle

(4) How many ways can a committee of two men and a woman be selected out of 7 men and 5 women?

Solution

Number of ways to select 2 men out of 7 men = ${}^{7}C_{2} = 21$

Number of ways to select a women out of 5 women = ${}^{5}C_{1} = 5$

According to the counting principle, the number of ways to form the committee = $21 \times 5 = 105$ ways

Critical thinking: How many ways can a committee of 4 men and 3 women be selected out of 6 men and 5 women ?

Try to solve

(4) How many ways can a five-member committee formed from 3 male students and 2 female students be selected out of a class contains 10 male students and 8 female students?

Activity Pascal's triangle

Blaise pascal (1623 - 1662):

Blaise pascal is a french philosopher, mathematician and physicist. He had stated the theory of probabilities and designed a triangular array of numbers called pascal's triangle in calculating the probabilities. Furthermore, he had invented a calculator to perform the addition and multiplication operations

Check the opposite number triangle , then answer the following questions:

1- What do you notice about how numbers are written in this triangle?

2- Is there a relation among the number of elements of each row and the row directly next to it?

3- Is there a symmetry among the numbers existed on the two sides of the triangle?

After performing the activity, we can notice that:

➢ **First row:** represents (n = 1) of the elements taken off from r = 0 or r = 1

So: ${}^{1}C_{0} = 1$, ${}^{1}C_{1} = 1$

- **Second row:** represents (n = 2) of the elements taken off from r = 0 or r = 1 or r = 2 in each time.

  then: ${}^2C_0 = 1$, ${}^2C_1 = 2$, ${}^2C_2 = 1$ and so on.

As we notice that:

- Each row starts with one because ${}^nC_0 = 1$, and ends in one because ${}^nC_n = 1$
- Each number in any row except for the first row equals the sum of the two numbers located above it in the row directly above it.
- **In the third row , we find:** 1 , 1 + 2 , 2 + 1, 1
- **In the fourth row , we find:** 1 , 1 + 3 , 3 + 3 , 3 + 1 , 1 and so on.
- There is a symmetry about the number located at the middle of the row (if n is even)
- There is a symmetry about the two numbers located at the middle of the row (if n is odd)
- This coincides the previous relation ${}^nC_r = {}^nC_{n-r}$

Application on the activity:

Prove that: ${}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 2^5$

Exercises (2 - 3)

Choose the correct answer :

1) The number of ways to choose 3 people out of five people equals.........

a) 15 b) 10 c) 20 d) 35

2) The number of ways to answer 4 questions only out of a 6 - question exam equals.........

a) 30 b) 15 c) 24 d) 10

3) The number of ways to choose a red ball and a white ball out of 5 red balls and 3 white balls equals.........

a) 15 b) 8 c) 60 d) 2

Answer the following questions:

4) Calculate the value of ${}^6C_3$, ${}^9C_1$, ${}^{12}C_{11}$ and ${}^{100}C_0$

5) If ${}^nC_3 = 120$, find the value of ${}^nC_{n-9}$

6) If ${}^{n+1}C_4 = \frac{5}{2}\ {}^nC_3$, find the value of n.

7) If ${}^nC_3 = \frac{1}{3}\,30n$, find the value of n.

8) How many ways can a five-people committee take the majority of a decision?

9) How many ways can a five - student activity committee formed from three male students and two female students out of a class contains 10 male students and 8 female students?

10) Write in terms of permutations each of:

a) ${}^8C_3$ b) ${}^{19}C_2$ c) ${}^5C_0$ d) ${}^xC_{x-y}$

11) Use the form ${}^nC_n$ to write each of:

a) $\frac{{}^8P_2}{\lfloor 2}$ b) $\frac{{}^9P_3}{\lfloor 3}$ c) $\frac{{}^{10}P_4}{\lfloor 4}$ d) $\frac{{}^8P_0}{\lfloor 0}$

Activities

You have previously learned that the diagonal of a geometric figure is the line segment connecting two non-consecutive vertices and you also knew that:

The number of the triangle diagonals = zero

The number of the quadrilateral diagonals = 2

The number of the pentagon diagonals = 5

Can you find the number of the diagonals of the following figures:

1) Hexagon, octagon and decagon.

2) Can you use combinations to write the rule by which you can find the diagonals of any geometric figure? Check with your instructor using internet.

Unit summary

Fundamental counting principle: If the number of ways to perform a certain task equals n way an the number of ways to perform another task equals m way, then the number of ways to perform the first and second tasks together = m × n ways.

Permutations: If X is a set of n - elements, then each order for some or all the elements of this set is called a permutation.

Let X be a set of n-elements, then each order of r-element can be formed from the set X is called a permutation where $0 \leqslant r \leqslant n$ and the number of the sub-permutations formed from r element taken off from a set of n elements is denoted by ${}^{n}p_{r}$ (and read as n p r)

Theory: The number of the possible permutations from a set of n - elements equals n(n - 1)(n - 2) ... × 3 × 2 × 1 the previous product is denoted by the symbol $\underline{|n}$ and read as (factorial of n)

Combinations: If X is a set of n- elements , then each choice of r- elements where $(0 \leqslant r \leqslant n)$ can be taken off from the elements of the set X disregarding the order is called a combination.

- The number of possible combinations formed from r element and taken off from n element is denoted by the symbol ${}^{n}C_{r}$ (and read as n C r)

Enrichment Information

Please visit the following links.

Accumulative test

Choose the correct answer:

1. The number of ways to form a different three -digit prime number of the set of the numbers 3, 4, 5 is............

   a) 6   b) 3   c) 1   d) zero

2. The number of ways to form the number 5476 from the numbers 4, 5, 6, 7 is ....

   a) 24   b) 16   c) 1   d) zero

3. The number of ways to form a 3-digit number of 5 numbers except zero is ....

   a) $5 \times 4 \times 3$ b) $5 + 4 + 3$ c) $5 \times 5 \times 5$ d) $3 \times 2 \times 1$

4. If the letters of the word CAT are used to write a different three- letter word, then number of the words resulted equals:

   a) 3   b) 6   c) 9   d) 1

5. $^{9}P_{6}$ equals

   a) $9 \times 8 \times 7$ b) $9 \times 8 \times 7 \times 6 \times 5 \times 4$ c) $\lfloor 9$ d) $\lfloor 6$

6. Find:

   a) $^{5}C_{0} + {}^{5}C_{1} + {}^{5}C_{2} + {}^{5}C_{3} + {}^{5}C_{4} + {}^{5}C_{5}$

   b) $^{5}C_{0} - {}^{5}C_{1} + {}^{5}C_{2} - {}^{5}C_{3} + {}^{5}C_{4} - {}^{5}C_{5}$

7. Find: $^{8}p_{3} + {}^{8}p_{2} + {}^{8}p_{1}$

8. 5 points are on a plane so that there are not any three points on a straight line. How many triangles can be formed from these points?

9. How many different three -digit numbers can be formed from the set of numbers {1, 2, 3, 4, 5, 6}?

10. **Find the value of each:**

    a) $\frac{\lfloor 99}{\lfloor 100}$ b) $\lfloor 0 + \lfloor 1 + \lfloor 2 + \lfloor 3$

    c) $\frac{\lfloor 19}{\lfloor 20}$ d) $\lfloor 5 - \lfloor 4$

    e) $\frac{\lfloor 4}{\lfloor 0}$

(11) If x = { - 2 , - 1 , 0 , 1 , 2 , 3}
and z = {{a , b}: a ≠ b , a , b ∈ x }, find the number of z elements.

(12) **Find the value of n in each of the following cases:**

a) $\lfloor n = 720$

b) $n \lfloor 2n-1 = 60$

c) ${}^{n}C_{n-2} = 28$

d) ${}^{n+2}P_4 = 56 \times {}^{n}P_2$

e) $\frac{1}{\lfloor n} = \frac{2}{\lfloor n+1}$

f) ${}^{m}P_n = 24\ {}^{m}C_n$

If you can not answer any of these questions , you can use the following table:

Question no	1	2	3	4	5	6	7	8	9	10	11	12
Back to	counting principle	counting principle	counting principle	counting principle	permutations	combinations	permutations	combinations	counting principle	factorial of a number	combinations	permutations combinations

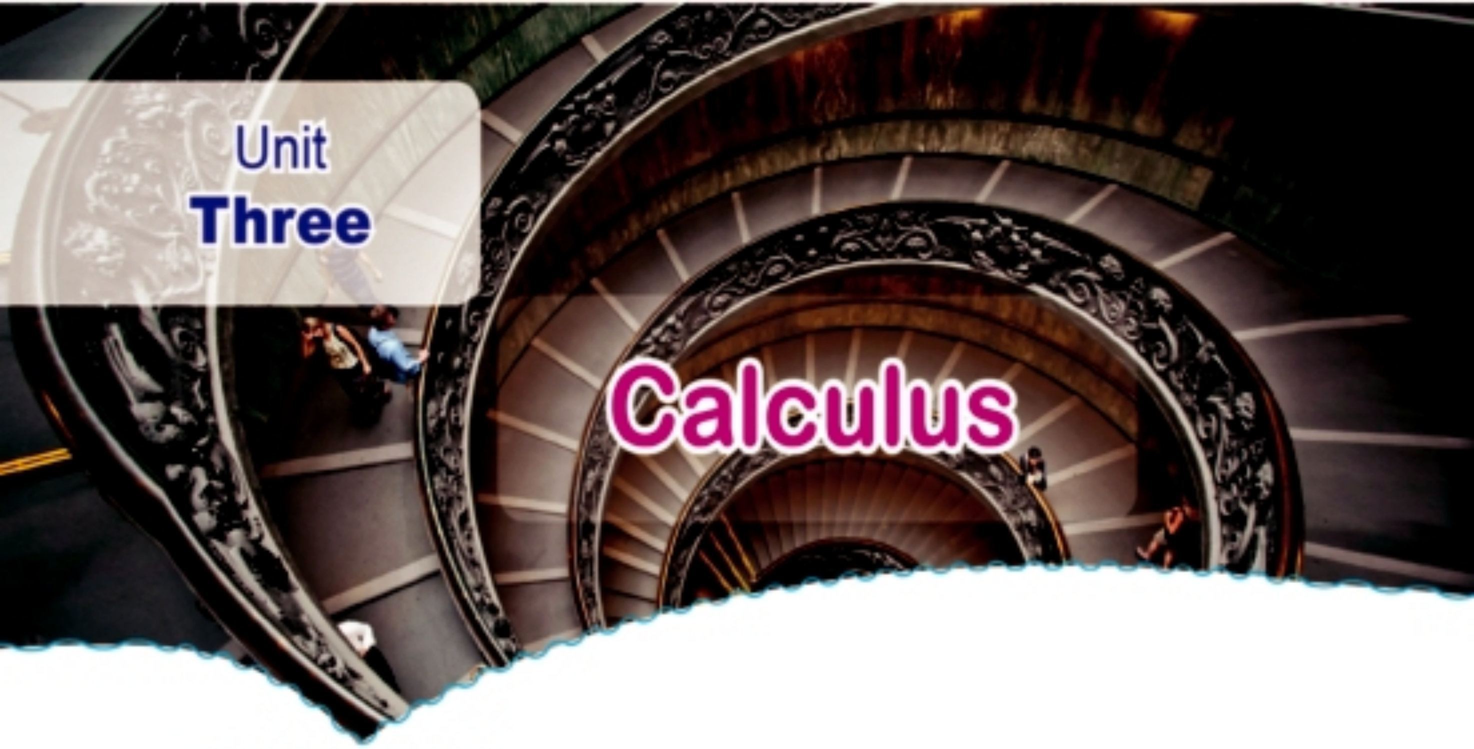
General Exercises

For more exercises, please visit the website of Ministry of Education.

Unit

Three

Calculus



Unit preface

Isaac Newton (1642 - 1727) had invented calculus and Leibeniz (1646 - 1716) had competed him independently. The result was that the differentiation or derivation had been originated to be related to the problem of finding the tangents of curves and to calculate the maximal and minimal values of the functions of pure mathematics or functions modeling social and economical problems . For integration, Newton had considered that it is a reverse operation of differentiation while Leibeniz considered it a limit of summations. The letter S to express integration from summation is related to him, then the letter S is changed into the letter known currently ∫ which is closer to S. Calculus has been developed by many scientists such as Berkeley, Lagrange , Laplace, Gauss and weierstrass and the process of inventing the analytic geometry has helped to develop calculus.

Unit objectives

By the end of the unit and doing its activities , the student should be able to:

- Identify the concept of the function of variation, average rate of change and rate of change.
- Deduce the first derivative of the function.
- Identify the geometric interpretation of the first derivative (slope of tangent) .
- Investigate the differentiability of the function (right derivative- left derivative)
- Deduce the relation between differentiation and continuity.
- Determine some differentiation rules
  - Derivative of the constant function.
  - Derivative of the function $f : f(x) = x^n$
  - Derivative of the function $f : f(x) = x$
  - Derivative of the function $f : f(x) = a\,x^n$
  - Derivative of the sum or the difference between two functions.
  - Derivative of the product of two function.
  - Derivative of the quotient of two functions.
  - Derivative of the composite function - chain rule.
  - Derivative of the function $y = (f(x))^n$
  - Derivative of the trigonometric functions.
- Use the derivatives in geometric applications such as finding the equations of the tangent and the normal of a curve at a point on it.
- Identify the concept of integration - antiderivative.
- Identify and deduce the following integration rules:
  - $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ where $n \neq -1$
  - $\int a f(x)\,dx = a\int f(x)$ E x. where a is a constant
  - $\int [f(x) \pm r(x)]\,dx = \int f(x)\,dx \pm \int r(x)\,dx$
  - $\int (a\,x + b)^n\,dx = \frac{1}{a(n+1)}(a\,x + b)^{n+1} + C$ where $n \neq -1$
  - $\int \sin a\,x\,dx = \frac{-1}{a}\cos a\,x + C$ where a is a constant
  - $\int \cos a\,x\,dx = \frac{1}{a}\sin a\,x + C$ where a is a constant
  - $\int \sec^2 a\,x\,dx = \frac{1}{a}\tan a\,x + C$ where a is a constant
  - $\int f'(x)\,[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + C$

Key terms

- Variation
- Average Rate of Change
- Rate of Change
- First Derivative
- Left Derivative
- Right Derivative
- Differentiation
- Differintiable Functions
- Product
- Quotient
- Chain Rule
- Trigonometric Functions
- Integration
- Antiderivative

Materials

Scientific calculator - Computer - Graphical program .

Lessons of the unit

Lesson (3 - 1): Rate of change.

Lesson (3 - 2): Differentiation.

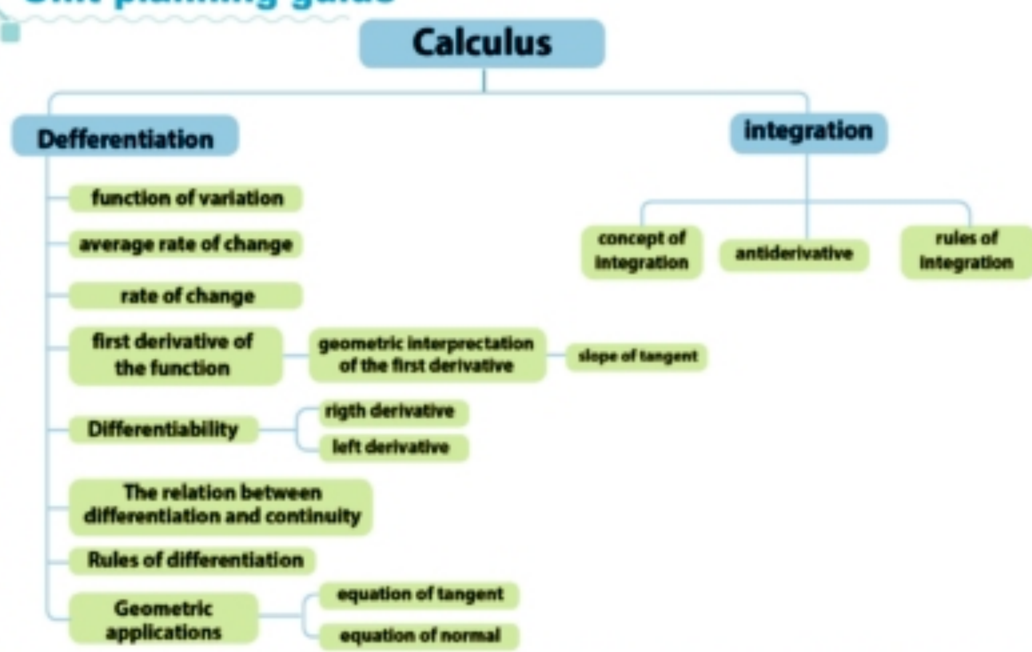
Lesson (3 - 3): Rules of differentiation.

Lesson (3 - 4): Derivatives of trigonometric functions.

Lesson (3 - 5): Applications on the derivative.

Lesson (3 - 6): Integration.

Unit planning guide



Unit three

3 - 1 Rate of Change

You will learn

- The concept of the function of variation.
- The concept of the average rate of change.
- The concept of the rate of change.

Key terms

- Function of Variation
- Average Rate of Change
- Rate of Change

aterials

- Scientific calculator

Think and discuss

If a table tennis ball and football fall down from the same height and at the same times disregarding the effect of air resistance, which ball will hit the ground first ? Explain.

Differentiation interests in learning the change occurring in a variable as a result of the change of another variable. The change occurred in the time of the ball motion $(t_2 - t_1)$ leads to a corresponding change in its velocity $(v_2 - v_1)$, so the average rate of change of velocity can be compared in regard to the time unit of the two balls by calculating the rate $\frac{v_2 - v_1}{t_2 - t_1}$

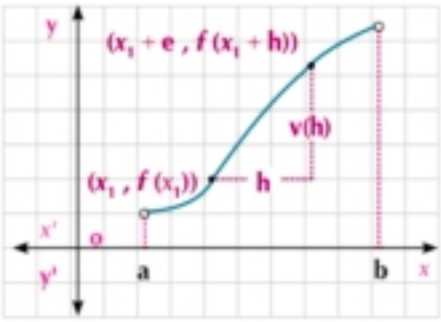
Learn **Function of Variation**

If $f :]a, b[\longrightarrow \mathbb{R}$ where $y = f(x)$, then any change in the value of x from x_1 to x_2 in the domain of f corresponde a change in the value of y from $f(x_1)$ to $f(x_2)$, thus:

The amount of change in $x = \triangle x$ (read as Delta x) $= x_2 - x_1$,

The amount of change in $y = \triangle y = f(x_2) - f(x_1)$

Let $(x_1 , f(x_1))$ be a point on the curve of the function f, then each change in its x coordinate from x_1 to $x_2 = x_1 + h$ where $x_1 + h \in]a , b[$ and $h \neq 0$. A corresponding change occurs in its y coordinates and it can be identified by the relation: $V(h) = f(x_1 + h) - f(x_1)$



the function V is called the function of variation in f when $x = x_1$

Notice: Both symbols $\triangle x$ or h represent the change in x

Example

(1) If $f(x) = 3x^2 + x - 2$

and x varies from 2 to 2 + h , find the function of variation V, then calculate the change in f when:

(a) h = 0.3 (b) h = - 0.1

Solution

$\because f(x) = 3x^2 + x - 2$, x varies from 2 to 2 + h

$\therefore x_1 = 2$, $f(2) = 3 \times 4 + 2 - 2 = 12$, then :

$f(2 + h) = 3(2 + h)^2 + (2 + h) - 2 = 12 + 12h + 3h^2 + 2 + h - 2$

$= 3h^2 + 13h + 12$

$V(h) = f(2 + h) - f(2)$

$= (3h^2 + 13h + 12) - 12 = 3h^2 + 13h$

(a) when h = 0.3

$V(0.3) = 3(0.3)^2 + 13 \times 0.3$

$= 4.17$

(b) when h = - 0.1

$V(-0.1) = 3(-0.1)^2 + 13(-0.1)$

$= -1.27$

Try to solve

(1) If $f(x) = x^2 - x + 1$, find the function of variation V when $x = 3$, then calculate:

(a) V(0.2) (b) V(- 0.3)

Learn Average Rate of Change Function

By dividing the function of variation v by h where h ≠ 0 , we get a new function a called the average rate of change function in f when $x = x_1$ where :

$$A(h) = \frac{v(h)}{h} = \frac{f(x_1 + h) - f(x_1)}{h} \quad \text{or} \quad \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example

(2) If $f : [0, \infty[\longrightarrow \mathbb{R}$ where $f(x) = x^2 + 1$, find :

(a) The average rate of change function in f when $x = 2$, then calculate A (0.3)

(b) The average rate of change in f when x varies from 3 to 4

Solution

(a) $f(x_1) = f(2) = (2)^2 + 1 = 5$, $f(x_1 + h) = f(2 + h)$

$\therefore f(2 + h) = (2 + h)^2 + 1 = h^2 + 4h + 5$

$\because A(h) = \frac{f(x_1 + h) - f(x_1)}{h}$

$\therefore A(h) = \frac{h^2 + 4h + 5 - 5}{h} = h + 4$ then $A(0.3) = 4.3$

b when x varies from 3 to 4 then $x_1 = 3$, $x_2 = 4$

and $f(3) = 9 + 1 = 10$, $f(4) = 16 + 1 = 17$

Average rate of change $= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{17 - 10}{4 - 3} = 7$

Try to solve

(2) If $f(x) = x^2 + 3x - 1$, find:

a The average rate of change function when $x = 2$, then find a (0.2)

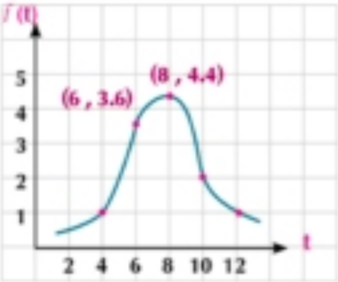
b The average rate of change when x varies from 4.5 to 3

Example

(3) The opposite figure shows the curve of $r = f(t)$ where r is the total sales of a computer store approximated in millions L.E and t is the time in months.

From the graph, find the average rate of change in the total sales when the time varies from:

a $t = 4$ to $t = 8$ **b** $t = 8$ to $t = 10$



Solution

a From the graph : $f(8) = 4.4$, $f(4) = 1$

the average rate of change in $f = \frac{f(8) - f(4)}{8 - 4} = \frac{4.4 - 1}{4} = 0.85$ of million L.E / month

I.e. the average of the total sales increases 0.85 of million L.E monthly during this period.

b From the graph : $f(10) = 2$ and $f(8) = 4.4$

the average rate of change in $f = \frac{f(10) - f(8)}{10 - 8} = \frac{2 - 4.4}{2} = -1.2$ million L.E / month

I.e. the average of the total sales decreases 1.2 million L.E monthly during this period.

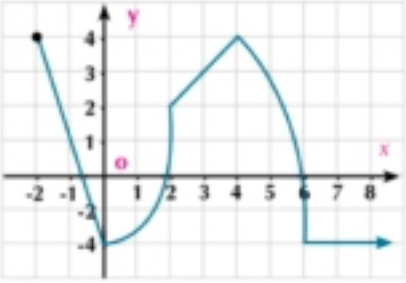
Try to solve

(3) Use the graph in example (3) to find the average rate of change in the total sales when time varies from :

a $t = 4$ to $t = 6$ **b** $t = 6$ to $t = 10$ **c** $t = 4$ to $t = 12$

Critical thinking :

The opposite figure shows the curve of the function f where $y = f(x)$. Determine the intervals in which the average rate of change in f is constant. Explain



Learn

Rate of Change Function

If $f :]a , b[\longrightarrow \mathbb{R}$ where $y = f(x)$ and $x_1 , x_1 + h \in]a , b[$, then :

the rate of change function in f when $x_1 = \lim\limits_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h} = \lim\limits_{h \to 0} A(h)$ in a condition the limit should be existed.

Example

4 Find the rate of change function in f when $x = x_1$ for each of the following , then find this rate at the given values of x .

a $f(x) = 3x^2 + 2$ when $x = 2$

b $f(x) = \frac{2}{x - 1}$ when $x = 3$

Solution

a $\because f(x) = 3x^2 + 2$ $\therefore$ when $x = x_1$ then $f(x_1) = 3x_1^2 + 2,$

$f(x_1 + h)^2 = 3(x_1 + h)^2 + 2 = 3x_1^2 + 6x_1 h + 3h^2 + 2$

The rate of change function in $f = \lim\limits_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$

$= \lim\limits_{h \to 0} \frac{6x_1 h + 3h^2}{h} = \lim\limits_{h \to 0} (6x_1 + 3h) = 6x_1$

When $x = 2$ $\therefore x_1 = 2$ and the rate of change in $f = 6 \times 2 = 12$

b $\because f(x) = \frac{2}{x - 1}$ $\therefore$ when $x = x_1$, then :

$f(x_1 + h) - f(x_1) = \frac{2}{x_1 + h - 1} - \frac{2}{x_1 - 1}$

$= \frac{2x_1 - 2 - 2x_1 - 2h + 2}{(x_1 + h - 1)(x_1 + h)} = \frac{-2h}{(x_1 + h - 1)(x_1 - 1)}$

the rate of change function in $f = \lim\limits_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$

$= \lim\limits_{h \to 0} \frac{1}{h} \times \frac{-2h}{(x_1 + h - 1)(x_1 - 1)} = \frac{-2}{(x_1 - 1)^2}$

when $x = 3$ $\therefore x_1 = 3$ and the rate of change in $f = \frac{-2}{(3 - 1)^2} = -\frac{1}{2}$

Try to solve

4 Find the average rate of change function in f where $f(x) = \frac{3}{x - 2}$ when x varies from x_1 to $x_1 + h$, then deduce the rate of change in f when $x = 5$.

Example

(5) Find the average rate of change function in f where $f(x) = \sqrt{x+3}$ when $x = x_1$, then find the rate of change in f when:

a $x = 6$ b $x = -2$

Solution

a $\because f(x) = \sqrt{x+3}$ $\therefore$ domain of $f = [-3, \infty[$

when $x = x_1$, then : $f(x_1) = \sqrt{x_1+3}$, $f(x_1 + h) = \sqrt{x_1 + h + 3}$

then:

Average rate of change function in f = $A(h) = \dfrac{f(x_1+h) - f(x_1)}{h}$

$$= \frac{\sqrt{x_1+h+3} - \sqrt{x_1+3}}{h}$$

Rate of change function in $f = \lim\limits_{h \to 0} A(h) = \lim\limits_{h \to 0} \dfrac{\sqrt{x_1+h+3} - \sqrt{x_1+3}}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{x_1+h+3} - \sqrt{x_1+3}}{h} \times \frac{\sqrt{x_1+h+3} + \sqrt{x_1+3}}{\sqrt{x_1+h+3} + \sqrt{x_1+3}}$$

$$= \lim_{h \to 0} \frac{x_1 + h + 3 - x_1 - 3}{h[\sqrt{x_1+h+3} + \sqrt{x_1+3}]}$$

$$= \frac{1}{2\sqrt{x_1+3}} \text{ where } x_1 > -3$$

a when $x = 6$ $\therefore$ Rate of change in $f = \dfrac{1}{2\sqrt{6+3}} = \dfrac{1}{6}$

b When $x = -2$ $\therefore$ Rate of change in $f = \dfrac{1}{2\sqrt{-2+3}} = \dfrac{1}{2}$

Try to solve

(5) Find the average rate of change function in f where $f(x) = \sqrt{x-5}$ when $x = x_1$, then deduce the rate of change in f when $x = 9$

Can the rate of change in f when $x = 5$ be calculated? **Explain**

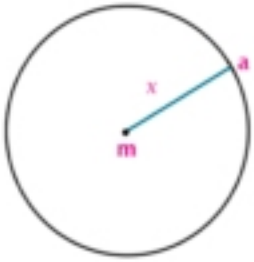
Life applications

Example

(6) A piece of stone has fallen in calm water, a circular wave has been formed and expanded regularly so that the wave has kept its circular Shope. Find the rate of change in its surface area with respect to its radius length when the radius length is 3.5 ($\pi = \frac{22}{7}$).

Solution

Modelling the problem:



Let the radius length of the wave = x cm

$\therefore$ Area of the circle a = $\pi\ x^2$ cm^2

Then a = $f(x) = \pi x^2$

When x varies from x_1 to $x_1 + h$

Then rate of change function in a = $\lim\limits_{h \to 0} \dfrac{f(x_1 + h) - f(x_1)}{h}$

$= \lim\limits_{h \to 0} \dfrac{\pi (x_1 + h)^2 - \pi x_1^2}{h}$

$= \pi \lim\limits_{h \to 0} \dfrac{(x_1 + h)^2 - x_1^2}{h} = 2\pi\ x_1$

When $x = x_1 = 3.5$ $\quad \therefore$ the rate of change in a $= 2 \times \frac{22}{7} \times 3.5 = 22$

Try to solve

(6) A squared lamina expands regularly keeping its shape. Calculate the average rate of change in its surface area when its side length varies from 3cm to 3.4cm, then calculate the rate of change in its surface area when its side length = 5 cm.

Example

(7) In a chemical reaction whose a final product is substance a, it's found that the product after t second is given by the relation $y = t^3$ mg. Find the instantaneous rate to produce the substance a when t = 2 seconds .

Solution

Let $y = f(t) = t^3$, then :

The instantaneous rate to produce substance a is the same rate of change in f.

When $t = t_1$, then the rate of change function in $f = \lim\limits_{h \to 0} \dfrac{f(t + h) - f(t)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(t + h)^3 - t^3}{h} = 3t^2$

when $t = t_1 = 2$ seconds

$\therefore$ the instantaneous rate to produce substance $a = 3(2)^2 = 12$ mg /sec

Try to solve

(7) The size of a bacterial culture at any time t (measured in minute) is given by the relation $f(t) = 2t^3 + 100$ mg, find the rate of the instantaneous growth of the function f when t = 5

Exercises 3 - 1

Choose the correct answer :

1. If the average rate of change in $f = 2.4$ when x varies from 3 to 3.2, then the change in f equals

   a) 0.32 b) 0.48 c) 3.6 d) 7.2

2. If the average rate of change in $f = 5$ when x varies from 2 to 4 and $f(2) = 6$, then $f(4)$ equals

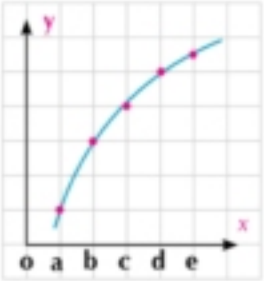
   a) - 4 b) 7 c) 8 d) 16

3. The average rate of change in the cube volume when its edge length varies from 5cm to 7cm equals ....

   a) 125 b) 343 c) 218 d) 109

4. **The opposite figure** shows the curve of the function f where $y = f(x)$. In which of the following intervals the average rate of change in f is the greatest?

   a) [a , b] b) [b , c] c) [c , d] d) [a , e]



Answer each of the following:

5. If $f(x) = x^2 + 2x - 1$, find the variation in f when

   a) x varies from 2 to 2.1 b) $x = -2$ and $h = 1$

6. Find the average rate of change function in f when $x = x_1$, then deduce the rate of change of f at the shown values of x_1 in the following :

   a) $f(x) = 2x^3, \quad x_1 = 2$ b) $f(x) = \frac{x+1}{x-1}, \quad x_1 = 0$

7. Find the average rate of change function in f where $f(x) = \sqrt{x-3}$ when $x = x_1$, then deduce the rate of change in f when $x = 7$

8. **Areas:** A squared lamina shrinks by cooling and maintaining its square shape. Calculate the rate of change in the surface of lamina with respect to its side length when the side length is 8cm.

9. **Volumes:** A metal sphere expands by heating and maintaining its spherical shape. Find the rate of change in the sphere volume with respect to its radius length when the radius length is 7 cm.

(10) **Discover the error :** A rectangular lamina its length is twice its width expands regularly by heating so that it maintains its shape and the same ratio among its dimensions. Calculate the rate of change in the area of the lamina with respect to its width when the width is 10cm.

First solution

Let the width be x, and length $2x$

Area = length × width. $\therefore f(x) = 2x^2$

Rate of change $= \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$

$= \lim_{h \to 0} \frac{2h(2x_1 + h)}{h}$

$= \lim_{h \to 0} (4x_1 + 2h) = 4x_1$

when $x_1 = 10$cm

rate of change = 40

Second solution

Let length be x, and width $\frac{1}{2}x$

Area = length × width $\therefore f(x) = \frac{1}{2}x^2$

Rate of change $= \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$

$= \lim_{h \to 0} \frac{h(x_1 + \frac{1}{2}h)}{h}$

$= \lim_{h \to 0} (x_1 + \frac{1}{2}h) = x_1$

when $x_1 = 10$cm

rate of change = 10

Agriculture:

(11) if y is the quantity (measured in kg) which represents the average production of an orange tree depends upon the number of kilograms x of insecticide used to spray the tree according to the relation $y = 100 - \frac{42}{x+1}$, calculate the average rate of change in y when x varies from 1 to 2.

Geometry:

(12) A sphere shaped soap bubble expands regularly to maintain its spherical shape. Calculate the average rate of change in its spherical surface area when its radius length varies from 0.5 cm to 0.6cm, known that the surface area of the sphere equals $4\pi r^2$ where r is the sphere radius length .

(13) A triangular lamina whose base length equals twice its corresponding height. it expands by heating maintaining its shape . Find the average rate of change in its area if its height varies from 8 cm to 8.4 cm.

Unit three

Differentiation

3 - 2

You will learn

- The first derivative of the function.
- The left derivative of the function .
- The right derivative of the function.
- The differentiability at a point.
- The relation between differentiation and continuity

Key terms

- First Derivative
- Left Derivative
- Right Derivative
- Differentiation
- Differintiable Function

aterials

- Scientific calculator
- Graphical programs

Think and discuss

1. The opposite figure (1) shows the curve of f :]a, b[$\longrightarrow \mathbb{R}$ where $y = f(x)$, $\overleftrightarrow{CD}$ intersects it at the two points C $(x_1, f(x_1))$, D $(x_1 + h, f(x_1 + h))$.

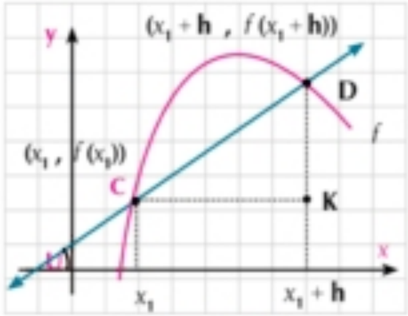


fig. (1)

Find the slope of the secant $\overleftrightarrow{CD}$.

2. let x varies from x_1 to x_1 + h, compare the average rate of change function in f, and the slope of the secant $\overleftrightarrow{CD}$. is the next relation true?

   The slope of the secant $\overleftrightarrow{CD}$ = tan U = $\frac{f(x_1 + h) - f(x_1)}{h}$ = A (h)

3. If C $(x_1, f(x_1))$ is a constant point on the curve of the function f and point D moves on the curve so that it gets near to point C to let $\overleftrightarrow{CD}$ take the position $\overleftrightarrow{CN}$ and become a tangent to the curve at C. i.e. h $\longrightarrow$ 0

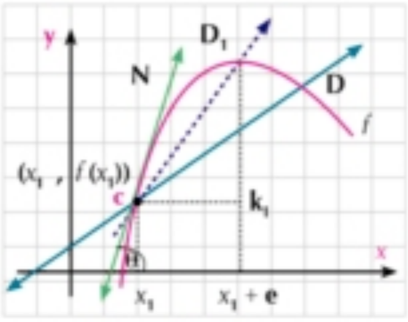


fig. (2)

Find the slope of the tangent to the curve of f at C

Notice :

The slope of the tangent at C = tan θ = $\lim\limits_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$, if found

i.e. :

> the slope of the tangent to the curve of the function f where $y = f(x)$ at point $(x_1, f(x_1))$ equals the rate of change in f when $x = x_1$

Example

1. Find the slope of the tangent to the curve of the function f where $f(x) = 3x^2 - 5$ at point A (2, 7), then find the measure of the positive angle which the tangent makes with the positive direction of x-axis at point A to the nearest minute .

Solution

$\because f(2) = 3(2)^2 - 5 = 7$ $\therefore$ point A (2, 7) belongs to the curve of f

The slope of the tangent at $(x = 2)$ = the rate of change in f when $(x = 2)$

$$= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$\therefore \text{ The slope of the tangent } = \lim_{h \to 0} \frac{3(2+h)^2 - 5 - 7}{h} = \lim_{h \to 0} \frac{12h + 3h^2}{h}$$

$$= \lim_{h \to 0} (12 + 3h) = 12$$

Then $\tan \theta = 12$ $\therefore \theta = \tan^{-1}(12) \simeq 85°14'$

Try to solve

1. Find the slope of the tangent to the curve of the function f where $f(x) = x^3 - 4$ at point A (1, - 3), then find the measure of the positive angle which the tangent makes with the positive direction of x-axis at point A to the nearest minute.

Learn

The Derivative Function

For each value of the variable x in the domain of f is corresponded by a unique value to the rate of change in f , thus , the rate of change is a function in the variable x and called the "derivative function" or the first derivative of the function or first differential coefficient.

Definition

If $f :]a, b[\longrightarrow \mathbb{R}$ and $x \in]a, b[$, then **the derivative function** f' :

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ in a condition that this limit is existed.

Symbols of the derivative function :

If $y = f(x)$, then the first derivative of the function f is denoted by

y' or f' and read as " derivative of y " or derivative of f "

$\frac{dy}{dx}$ read as " dy by dx " or " derivative of y with respect to x"

Notice that the slope of the tangent to the curve of $y = f(x)$ at point $(x_1, f(x_1))$ is $f'(x_1)$

Example

(2) Find the derivative function of the function f where $f(x) = x^2 - x + 1$ using the definition of the derivative, then find the slope of the tangent at the point (- 2 , 7)

Solution

$\because f(x) = x^2 - x + 1$

$\therefore f(x + h) = (x + h)^2 - (x + h) + 1 \quad = x^2 + 2x\,h + h^2 - x - h + 1$,

$f(x + h) - f(x) = (2x + h - 1)\,h$

$\because f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$ $\qquad \therefore f'(x) = \lim\limits_{h \to 0} \dfrac{(2x + h - 1)\,h}{h}$

$f'(x) = \lim\limits_{h \to 0} (2x + h - 1) \quad f'(x) = 2x - 1$

$\because f(-2) = (-2)^2 - (-2) + 1 = 7$ $\qquad \therefore$ point (- 2, 7) lies on the curve of f

the slope of the tangent at point $(-2, 7) = f'(-2) = 2(-2) - 1 = -5$

Try to solve

(2) If $f(x) = 3x^2 + 4x + 7$, find the derivative of the function f using the definition of the derivative, then find the slope of the tangent at the point (- 1 , 6)

Learn

Differentiability of a function at a Point

It is said that the function f is differentiable when $x = a$ (where a belongs to the domain of the function) if and only if $f'(a)$ is existed where $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$

If a derivative is found for the function f at each point belongs to the interval] c , d [, we say that the function is differentiable in this interval.

In example (2) : we find that for each $x \in \mathbb{R}$ there is a derivative to the function f where $f'(x) = 2x - 1$ so the polynomial function is differentiable on $\mathbb{R}$

Example

(3) Prove that $f(x) = \dfrac{x - 1}{x + 1}$ is differentiable when $x = 2$

Solution

$\because$ domain of $f = \mathbb{R} - \{-1\}$ $\therefore f$ is defined when $x = 2$, $f(2) = \frac{1}{3}$

$$f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{2 + h - 1}{2 + h + 1} - \frac{1}{3}}{h}$$

$$= \lim_{h \to 0} \frac{3 + 3h - 3 - h}{3\,h\,(3 + h)} = \lim_{h \to 0} \frac{2\,h}{3h(3 + h)} = \frac{2}{9} \in \mathbb{R}$$

$\therefore f$ is differentiable when $x = 2$

Try to solve

(3) Prove that $f(x) = x^2 - x + 1$ is differentiable when $x = 1$

Right and Left Derivatives

If the function f is defined when $x = a$ (where a belongs to the domain of the function), and the function rule on the right of a differs from its rule on the left of a , we discuss the differentiability when $x = a$ by finding the right derivative of the function which is denoted by the symbol $f'(a^+)$ and the left derivative denoted by $f'(a^-)$ where :

The right derivative $f'(a^+) = \lim\limits_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$, **left derivative** $f'(a^-) = \lim\limits_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$

The function is differentiable at a if and only if $f'(a^+) = f'(a^-)$, and the derivative of the function is denoted by the symbol $f'(a)$

Example

(4) Show that the function f where $f(x) = \begin{cases} x^2 & \text{when } x \leqslant 2 \\ x+2 & \text{when } x < 2 \end{cases}$ is not- differentiable when $x = 2$

Solution

$\because$ Domain of $f = \mathbb{R}$

$\therefore$ The function is defined when $x = 2$ and $f(2) = (2)^2 = 4$

$\therefore$

$$f'(2^-) = \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^2 - 4}{h} = \lim_{h \to 0} \frac{4h + h^2}{h}$$

$\therefore f'(2^-) = 4$

$$f'(2^+) = \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h+2) - 4}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

$\therefore f'(2^+) = 1$

$\because f'(2^+) \neq f'(2^-)$ $\therefore f'(2)$ is not existed. I.e. the function is not-differentiable when $x = 2$

Try to solve

(4) Discuss the differentiability of the function f when $x = 2$ where $f(x) = \begin{cases} x^2 - 5 & \text{when } x < 2 \\ 4x - 9 & \text{when } x \geqslant 2 \end{cases}$

Critical thinking :

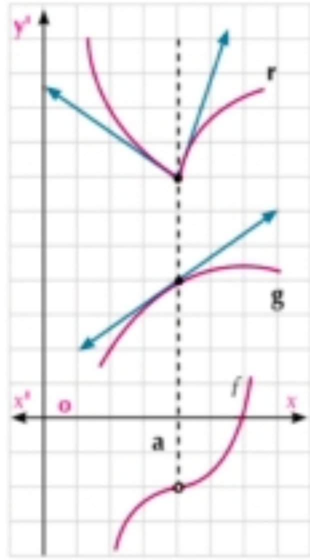
- Discuss the continuity of the two functions in Example (4) and try to solve (4) and deduce the relation between the differentiability of the function at a point in its domain and its continuity at the same point.
- Is the function f where $f(x) = |x - 2|$ differentiable when $x = 2$?

Differentiation and Continuity

Definition

➢ If the function f where $y = f(x)$ is differentiable when $x = a$, then it is continuous at this point.

The opposite figure shows that:



1 - The continuity of a function at a point does not necessarily mean it is differentiable at the same point as in the two functions r and g.

2 - If the function is discontinuous when $x = a$, then the function is not-differentiable when $x = a$ as in the function f .

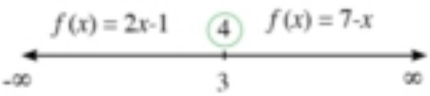
Important note: when discussing the differentiability of a function at a point in its domain, it is favourable to discuss its continuity at this point first. If it is continuous, we discuss the differentiation and if it is discontinuous, then the function is not-differentiable

Example

(5) Discuss the differentiability of the function f at $x = 3$ where $f(x) = \begin{cases} 2x - 1 & \text{when } x < 3 \\ 7 - x & \text{when } x \geqslant 3 \end{cases}$

Solution

Discuss the continuity at $x = 3$



(1) $f(3) = 7 - 3 = 4$

(2) $f(3^-) = \lim_{x \to 3} (2x - 1) = 5$, $f(3^+) = \lim_{x \to 3} (7 - x) = 4$

$\because f(3^-) \neq f(3^+)$ $\therefore \lim_{x \to 3} f(x)$ is not existed and f is discontinuous at $x = 3$

$\because f$ is discontinuous at $x = 3$ $\therefore f$ is not-differentiable at $x = 3$

Try to solve

(5) Discuss the differentiability of the function f where $f(x) = \begin{cases} x^2 + 2 & \text{when } x \geqslant 1 \\ 2x + 1 & \text{when } x < 1 \end{cases}$ $x = 1$

(6) If $f(x) = \begin{cases} x^2 + a & \text{when } x \leqslant 2 \\ ax + b & \text{when } x > 2 \end{cases}$ is differentiable at $x = 2$ then a + b equals:

a 4   b -4   c -8   d 8

Exercises (3 - 2)

Answer the following:

1. Using the definition of derivative, find the derivative of the function f where $f(x) = x^2 - 5$ when $x = 3$ and show the geometric meaning of the derivative of the function when $x = 3$.

2. Using the definition of derivative, find the derivative of the function f where $f(x) = 1 - 5x - 3x^2$ at point $(-1, 3)$, then find the measure of the positive angle which this tangent makes with the positive direction of x-axis to the nearest minute.

3. Find the derivative function for each of the following functions using the definition:
   a. $f(x) = \sqrt{3x + 1}$
   b. $f(x) = \frac{1}{x}$

4. Discuss the differentiability of the function f where $f(x) = \begin{cases} 4 - x^2 & \text{when } x \leqslant 1 \\ 2x + 1 & \text{when } x > 1 \end{cases}$ $x = 1$

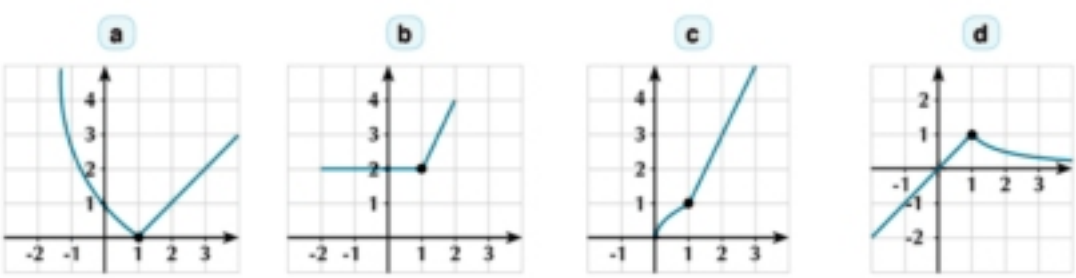
5. Find the value of the constant a if the function f is differentiable at $x = 2$ where

$$f(x) = \begin{cases} 2x + 3 & \text{when } x < 2 \\ a\,x^2 + 8\,x - 1 & \text{when } x \geqslant 2 \end{cases}$$

6. If the function f where $f(x) = \begin{cases} a\,x^2 + 1 & \text{when } x \leqslant 2 \\ 4\,x - 3 & \text{when } x < 2 \end{cases}$ is continuous at $x = 2$, find the value of the constant a , then discuss the differentiability of the function when $x = 2$

7. If $f(x) = a\,x^2 + b$ where a and b are two constants, find :
   a. The first derivative of the function f at any point (x, y).
   b. The two values of a and b if the slope of the tangent to the curve of the function at point $(2, -3)$ lying on it equals 12.

8. Compare the right derivative and the left derivative for each of the following functions and prove that each of them is not-differentiable at point $x = 1$.



Unit three

3 - 3 Rules of Differentiation

You will learn

- Derivative of the constant function.
- Derivative of $f(x) = x^n$
- Derivative of $f(x) = x$
- Derivative of $f(x) = ax^n$
- Derivative of the sum and difference of two functions.
- Derivative of the product of two functions.
- Derivative of the quotient of two functions.
- Derivative of the composite function (chain rule).
- Derivative of $y = (f(x))^n$.

Key terms

- First Derivative
- Product
- Quotient
- Chain Rule

aterials

- Scientific calculator
- Graphical programs.

Explore

1 - Find by using the definition of the first derivative of the function the derivative for each of :

$f(x) = x^3$ $\qquad$ $f(x) = x^5$

2 - Can you discover the derivative of $f(x) = x^7$ without using the definition?

3 - Can you deduce a rule for the derivative of the function f where $f(x) = x^n$?

Learn

Derivative of a Function

1 - Derivative of the constant function

If $y = c$ $\quad$ **where:** $c \in \mathbb{R}$ $\quad$ **then:** $\frac{dy}{dx} = 0$

Notice :

$y = f(x) = c\ ,\quad f(x + h) = c$

$\because \frac{dy}{dx} = f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\therefore \frac{dy}{dx} = f'(x) = \lim\limits_{h \to 0} \frac{c - c}{h} = \text{zero} \ (h \neq 0)$

2 - Derivative of the funciton $f(x) = x^n$

If $y = x^n$ $\quad$ **where:** $n \in \mathbb{R}$ $\quad$ **then:** $\frac{dy}{dx} = n x^{n-1}$

If $y = x$ $\quad$ **then:** $\frac{dy}{dx} = 1$

If $y = a x^n$ $\quad$ **where:** $a, n \in \mathbb{R}$ $\quad$ **then:** $\frac{dy}{dx} = a n x^{n-1}$

Example

(1) Find $\frac{dy}{dx}$ in each of the following:

(a) $y = -3$ $\qquad$ (b) $y = x^4$ $\qquad$ (c) $y = 5x$

(d) $y = \frac{3}{x^2}$ $\qquad$ (e) $y = \sqrt{x^3}$

Solution

(a) $\because y = -3$ $\quad \therefore \frac{dy}{dx} = 0$ $\qquad$ (b) $\because y = x^4$ $\quad \therefore \frac{dy}{dx} = 4x^3$

c $\because y = 5x$ $\quad\therefore \frac{dy}{dx} = 5$

d $\because y = \frac{3}{x^2} = 3x^{-2}$ $\quad\therefore \frac{dy}{dx} = -6x^{-3}$

e $\because y = \sqrt{x^3} = x^{\frac{3}{2}}$ $\quad\therefore \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$ where $x \geqslant 0$

Try to solve

1) Find $\frac{dy}{dx}$ in each of the following:

a $y = -\sqrt{2}$ b $y = \frac{4}{3}\pi x^3$ c $y = \frac{-4}{x^5}$ d $y = \sqrt[3]{x^5}$

Derivative of the sum or difference between two functions

If z and g are two differentiable functions with respect to the variable x, then $z \pm g$ is also differentiable with respect to x and $\frac{d}{dx}(z \pm g) = \frac{dz}{dx} \pm \frac{dg}{dx}$, and in general :

If $f_1, f_2, ..., f_n$ **are differentiable functions with respect to the variable x then:**

$$\frac{d}{dx}(f_1 \pm f_2 \pm f_3 \pm ... \pm f_n)(x) = f_1'(x) \pm f'_2(x) \pm f'_3(x) \pm ... \pm f'_n(x)$$

Example

2) Find $\frac{dy}{dx}$ in each of the following:

a $y = 2x^6 + x^{-9}$ b $y = \frac{\sqrt{x} - 2x}{\sqrt{x}}$

Solution

a $\because y = 2x^6 + x^{-9}$

$\therefore \frac{dy}{dx} = 12x^5 - 9x^{-10}$

b $\because y = \frac{\sqrt{x} - 2x}{\sqrt{x}}$

$= 1 - 2\sqrt{x} = 1 - 2x^{\frac{1}{2}}$

$\therefore \frac{dy}{dx} = 0 - 2 \times \frac{1}{2}x^{1-\frac{1}{2}} = -x^{-\frac{1}{2}}$

Try to solve

2) Find $\frac{dy}{dx}$ if:

a $y = 3x^8 - 2x^5 + 6x + 1$ b $y = \frac{5}{x} + x\sqrt{x} + \sqrt{3}\,x - 4$

The derivative of the product of two functions:

if z and g are two differentiable functions with respect to the variable x, then the function (z . g) is also differentiable with respect to the variable x and $\frac{d}{dx}(z \cdot g) = z\,\frac{dg}{dx} + g \cdot \frac{dz}{dx}$

Example

(3) Find $\frac{dy}{dx}$ if $y = (x^2 + 1)(x^3 + 3)$, then find $\frac{dy}{dx}$ when $x = -1$

Solution

$\because y = (x^2 + 1)(x^3 + 3)$

$$\therefore \frac{dy}{dx} = (x^2 + 1) \times 3x^2 + (x^3 + 3) \times 2x$$
$$= 3x^4 + 3x^2 + 2x^4 + 6x$$
$$= 5x^4 + 3x^2 + 6x$$

when $x = -1$ $\quad \therefore \frac{dy}{dx} = 5(-1)^4 + 3(-1)^2 + 6(-1) = 2$

Try to solve

(3) Find $\frac{dy}{dx}$ if $y = (4x^2 - 1)(7x^3 + x)$, then find $\frac{dy}{dx}$ when $x = 1$

Derivative of the quotient of two functions:

If z and g are two differentiable functions with respect to the variable x and $g(x) \neq 0$, then the function $(\frac{z}{g})$ is also differentiable with respect to the variable x

and $$\frac{d}{dx}\left(\frac{z}{g}\right) = \frac{g\frac{dz}{dx} - z\frac{dg}{dx}}{g^2}$$

i.e $$\left(\frac{z}{g}\right)' = \frac{g z' - z g'}{g^2}$$

Example

(4) Find $\frac{dy}{dx}$ If $y = \frac{x^2 - 1}{x^3 + 1}$

Solution

$\because y = \frac{x^2 - 1}{x^3 + 1}$

$$\therefore \frac{dy}{dx} = \frac{(x^3 + 1) \times 2x - (x^2 - 1) \times 3x^2}{(x^3 + 1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2x^4 + 2x - 3x^4 + 3x^2}{(x^3 + 1)^2}$$
$$= \frac{-x^4 + 3x^2 + 2x}{(x^3 + 1)^2}$$

Try to solve

(4) Find $\frac{dy}{dx}$ if $y = \frac{x^3 + 2x^2 - 1}{x + 5}$

the composite function (Chain rule)

Work together

Work with a classmate

If $y = (3x^2 - 1)^4$, find $\frac{dy}{dx}$

Do you need long mathematical operations? Have you faced difficulties to find the derivative function?

You have previously learned the composition of the function and known that :

$(f \circ r)(x) = f[r(x)]$

If f and r are two functions where $y = f(z)$ and $z = r(x)$, then $y = f[r(x)]$

and we say that y is the composite function of x

Theorem

If $y = f(z)$ is differentiable with respect to the variable z , and $z = r(x)$ is differentiable with respect to the variable x, then $y = f(r(x))$ is differentiable with respect to the variable x and: $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$

This theorem is known as the chain rule

Example

5 If $y = (x^2 - 3x + 1)^5$, find $\frac{dy}{dx}$

Solution

let $z = x^2 - 3x + 1$ $\therefore y = z^5$

it is clear that y is differentiable with respect to z (polynomial at z) and $\frac{dy}{dz} = 5z^4$

and also z is differentiable with respect to x (polynomial at x) and $\frac{dz}{dx} = 2x - 3$,

By applying the chain rule $\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = 5z^2 \times (2x - 3)$

By substituting z

$$\therefore \frac{dy}{dx} = 5(x^2 - 3x + 1)^4 \times (2x - 3)$$

Try to solve

(5) Find $\frac{dy}{dx}$ using the chain rule in **work together** and check your previous work.

Example

(6) If $y = \sqrt[3]{z}$, $z = x^2 - 3x + 2$, find $\frac{dy}{dx}$

Solution

$\because y = z^{\frac{1}{3}}$ $\qquad \frac{dy}{dz} = \frac{1}{3} z^{-\frac{2}{3}}$

$\because z = x^2 - 3x + 2$ $\qquad \frac{dz}{dx} = 2x - 3$

a $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$ $\quad = \frac{1}{3} z^{-\frac{2}{3}} (2x - 3)$ $\qquad \therefore \frac{dy}{dx} = \frac{1}{3} (2x - 3)(x^2 - 3x + 2)^{-\frac{2}{3}}$

Try to solve

(6) If $y = 3z^2 - 1$, $z = \frac{5}{x}$, find $\frac{dy}{dx}$

Derivative of the function $[f(x)]^n$

If $z = [f(x)]^n$ where f is differentiable with respect to x and n is a real number ,

then: $\frac{dz}{dx} = n\,[f(x)]^{n-1} \times f'(x)$

Example

(7) Find $\frac{dy}{dx}$ if

a $y = (6x^3 + 3x + 1)^{10}$ $\qquad$ **b** $y = (\frac{x-1}{x+1})^5$

Solution

a $y = (6x^3 + 3x + 1)^{10}$

$$\therefore \frac{dy}{dx} = 10(6x^3 + 3x + 1)^9 \times \frac{d}{dx}(6x^3 + 3x + 1)$$
$$= 10(18x^2 + 3)(6x^3 + 3x + 1)^9$$
$$= 30(6x^2 + 1)(6x^3 + 3x + 1)^9$$

b $y = (\frac{x-1}{x+1})^5$ $\qquad \therefore \frac{dy}{dx} = 5(\frac{x-1}{x+1})^4 \times \frac{(x+1) \times 1 - (x-1) \times 1}{(x+1)^2}$

$$= 5(\frac{x-1}{x+1})^4 \times \frac{x+1-x+1}{(x+1)^2}$$
$$= \frac{10}{(x+1)^2} \times (\frac{x-1}{x+1})^4 = \frac{10(x-1)^4}{(x+1)^6}$$

Try to solve

(7) Find $\frac{dy}{dx}$ if $y = (\frac{5x^2}{3x^2+2})^3$

Example

(8) If $f(x) = \frac{1}{3}x^3 - 2x^2 + 5x - 4$, find the values of x which make $f'(x) = 2$

Solution

$f'(x) = \frac{1}{3} \times 3x^2 - 2 \times 2x + 5 \times 1$

$= x^2 - 4x + 5$

When $f'(x) = 2$ $\therefore x^2 - 4x + 5 = 2$ and $x^2 - 4x + 3 = 0$

$\because (x-1)(x-3) = 0$ $\therefore x = 1$ or $x = 3$

Try to solve

(8) Find the values of x which make $f'(x) = 7$ in each of the following:

a $f(x) = x^3 - 5x + 2$

b $f(x) = (x - 5)^7$

Exercises 3 - 3

Complete:

(1) $\frac{d}{dx}(\frac{1}{x^3}) =$ ....

(2) $\frac{d}{dx}(5\pi) =$ ....

(3) $\frac{d}{dx}(x^{\frac{1}{2}}) =$ ....

(4) $\frac{d}{dx}(\frac{1}{\sqrt[3]{x}}) =$ ....

(5) $\frac{d}{dx}(5x^2 + 3x + 2) =$ ....

(6) $\frac{d}{dx}(\sqrt{2}\,x^7 - \frac{x^5}{5} + \pi) =$ ....

Find the first derivative for each of the following functions with respect to x.

(7) $y = 2x^6 + 3\sqrt{x}$

(8) $y = \frac{4x^2 - 3x + 2\sqrt{x}}{x}$

(9) $y = x(3x^2 - \sqrt{x})$

(10) $y = \frac{4x^2 - x + 3}{\sqrt{x}}$

(11) $y = (x-1)(x+1)(x^2+1)(x^4+1)(x^8+1)(x^{16}+1)$

Find the first derivative for each of the following functions

(12) $y = (x^2 + 3)(x^3 - 3x + 1)$

(13) $y = (x^2 - \sqrt{x})(x^2 + 2\sqrt{x})$

(14) $y = (2x^4 - 3x + 4)(x^2 - + \sqrt{x} + \frac{2}{x})$

(15) $y = \frac{5x - 2}{5 \quad x + 1}$

(16) $y = \frac{x^2 + 2x + 5}{x^2 - 5x + 1}$

(17) $y = \frac{x - 2}{x + 5}$

(18) Find $\frac{dy}{dx}$ when $x = 2$ for each of the following:

a) $y = (x^3 + x - 9)^5$

b) $y^3 = 3x^2 - 4$

c) $y = \sqrt[3]{(2x^3 + 4x + 3)^2}$

d) $y = z^2 \quad , \quad z = 3x^2 - 2$

e) $y = 2z^3 \quad , \quad z = 8x - 11$

f) $y = \frac{z - 1}{z + 1}, z = \frac{x + 1}{x - 1}$

(19) If $y = a\,x^3 + b\,x^2$ and $\frac{dy}{dx} = 8$ when $x = 1$, and the average rate of change of y when x varies from - 1 to 2 equals 7, find the values of the two constant a and b.

(20) Find the value of $\frac{dy}{dx}$ at the point shown in each of the following:

a) $y = (\frac{x^2 - 2}{3 + x^2})^7$ when $x = 0$

b) $y = (x^2 + 1)^5 (x^2 - x + 1)^{-4}$ when $x = 1$

Activity

(21) **Volumes** if the oil is poured at a rate of 10 cm^3/sec in a cylindric barrel whose radius length of its base is 90 cm, find the rate of rising the oil in the barrel.

Creative thinking

(22) Find $\frac{d}{dx}(y^n)$, where y is a function of x.

(23) If $y = 3z^2 + 1$ and $z = \sqrt{x^2 - 2}$, find $\frac{dy}{dx}$. Is one of the following two solution wrong or both are true?

First solution

$y = 3z^2 + 1$

$z = \sqrt{x^2 - 2}$

By substituting , we find that

$y = 3x^2 - 5$

$\frac{dy}{dx} = 6x$

Second solution

$\because y = 3z^2 + 1 \qquad \therefore \frac{dy}{dz} = 6z$

$z = (x^2 - 2)^{\frac{1}{2}}$

$\frac{dz}{dx} = \frac{1}{2}(x^2 - 2)^{-\frac{1}{2}} \times 2x = \frac{x}{\sqrt{x^2 - 2}}$

$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$

$= 6\sqrt{x^2 - 2} \times \frac{x}{\sqrt{x2 - 2}} = 6x$

Derivatives of Trigonometric Functions

3 - 4

Explore

In our learning here, the measures of the angles are in radian measurement.

the opposite figure shows the curve of the sine function $y = \sin x$ and when it is translated to the left of a magnitude of $\frac{\pi}{2}$, the curve of the cosine function $y = \cos x$ is drawn also : $\cos x = \sin(\frac{\pi}{2} + x) = \sin(\frac{\pi}{2} - x)$, so, it is enough to find the derivative of $\sin x$ using the definition and deducing the rest of the other trigonometric derivatives.

You will learn

- Derivative of the trigonometric functions
  $f(x) = \sin x$
  $f(x) = \cos x$
  $f(x) = \tan x$

Key terms

- Derivative
- Trigonometric Function

aterials

- Scientific calculator
- Graphic programs

Derivatives of Trigonometric Function

Learn **The derivative of sine function**

If $f(x) = \sin x$ **then** $f'(x) = \cos x$

$\because f(x) = \sin x, \quad f(x + h) = \sin(x + h)$

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\cos x \sin h}{h} + \frac{\sin x(\cos h - 1)}{h} \right]$$

$$= \cos x \lim_{h \to 0} \frac{\sin h}{h} + \sin x \times \lim_{h \to 0} \frac{\cos h - 1}{h}$$

$$= \cos x \times 1 + \sin x \times 0 = \cos x$$

i.e : $\frac{d}{dx}(\sin x) = \cos x$

✱ **(The proof is not required from students)**

In general If z is a differentiable function with respect to the variable x, then :

$\frac{d}{dx}[\sin z] = \cos z \cdot \frac{dz}{dx}$ [**chain rule**]

Example

1 Find $\frac{dy}{dx}$ for each of the following :

a $y = 5 \sin x$ b $y = x^3 \sin x$ c $y = 2 \sin (3x + 4)$

Solution

a $\because y = 5 \sin x$ $\therefore \frac{dy}{dx} = 5 \times \frac{d}{dx}(\sin x) = 5 \cos x$

b $\because y = x^3 \sin x$ $\therefore \frac{dy}{dx} = x^3 \times \frac{d}{dx}(\sin x) + \sin x \times \frac{d}{dx}(x^3)$

$= x^3 \cos x + 3x^2 \sin x$

c $\because y = 2 \sin (3x + 4)$

putting $z = 3x + 4$ then : $y = 2 \sin z$

$\therefore \frac{dz}{dx} = 3$, $\frac{dy}{dx} = 2 \cos z$ and by applying the chain rule ($\frac{dy}{dx} = \frac{dy}{dx} \times \frac{dz}{dx}$)

$\therefore \frac{dy}{dx} = 2 \cos z \times 3 = 2 \cos (3x + 4) \times 3 = 6 \cos (3x + 4)$

We can find $\frac{dy}{dx}$ directly using the generalization above as follows:

$\frac{dy}{dx} = 2 \times \cos (3x + 4) \times 3 = 6 \cos (3x + 4)$

Try to solve

1 Find $\frac{dy}{dx}$ for each of the following :

a $y = x^2 + \sin x$ b $y = \sin \frac{\pi}{4} - 7 \sin x$ c $y = 5 \sin (3 - 2x)$

Learn

1- Derivative of the cosine function

If $y = \cos x$ then $\frac{dy}{dx} = -\sin x$

2- Derivative of the tangent function

If $y = \tan x$ then $\frac{dy}{dx} = \sec^2 x$

Notice:

(1) $\frac{d}{dx}(\cos x) = \frac{d}{dx}[\sin(\frac{\pi}{2} - x)]$

$= \cos(\frac{\pi}{2} - x) \times -1$

$= -\sin x$

(2) $\frac{d}{dx}(\tan x) = \frac{d}{dx}(\frac{\sin x}{\cos x})$

$= \frac{\cos x \times \cos x - (-\sin x) \times \sin x}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

Example

(2) **Find the first derivative for each of the following :**

(a) $y = 2\cos x - \tan 5x$ (b) $y = \tan(1 - x^2)$ (c) $y = \cos^2(4x^2 - 7)$

Solution

(a) $\because y = 2\cos x - \tan 5x$

$\therefore \frac{dy}{dx} = 2(-\sin x) - \sec^2 5x \times 5$ $= -2\sin x - 5\sec^2 5x$

(b) $\because y = \tan(1 - x^2)$ $\therefore \frac{dy}{dx} = \sec^2(1 - x^2) \times -2x = -2x\sec^2(1 - x^2)$

(c) $\because y = \cos^3(4x^2 - 7)$ let $y = \cos^3 z$ where $z = 4x^2 - 7$

$\therefore \frac{dy}{dz} = 3\cos^2 z \times -\sin z = -3\sin z\cos^2 z$, $\frac{dz}{dx} = 8x$

$\because \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = -24x\sin z\cos^2 z$

$\therefore \frac{dy}{dx} = -24x\sin(4x^2 - 7)\cos^2(4x^2 - 7)$

Try to solve

(2) find $\frac{dy}{dx}$ for each of the following :

(a) $y = 2\tan 3x$ (b) $y = 2\cos(4 - 3x^2)$ (c) $y = 2\sin x\cos x$

(d) $y = 2x\tan x$ (e) $y = \tan^2 3x$ (f) $y = \tan 4x^3$

Example

(3) If $y = \frac{\cos x}{1 - \sin x}$ prove that $(1 - \sin x)\frac{dy}{dx} = 1$, then find $\frac{dy}{dx}$ when $x = \frac{\pi}{6}$

Solution

$\because y = \frac{\cos x}{1 - \sin x}$ $\therefore \frac{dy}{dx} = \frac{(1 - \sin x)(-\sin x) - \cos x(-\cos x)}{(1 - \sin x)^2}$

$$\frac{dy}{dx} = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1-\sin x} \qquad \text{i.e}: (1-\sin x)\frac{dy}{dx} = 1$$

Remember

$\sin^2 x + \cos^2 x = 1$

when $x = \frac{\pi}{6}$, then $(1 - \sin \frac{\pi}{6})\frac{dy}{dx} = 1$

$$(1 - \frac{1}{2})\frac{dy}{dx} = 1 \qquad \therefore \frac{dy}{dx} = 2$$

Try to solve

3) If $y = 2x \sin x \cos x$ prove that: $\frac{dy}{dx} = \sin 2x + 2x \cos 2x$

Activity

4) **Mechanics:** A force of a magnitude F has acted on an object of weight (w) in the direction that creates an angle of measurement θ with the direction of the motion and the magnitude of the force is given by the rule $F = \frac{m\,w}{m \sin\theta + \cos\theta}$ where m is constant called the friction coefficient

a) Find the rate of change of the force F with respect to the angle θ

b) When does the rate of change equal zero?

Answer of the Activity:

a) $F = \frac{m\,w}{m \sin\theta + \cos\theta} = m\,w\,(m \sin\theta + \cos\theta)^{-1}$

the rate of change of the force with respect to $\theta = \frac{dF}{d\theta}$

$$= -mw\,(m \sin\theta + \cos\theta)^{-2}\,(m \cos\theta - \sin\theta) = \frac{-mw\,(m\cos\theta - \sin\theta)}{(m \sin\theta + \cos\theta)^2}$$

b) when $\frac{dF}{d\theta} = 0 \qquad \therefore m\,w\,(m\cos\theta - \sin\theta) = 0$

$\therefore m\cos\theta - \sin\theta = 0 \qquad m\cos\theta = \sin\theta \qquad m = \frac{\sin\theta}{\cos\theta} = \tan\theta$

Application on the activity

4) If $y = \sin^2 x - \cos 2x - 4x$:

a) Find the rate of change of y with respect to the variable x.

b) Find the values of $x \in]0, \pi[$ when the rate of change equals -1

Creative thinking: Find $\frac{dy}{dx}$ if:

a) $y = \sin(\tan 3x)$ b) $y = \sin x$ where x is measured by the degree measurement

Exercises 3 - 4

Choose the correct answer:

1) If $y = \sin(2x+5)$, then $y' =$

a) $2\cos 2x$ b) $-2\cos 2x$ c) $\cos(2x+5)$ d) $2\cos(2x+5)$

2) If $y = 3x - \cos 2x$, then $y' =$

a) $3 - \sin 2x$ b) $3 + \sin 2x$ c) $3 + 2\sin 2x$ d) $3 - 2\sin 2x$

3) If $y = 3\cos(2 - 4x)$, then $y' =$

a) $4\sin(2 - 4x)$ b) $12\sin(2 - 4x)$ c) $-6\sin(2 - 4x)$ d) $-12\sin(2 - 4x)$

4) If $f(x) = \tan(5x - \pi)$, then $f'(\frac{\pi}{4}) = ...$

a) 5 b) $5\sqrt{2}$ c) 10 d) $10\sqrt{3}$

Complete :

5) $\frac{d}{dx}(\cos x - \sin x) =$ ....

6) $\frac{d}{dx}(\tan 3x^2) =$ ....

7) $\frac{d}{dx}(\cos 2\pi) =$ ....

8) $\frac{d}{dx}(\cos^2 x + \sin^2 x) =$ ....

9) $\frac{d}{dx}(x\cos x) =$ ....

10) Find $\lim_{x \to a} \frac{\sin x - \sin a}{x - a} =$ ....

Find $\frac{dy}{dx}$ in each of the following:

11) $y = \sin(4x + 7)$

12) $y = \sin(x^2 + 3)$

13) $y = \cos(5x + 3)$

14) $y = 3\tan(2x + 3)$

15) $y = \tan(-5x^2 + 19)$

16) $y = \sin(\cos^2 x)$

17) $y = \frac{\tan x}{x}$

18) $y = x\sin(3x - 2)$

19) $y = \frac{x}{\cos x}$

20) $y = x^2\sin(2x^2 + 5)$

21) $y = \frac{\sin x}{1 + \cos x}$

22) $y = \sec^2 x - 1$

(23) $y = \sin\left(\frac{1}{x^2}\right)$

(24) $y = \tan\sqrt{x}$

(25) $y = 4\cos^5 x$

(26) $y = 4x + 5\sin 4x$

(27) $y = \sqrt{3x - \sin^2 4x}$

(28) $y = \cos^3\left(\frac{x}{x+1}\right)$

(29) $y = \sqrt{\cos(5x)}$

(30) $y = \cos(\cos x)$

(31) $y = \frac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}}$

Find the slope of the tangent for each of the following curves:

(32) $y = 5 - \sin x$ when $x = \frac{\pi}{4}$

(33) $y = \sin x + \sin 2x$ when $x = \frac{\pi}{2}$

(34) $y = x\cos 2x$ when $x = \frac{\pi}{4}$

(35) $y = x\sqrt{\sin x}$ when $x = \frac{\pi}{2}$

(36) Prove that the tangent to the curve of $y = \cos x$ when $x = \frac{\pi}{2}$ makes with the positive direction of x-axis a positive angle of measurement $\frac{3\pi}{4}$.

(37) If $y = \sin^2 x - \cos^2 x$, Prove that $\frac{dy}{dx} = 2\sin 2x$

(38) If $y = (\sin x + \cos x)^2$, Prove that $\frac{dy}{dx} = 2\cos 2x$

(39) If $y = \frac{\sin x}{1 + \cos x}$, Prove that $(1 + \cos x)\frac{dy}{dx} = 1$

(40) If $y = \sec 4x$, find the rate of change of y with respect to x when $x = \frac{\pi}{4}$

Applications on Derivatives

3 - 5

Introduction

Geometric applications on the derivative of the function require finding the equation of the straight line in terms of its slope and a point lies on it. Remember the relation between the two slopes of the two parallel straight lines and the two perpendicular straight lines.

Slope of the Tangent and the Normal to a Curve

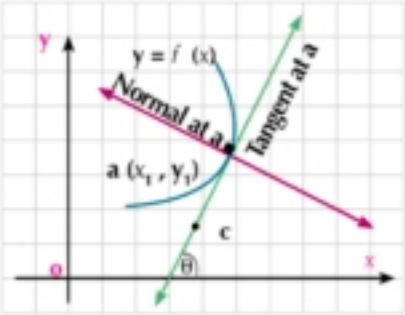
In this unit, you knew that:

- The first derivative of the function f where $y = f(x)$ means the slope of the tangent to the curve of this function at any point (x, y) lying on it.

i.e.: The slope of the tangent to the curve of $y = f(x)$ at point $a(x_1, y_1)$ lying on it $= [\frac{dy}{dx}]_{(x_1, y_1)}$

then

$\tan\theta = [\frac{dy}{dx}]_{(x_1, y_1)}$



where θ is the measurement of the positive angle which the tangent makes with the positive direction of x-axis .

Notice :

- If m_1 and m_2 are the two slopes of two known straight lines ℓ_1 and ℓ_2, then:
- $\ell_1 \parallel \ell_2$ if and only if $m_1 = m_2$ (parallel condition)
- $\ell_1 \perp \ell_2$ if and only if $m_1 m_2 = -1$ (perpendicular condition)

So :

slope of the normal on the curve of $y = f(x)$ at point (x_1, y_1) lying on it $= -\frac{1}{[\frac{dy}{dx}]_{(x_1, y_1)}}$

You will learn

- The slope of the tangent to the curve of the function.
- Slope of the normal

Key terms

- Slope of the tangent
- Slope of the normal

aterials

- Scientific calculator
- Graphic program

Remember

Slope of the straight line $ax + by + c = 0$ **equals** $\frac{-a}{b}$

Example

(1) Find the points which lie on the curve of $y = x^3 - 4x + 3$ at which the tangent makes a positive angle of measure 135° with the positive direction of x axis .

Solution

$\because \ y = x^3 - 4x + 3 \qquad \therefore \frac{dy}{dx} = 3x^2 - 4$

$\therefore$ the tangent makes an angle of measure 135° with the positive direction of x-axis

$\therefore$ the slope of the tangent = tan 135 = -1

$\therefore \frac{dy}{dx} = 3x^2 - 4 = -1 \qquad \therefore 3x^2 = 3 \qquad \therefore x = \pm 1$

when $x = -1 \qquad \therefore y = (-1)^3 - 4(-1) + 3 = 6$

, when $x = 1 \qquad \therefore y = 1 - 4 + 3 = 0$

$\therefore$ the points are (-1 , 6) , (1, 0)

Try to solve

(1) Find the points which lie on the curve of $y = x^2 - 2x + 3$ at which the tangent to the curve is :

a Parallel to x-axis   b Perpendicular to the straight line $x - 4y + 1 = 0$

Example

(2) Find the slope of the normal on the curve of $y = \tan(\pi - \frac{2}{3}x)$ at point $(\pi, \sqrt{3})$

Solution

$\because \ y = \tan(\pi - \frac{2}{3}x) \qquad \therefore \frac{dy}{dx} = -\frac{2}{3} \sec^2(\pi - \frac{2}{3}x)$

the slope of the tangent to the curve at point $(\pi, \sqrt{3}) = -\frac{2}{3} \sec^2(\pi - \frac{2\pi}{3})$

$$= -\frac{2}{3} \sec^2 \frac{\pi}{3} = \frac{-2}{3} \times 4 = \frac{-8}{3}$$

the slope of the normal at point $(\pi, \sqrt{3}) = \frac{3}{8}$

Try to solve

(2) Find the measurement of the positive angle which the normal on the curve of $y = \sqrt{2x^2 + 7}$ makes with the positive direction of x-axis at point (-3, 5) to the nearest minute .

Creative thinking :

Find the value of a which makes the straight line $y = 4x + a$ is a tangent to the curve of $y = x^2 + 5$

Learn

The equations of the Tangent and Normal to a Curve

If (x_1, y_1) is a point lying on the curve of the function f where $y = f(x)$,and m is the slope of the tangent at this point, then:

1 - The equation of the tangent to the curve at point (x_1, y_1) is :

$y - y_1 = m(x - x_1)$

2 - The equation of the normal to the curve at point (x_1, y_1) is :

$y - y_1 = \frac{-1}{m}(x - x_1)$

Example

(3) Find the two equations of the tangent and normal to the curve of $y = 2x^3 - 4x^2 + 3$ at the point lying on the curve and whose abscissa = 2

Solution

$\because y = 2x^3 - 4x^2 + 3$

$\therefore$ when $x = 2$ $\qquad \therefore y = 2(2)^3 - 4(2)^2 + 3 = 3$

$\therefore$ point (2 , 3) lies on the curve

$\because \frac{dy}{dx} = 6x^2 - 8x$ $\qquad \therefore [\frac{dy}{dx}]_{(2,3)} = 6(2)^2 - 8(2) = 8$

` the slope of the tangent = 8 and its equation is

$y - 3 = 8(x - 2)$ $\qquad$ i.e $y - 8x + 13 = 0$

the slope of the normal = $\frac{-1}{8}$ and its equation is

$y - 3 = \frac{-1}{8}(x - 2)$ $\qquad$ i.e. $8y + x - 26 = 0$

Try to solve

(3) Find the two equations of the tangent and normal to the curve of $y = \frac{x+3}{x+1}$ at the point lying on the curve and whose abscissa = 1.

Does the point A (-3, 4) lie on the tangent ? **Explain**

Example

(4) Find the equation of the tangent to the curve of $y = 4x - \tan x$ at point $(\frac{\pi}{4}, f(\frac{\pi}{4}))$

Solution

$\because$ $y = 4x - \tan x$ $\qquad\therefore \frac{dy}{dx} = 4 - \sec^2 x$

when $x = \frac{\pi}{4}$ $\qquad m = 4 - \sec^2 \frac{\pi}{4} = 2$

and by substituting $x = \frac{\pi}{4}$ in the equation of the curve **we find that:** $y = \pi - \tan \frac{\pi}{4} = \pi - 1$

i.e.: point $(\frac{\pi}{4}, \pi - 1)$ lies on the curve

$\therefore$ the equation of the tangent at point $(\frac{\pi}{4}, \pi - 1)$ is: $y - (\pi - 1) = 2(x - \frac{\pi}{4})$

$y - \pi + 1 = 2x - \frac{\pi}{2}$ $\qquad y = 2x + \frac{\pi}{2} - 1$

Try to solve

(4) Find the equations of the tangent and normal to the curve of $y = x \sin 2x$ at point $(\frac{\pi}{4}, \frac{\pi}{4})$

Example

(5) If the curve $y = ax^3 + bx^2$ touches the straight line $y = 8x + 5$ at point $(-1, -3)$, find the two values of a and b.

Solution

$\because$ point $(-1, -3)$ lies on the curve $y = ax^3 + bx^2$

$\therefore -3 = a(-1)^3 + b(-1)^2$ i.e. $a - b = 3$ (1)

the slope of the tangent to the curve at any point on it $= \frac{dy}{dx} = 3ax^2 + 2bx$

$\because$ the straight line $y = 8x + 5$ is a tangent to the curve at point $(-1, -3)$

$\therefore [\frac{dy}{dx}]_{(-1, -3)} =$ the slope of the straight line $= 8$

$\therefore 3a(-1)^2 + 2b(-1) = 8$ i.e. $3a - 2b = 8$ (2)

By solving the two equations (1) and (2) **we find that:** $a = 2$, $b = -1$

Try to solve

(5) Find the value of the two constants a and b if the slope of the tangent to the curve of $y = x^2 + ax + b$ at point $(1, 3)$ lying on it equals 5

Example **Areas**

(6) If the tangent to the curve of $y = \frac{4}{x}$ at point C at the first quadrant intersects the two coordinate axes at the two points M and N, prove that the area of the triangle M O N is constant and doesnot depend on the position of point C lying on the curve of the function.

Solution

$\because$ $y = 4x^{-1}$

Let C $(a, \frac{4}{a})$ $\qquad \because \frac{dy}{dx} = \frac{-4}{x^2}$

$\therefore$ the slope of the tangent at C $(a, \frac{4}{a})$ equals $\frac{-4}{a^2}$

the equation of the tangent at C is : $y - \frac{4}{a} = -\frac{4}{a^2}(x - a)$

By multiplying $\times a^2$ $\quad \therefore a^2 y - 4a = -4x + 4a$

$\therefore$ the equation of $\overleftrightarrow{MN}$ is $a^2 y + 4x = 8a$

to find the intersection point of the straight line $\overleftrightarrow{MN}$ with x-axis, then $y = 0$

$\therefore x = 2a$ $\qquad$ i.e. O M = 2a of units

to find the intersection point of the straight line $\overleftrightarrow{MN}$ with y-axis, then $x = 0$

$\therefore y = \frac{8}{a}$ $\qquad$ i.e. O N $= \frac{8}{a}$ of units

$\therefore$ Area $\triangle$ M O N $= \frac{1}{2} \times 2a \times \frac{8}{a} = 8$ square units

it is a constant amount which doesnot depend on the coordinate of point C lying on the curve.

Try to solve

Areas:

(6) Find the surface area of the triangle formed from x-axis , the tangent and the normal to the curve of $y = x^2 - 6x + 13$ at point (4, 5) lying on it.

Exercises 3 - 5

(1) **Complete each of the following:**

a) The slope of the tangent to the curve of the function f where $y = f(x)$ at any point on it is ....

b) The slope of tangent to the curve of $y = \cos x$ when $x = \frac{\pi}{3}$ equals ....

c) If the straight line $y = 8 - 3x$ is a tangent to the curve of the function f at point (3, -1) then $f'(3)$ equals ....

d) The tangent to the curve of $y = (3x - 5)^3$ at point (1, 2) makes with the positive direction of x-axis a positive angle of tangent equals ....

e The slope of the normal to the curve of $y = \sin 2x$ at the point which lies on the curve and its x-coordinate $= \frac{\pi}{6}$ equals ....

f The equation of the tangent to the curve of $y = (x - 1)^2$ at point $(2, 1)$ is ....

Answer the following:

(2) Find the measure of the positive angle, which the tangent to the curve of $y = x^2 + \frac{1}{x} - 1$ makes with the positive direction of x-axis when $x = 1$

(3) Find the measure of the positive angle, which the tangent to the curve of $y = \frac{x+3}{x-2}$ makes with the positive direction of x-axis at point $(3, 6)$ ....

(4) Find the point lying on the curve of $y = x^3 - 6x^2 - 15x + 20$ at which the tangent is parallel to x-axis .

(5) Find the points on the curve of $y = 3x^3 - 11x + 5$ at which the tangent

a Parallel to the straight line $2x + y - 5 = 0$

b Perpendicular to the straight line $25y + x = 6$

c Makes a positive angle with the positive direction of x-axis whose tangent $= -11$

(6) Find the equation of the tangent to the curve of the function $y = (x - 2)(x + 1)$ at its two intersection points with x-axis.

(7) Find the equation of the normal on the tangent to the curve $y = \frac{x^2 - 1}{2 - x^2}$ when $x = 0$

(8) Find the equation of the tangent to the curve of $y = 2\sin x + \cos x$ at point $(0, 1)$.

(9) Prove that the tangent drawn to the curve of $y = x^2 + x - 1$ at point $(1, 1)$ is perpendicular to the tangent drawn to the curve of $y = 2 - \sqrt[3]{x}$ at the same point .

(10) If the curve of $y = (x^2 - 2x)(ax + b)$ touches the x-axis at point $(2, 0)$, and touches the straight line $y = 2x$ at the origin point, find the two values of a and b.

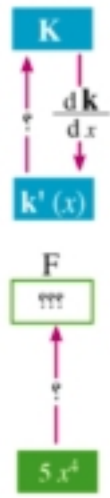
Integration

3 - 6

Explore

You have already learned the differentiation and now you are able to find the derivative k' if you know the function k where $k'(x) = \frac{d}{dx} k(x)$. In this lesson , we are going to learn the antidifferentiation operation. In other words, How can we get the original function if the derivative k' is known?

K is continous



to find the original function whose derivative with respect to x is $5x^4$, Let $f(x) = 5x^4$

Let's start with an inversed operation to the differentiation operation

$n\,x^{n-1} = 5x^4$ $\therefore n - 1 = 4 , n = 5$

then $F(x) = x^5$ or $x^5 + 3$ or $x^5 - 2$

the function f is called the antiderivative function or the original function of the function f

You will learn

- Antiderivative of the function
- Integration of some algebraic functions.
- Integration of some trigonometric functions.

Key terms

- Antiderivative
- Integraation

aterials

- Scientific calculator
- Graphic programs

Learn

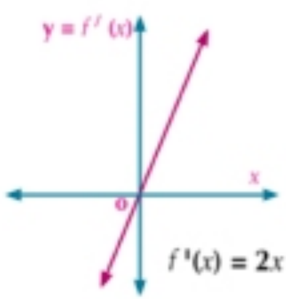
Antiderivative

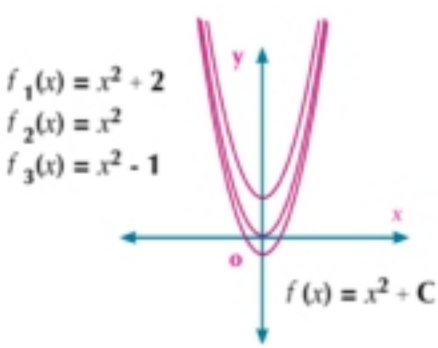
If $y = x^2$ **then the first derivative** is $\frac{dy}{dx} = 2x$

But deducting the function y from the derivative function $\frac{dy}{dx}$ is called the integration operation or the antiderivative .

For example x^2 is the antiderivative to the function $2x$, notice that $2x$ has several antiderivatives such as $x^2 - 1$, $x^2 + 2$, $x^2 + C$ where $C \in \mathbb{R}$, ... and all its derivatives are $2x$.

$\therefore \frac{d}{dx}(x^2 + C) = 2x$ where (C) is a constant and the following figures illustrate that.





Definition

It is said that the function F is antiderivative to the function f, if $F'(x) = f(x)$ for each x in the domain of f.

Example

1 Prove that the function f where $F(x) = \frac{1}{2}x^4$ is an antiderivative to the function f where $f(x) = 2x^3$.

Solution

Find the derivative of the function F, then $F'(x) = \frac{1}{2} \times 4x^3 = 2x^3$

$\therefore F'(x) = f(x)$ **i.e.** the function(F) is antiderivative to the function f

Try to solve

1 Show that the function F where $F(x) = \frac{1}{2}x^6$ is an antiderivative to the function f where $f(x) = 3x^5$

Critical thinking:

What is the relation between F_1 and F_2 if each of them is an antiderivative to the function f?

Indefinite Integral

The set of the antiderivatives to the function f is called an indefinite integral of this function and is denoted by the symbol $\int f(x)\,dx$ [and read: integral of $f(x)$ with respect to x]

Definition

If $F'(x) = f(x)$, then $\int f(x)\,\text{f}\,x = F(x) + C$

where C is an arbitrary constant.

Notice:

$\frac{d}{dx}(x^3 + 5) = 3x^2$ $\therefore \int 3x^2\,dx = x^3 + C$

$\frac{d}{dx}(x^5 - 3) = 5x^4$ $\therefore \int 5x^4\,dx = x^5 + C$

$\frac{d}{dx}(2x^7) = 14x^6$ $\therefore \int 14x^6\,dx = 2x^7 + C$

To identify the value of the constant C, it is necessary to know the value of integration at a certain value of the independent variable x it is out side your study.

Example

2 **Check the correctness for each of the following:**

a $\int x^7\,dx = \frac{1}{8}x^8 + C$

b $\int \frac{x}{\sqrt{1+x^2}}\,dx = \sqrt{1+x^2} + C$

Solution

(a) $\because \frac{d}{dx}\left(\frac{1}{8}x^8 + C\right) = x^7 \quad \therefore \int x^7\, dx = \frac{1}{8}x^8 + C$

(b) $\because \frac{d}{dx}\left(\sqrt{1+x^2} + C\right) = \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$

$\therefore \int \frac{x}{\sqrt{1+x^2}}\, dx = \sqrt{1+x^2} + C$

Try to solve

(2) **Check the correctness for each of the following:**

(a) $\int x^{-4}\, dx = \frac{-1}{3}x^{-3} + C$

(b) $\int x\sqrt{1+x^2}\, dx = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$

Rule:

$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$ where **C is a constant , and n ≠ -1**

Example find:

(3) (a) $\int x^5\, dx$

(b) $\int x^{-3}\, dx$

(c) $\int x^{\frac{2}{5}}\, dx$

(d) $\int \frac{1}{\sqrt[5]{x^3}}\, dx$

Solution

(a) $\int x^5\, dx = \frac{x^{5+1}}{5+1} + C = \frac{1}{6}x^6 + C$

(b) $\int x^{-3}\, dx = \frac{x^{-3+1}}{-2} + C = -\frac{1}{2}x^{-2} + C$

(c) $\int x^{\frac{2}{5}}\, dx = \frac{1}{\frac{7}{5}}\, x^{1+\frac{2}{5}} + C$

$= \frac{5}{7}x^{\frac{7}{5}} + C$

(d) $\int \frac{1}{\sqrt[5]{x^3}}\, dx = \int x^{\frac{-3}{5}}\, dx$

$= \frac{1}{\frac{2}{5}}\, x^{1+\frac{-3}{5}} + C = \frac{5}{2}x^{\frac{2}{5}} + C$

Try to solve

(3) Find:

(a) $\int x^8\, dx$

(b) $\int x^{\frac{2}{3}}\, dx$

(c) $\int \sqrt[7]{x^5}\, dx$

(d) $\int 7x^{\frac{7}{9}}\, dx$

Properties of Integration

If f and g are integrable functions on an interval , then:

1- $\int a f(x)\,dx = a \int f(x)\,dx$ where a is a constant $\neq 0$

2- $\int [f(x) \pm r(x)]\,dx = \int f(x)\,dx \pm \int r(x)\,dx$

Example

(4) Find: a $\int (4x + 3x^2)\,dx$ b $\int \frac{(x^2+2)^2}{x^2}\,dx$

Solution

a $\int (4x + 3x^2)\,dx$

$= \int 4x\,dx + \int 3x^2\,dx$

$= 4\int x\,dx + 3\int x^2\,dx$

$= \frac{4}{2}x^2 + 3 \times \frac{1}{3}x^3 + C$

$= 2x^2 + x^3 + C$

b $\int \frac{(x^2+2)^2}{x^2}\,dx$

$= \int \frac{x^4 + 4x^2 + 4}{x^2}\,dx$

$= \int x^2\,dx + \int 4\,dx + \int 4x^{-2}\,dx$

$= \frac{1}{3}x^3 + 4x - 4x^{-1} + C$

Try to solve

(4) Find:

a $\int (2 + \sqrt{x} + \frac{1}{\sqrt{x}})\,dx$ b $\int (\frac{1}{x^2} + \sqrt{x} + 3)\,dx$

Some rules of integration

1- $\int (ax+b)^n\,dx = \frac{1}{a} \times \frac{(ax+b)^{n+1}}{n+1} + C, n \neq -1$

2- $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$, where C is a constant and n is a rational number $\neq -1$

Critical thinking:

1- Can you check the correctness of the two previous relations through the definition of the antiderivative? Explain.

Example

(5) Find:

a $\int \left((3-2x)^5 + 3\right)\,dx$ b $\int \frac{x+3}{(x-2)^4}\,dx$

c $\int (x^2 - 3x + 5)^{-7}(2x-3)\,dx$ d $\int (3x^2 - 2x + 1)^{11}(3x-1)\,dx$

Solution

a $\int ((3-2x)^5+3)\,dx = \int (3-2x)^5\,dx + 3\int dx$

$= \frac{(3-2x)^6}{6\times -2} + 3x + C = \frac{-1}{12}(3-2x)^6 + 3x + C$

b $\int \frac{x+3}{(x-2)^4}\,dx = \int \frac{(x-2)+5}{(x-2)^4}\,dx = \int \frac{x-2}{(x-2)^4}\,dx + \int \frac{5}{(x-2)^4}\,dx$

$= \int (x-2)^{-3}\,dx + 5\int (x-2)^{-4}\,dx$

$= \frac{(x-2)^{-2}}{-2\times 1} + \frac{5(x-2)^{-3}}{-3\times 1} + C = \frac{-1}{2(x-2)^2} - \frac{5}{3(x-2)^3} + C$

c $\int (x^2-3x+5)^{-7}(2x-3)\,dx$

$\because f(x) = x^2-3x+5, \quad \because f'(x) = 2x-3$

$\therefore \int (x^2-3x+5)^{-7}(2x-3)\,dx = \frac{(x^2-3x+5)^{-6}}{-6} + C$

$= \frac{-1}{6}(x^2-3x+5)^{-6} + C$

d $\int (3x^2-2x+1)^{11}(3x-1)\,dx$

$\because f(x) = 3x^2-2x+1,$

$\because f'(x) = (6x-2)$

$\therefore \frac{1}{2}\int (3x^2-2x+1)^{11}(6x-2)\,dx = \frac{1}{2}\,\frac{(3x^2-2x+1)^{12}}{12} + C$

$= \frac{1}{24}(3x^2-2x+1)^{12} + C$

Try to solve

5 Find:

a $\int (3x+5)^{-9}\,dx$

b $\int (x+1)(x+3)^7\,dx$

c $\int (x^2+3x-2)^9(2x+3)\,dx$

d $\int x(4x^3-3x^2+4)^5(2x-1)\,dx$

Search: use the resources of knowledge and internet to find: $\int \frac{1}{x}\,dx$

Integration of Some Trignometric Functions

Remember

Some trigonometric relations

a. $\cos^2 x + \sin^2 x = 1$

b. $\cos^2 x - \sin^2 x = \cos 2x$

c. $1 + \tan^2 x = \sec^2 x$

d. $2 \sin x \cos x = \sin 2x$

1- $\int \sin x \, dx = -\cos x + C$

2- $\int \cos x \, dx = \sin x + C$

3- $\int \sec^2 x \, dx = \tan x + C$

where C is an arbitrary constant

Example

(6) Find the following integrations:

a. $\int (x - \sin x) \, dx$

b. $\int (4 \cos x + \frac{1}{\cos^2 x} + 1) \, dx$

Solution

a. $\int (x - \sin x) \, dx = \frac{1}{2} x^2 + \cos x + C$

b. $\int (4 \cos x + \frac{1}{\cos^2 x} + 1) \, dx = \int (4 \cos x + \sec^2 x + 1) \, dx$
$= 4 \sin x + \tan x + x + C$

Try to solve

(6) Find the following integrations:

a. $\int (6 \cos x - 8 \sin x) \, dx$

b. $\int (3 + 4 \tan^2 x) \, dx$

Important Corollaries :

1 - $\int \sin (ax + b) \, dx = -\frac{1}{a} \cos (ax + b) + C$

2 - $\int \cos (ax + b) \, dx = \frac{1}{a} \sin (ax + b) + C$

3 - $\int \sec^2 (ax + b) \, dx = \frac{1}{a} \tan (ax + b) + C$ where C is an arbitrary constant

Critical thinking:

1- Check the correctness of the previous relations by finding the antiderivative using the chain rule.

Example

(7) Find:

a. $\int \cos (2x + 3) \, dx$

b. $\int (\sec^2 \frac{x}{2} - \sin (\frac{\pi}{4} - x)) \, dx$

Solution

a $\int \cos(2x+3)\, dx = \frac{1}{2}\sin(2x+3) + C$

b $\int \sec^2(\frac{1}{2}x) dx - \int \sin(\frac{\pi}{4} - x)\, dx$

$= \frac{1}{\frac{1}{2}} \tan\frac{x}{2} - [\frac{-1}{-1}\cos(\frac{\pi}{4} - x)] + C$

$= 2\tan\frac{x}{2} - \cos(\frac{\pi}{4} - x) + C$

Try to solve

7) Find :

a $\int \sin(3x-5)f\ dx$

b $\int \cos(\frac{x}{3} - 2)\, dx$

Exercises 3 - 6

Complete each of the following:

1) The antiderivative of the function $(3x^2 - 2x + 5)$ is ....

2) $\int (\frac{5}{\sqrt{x}} + \frac{\sqrt{x}}{5})\, dx =$ ....

3) $\int (4x - \sin\frac{\pi}{3})\, dx =$ ....

4) $\int (x+5)^3\, dx =$ ....

5) $\int (x+2)(x^2+4x-7)^{11}\, dx =$ ....

6) $\int (\cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4})\, dx =$ ....

7) $\int \frac{\cos^2 x}{1 + \sin x}\, dx =$ ....

8) $\int 2\tan x \sec^2 x\, dx =$ ....

Calculate the following integrations :

9) $\int 9x^8\, dx$

10) $\int 12x^{-2}\, dx$

11) $\int 5\sqrt{x^3}\, dx$

12) $\int -\frac{8}{5} c^{-3}\, dc$

13) $\int 2(3z^2 - 5)\, dz$

14) $\int 3x^2 + x - 2)\, dx$

15) $\int (3\sqrt{x} - \frac{2}{x^3})\, dx$

16) $\int (ax^2 + 5x + c)\, dx$

17) $\int x(x+3)\, dx$

18) $\int x^2(3x - \frac{5}{x})\, dx$

19) $\int (x-2)(x+2)\, dx$

20) $\int (x-5)(x+1)\, dx$

21) $\int (\sqrt{x} - 1)^2\, dx$

22) $\int (x + \frac{1}{x})^2\, dx$

23. $\int \frac{2x^2+3}{x^2}\, dx$

24. $\int \frac{x^7-4x^5+x^3}{x^3}\, dx$

25. $\int \frac{x^2-1}{x-1}\, dx$

26. $\int \frac{x^3-27}{x-3}\, dx$

27. $\int \frac{x^3+8}{x^2-2x+4}\, dx$

28. $\int \frac{x^2-5x+6}{x-2}\, dx$

29. $\int 6(x-2)^5\, dx$

30. $\int 3(x+3)^{-4}\, dx$

31. $\int (8-3x)^4\, dx$

32. $\int (2z-3)^{\frac{2}{3}}\, dz$

33. $\int \frac{12}{(2x-5)^4}\, dx$

34. $\int \frac{7}{\sqrt{x+4}}\, dx$

35. $\int (x-1)(x^2-2x+1)\, dx$

36. $\int \sqrt{x-3}\,(x-3)^4\, dx$

37. $\int (x^2-2)^2\, dx$

38. $\int x(3x^2-5)^3\, dx$

39. $\int x(5x^2+2)^{-7}\, dx$

40. $\int \frac{x^3}{(x^4+1)^6}\, dx$

41. $\int (7\sin x - 2\cos x)\, dx$

42. $\int (\sin 2x - 3\cos x)\, dx$

43. $\int 9(\cos x - \sec^2 3x)\, dx$

44. $\int (3+5\sec^2 x)\, dx$

45. $\int \sin(5x-1)\, dx$

46. $\int \cos(2-x)\, dx$

47. $\int \sin\left(\frac{x}{4}+\frac{\pi}{3}\right) dx$

48. $\int (\cos^2 x - \sin^2 x)\, dx$

49. $\int 2\cos^2 x\, dx$

50. $\int (4-\sin^2 x)\, dx$

Unit summary

Function of variation & rate of change

- If $y = f(x)$ and x varies from x to $x + h$, then the function of variation $v(h) = f(x + h) - f(x)$
- The average rate of change function $A(h) = \frac{f(x+h) - f(x)}{h}$
- The rate of change function = $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- The rate of change is called the first derivative of the function and is denoted by the symbol $f'(x)$ where :

  $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ [the first derivative is denoted by one of these symbols : $\frac{dy}{dx}$ or y' or f' (x)]

Differentiability:

- It is said that: the function $f(x)$ is differentiable when $x = a$ if and only if:

$f'(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$ is existed, $f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ is existed,

$f'(a^-) = f'(a^+)$.

If $f(x)$ is differentiable at $x = a$, then its continuous at $x = a$

The continuity of the function at a point doesnot necessarily mean that its is differentiable at this point.

If the function is discontinuous at point $x = a$, then it is not - differentiable at $x = a$

The table of the derivatives of some functions:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	$n x^{n-1}$	$\sin a x$	$a \cos a x$
ax^n	$an x^{n-1}$	$\cos a x$	$- a \sin a x$
a	0	$\tan a x$	$a \sec^2 a x$

Derivative of the product of two functions:

- If $y = g(x) . z(x)$
- then $\frac{dy}{dx} = g . \frac{dz}{dx} + z . \frac{dg}{dx}$

Derivative of the quotient of two functions:

- If $y = \frac{g(x)}{z(x)}$ where $z(x) \neq 0$

then $\frac{dy}{dx} = \frac{z \times \frac{dg}{dx} - g \times \frac{dz}{dx}}{[z(x)]^2}$

Derivative of the composite function

- If $y = [f(x)]^n$ then $\frac{dy}{dx} = n\,[f(x)]^{n-1} f'(x)$

Chain rule

- If $y = f(z)$, $z = f(x)$ then $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$

Geometric applications:

- If $y = f(x)$, then the slope of the tangent to the curve of the function f at any point on it $= \frac{dy}{dx}$.

If (x_1 , y_1) is a point on the curve of the function $y = f(x)$, then the equation of the tangent at point (x_1 , y_1) is given by the relation $y - y_1 = m\,(x - x_1)$.

where $m = (\frac{dy}{dx})$ is the slope of the tangent at point (x_1 , y_1).

- The equation of the normal on the tangent at point (x_1 , y_1) is given by the relation $y - y_1 = \frac{-1}{m}(x - x_1)$

where $m = (\frac{dy}{dx})$ is the slope of the curve at point (x_1 , y_1) .

Antiderivative of the function:

- It is said that : the function F is an antiderivative to the function f if $F'(x) = f(x)$ for each x in the domain of f.
- If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + C$ where C is an arbitrary constant.

Properties of integration:

If f and g are integrable functions on an interval , then :

1- $\int a f(x)\,dx = a \int f(x)\,dx$ where a is a constant $\neq 0$

2- $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Standard formula for integration:

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

c $\int (ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{a\,(n + 1)} + C$ where $n \neq -1$

d $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{(n + 1)} + C$ where $n \neq -1$

e $\int \sin x\,dx = -\cos x + C$

f $\int \cos x\,dx = \sin x + C$

g $\int \sec^2 x\,dx = \tan x + C$

Accumulative test

Complete:

(1) $\frac{d}{dx}(x\cos x) =$ .... (2) $\int (3x^2 - 4x + 5)\,dx =$ ....

Choose the correct answer:

(3) $\frac{d}{dx}(\tan\frac{\pi}{4}) =$ ....

a) 1 b) $\sec^2\frac{\pi}{4}$ c) 0 d) 2

(4) The slope of the tangent to the curve of the function $y = \sin x \cos x$ equals

a) $\cos x \sin x$ b) $\cos^2 x - \sin^2 x$ c) $\sin x - \cos x$ d) $\sin^2 x + \cos^2 x$

(5) The function $y = |x - 2|$ when $x = 2$ is :

a) Continuous and differentiable b) Continuous and not-differentiable

c) Discontinuous and not-differentiable d) discontinuous.

(6) $\int 4(x-2)^3\,dx =$ ....

a) $x^4 - 32x + C$ b) $4(x-2)^4 + C$ c) $(x-2)^4 + C$ d) $\frac{(x-2)^4}{4} + C$

(7) Find $\frac{dy}{dx}$ in each of the following:

a) $y = z + \frac{1}{z}, xz = 1$ b) $y^3 = (x^2 - 4x + 4)^{-2}$

c) $y = x\tan 3x + \cos x$ d) $y = \frac{x^2}{x^2 - 1}$

(8) Find the following integrations:

a) $\int (x-1)(x+1)\,dx$ b) $\int (2\sqrt{x} - 6x^2)\,dx$

c) $\int (1 + \cos x)^2\,dx$ d) $\int 4x(3 - 2x^2)^5\,dx$

(9) Find the equation of the tangent to the curve of the function f where $f(x) = x^2 - x^3$ at point $(2, -4)$.

(10) If $y = \frac{x^3 - 1}{x^3}$, prove that $x^4\frac{dy}{dx} = 3$

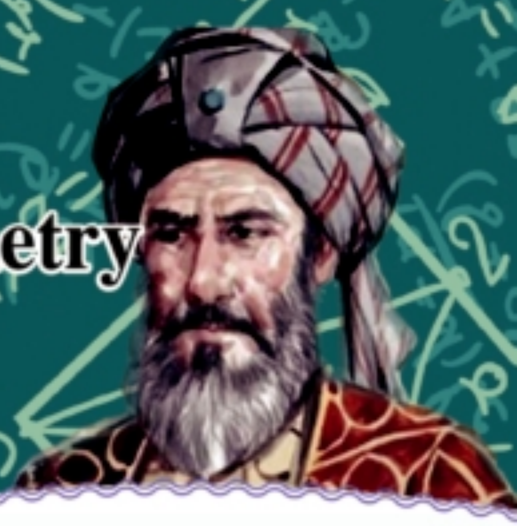
(11) If the tangent to the curve of the function f where $f(x) = ax^2 + bx + 5$ at point $(-1, 3)$ lying on it makes with the positive direction of x-axis a positive angle of measure 45°, find the two values of a and b.

General Exercises

For more exercises, please visit the website of Ministry of Education.

Unit Four

Trigonometry



Unit introduction

Mohammed bin Jaber Bin Sinan Abu Abdullah Al-Battani (850 - 929) was one of the most famous astronomers and mathematicians . He had cared much about trigonometry and had separated it as an independent science from the astronomy. He had been the first to use algebra in trigonometry. Furthermore, he had originated the cosine tables (cos) , sin angle (sin) , cotangent angle (cot), and tangent angle (tan) from zero to 90°. These tables are still used up to now with little amendments. He had developed a lot of trigonometric identities. on the other hand Mohammed Bin Yahya Bin Ismail Bin Abbas Abou El-Wafa Al-Buzjani (940 - 998) learned the works of Al-Battani in trigonometry well and he had explained the mysterious points. He had invented the inverse of cosine (sec) and inverse of sine(csc). He had originated tables for sin and tan for each 10 minutes and he had known trigonometric identities named after him such as :

$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$, $\tan A = \frac{\sin A}{\cos A}$, $\sec^2 A = 1 + \tan^2 A$, $\csc^2 A = 1 + \cot^2 A$ **and others** .

This unit involves the study of life applications including angles of elevation and depression as a practical applications on the cosine and sine rules and also the study of the special trigonometric identities of the sum and difference of the measures of two angles and the double angle. At the end of the unit, students are going to identify Heron's Formula to solve the triangle in terms of its side lengths and to solve life applications on them

Unit objectives

By the end of the unit and doing the activities, students should be able to:

- Solve applications on solving the triangle include angles of elevation and depression and cardinal points (main directions).
- Deduce the special trigonometric identities of sum and difference of the measures of two angles.
- Identify and deduce the special trigonometric identities of the double and half - angle.
- Deduce (*Heron's Formula*) and use it to find the area of the triangle and solve applications on it.
- Use the calculator to solve problems on the trigonometric identities.

Key terms

- Angle
- Angle of Elevation
- Angle of Depression
- Trigonometric Function
- Sine Function
- Cosine Function
- Trigonometric Functions of the sum of Measures of Two Angles
- Trigonometric Function of the Difference Between the Measures of Two Angles
- Trigonometric Functions of Double – Angle
- Trigonometric Functions of Half – an Angle
- Heron's Formula

Unit Lessons

Lesson (4 - 1): Angles of elevation and depression (Application on solving the triangle).

Lesson (4 - 2): Trigonometric functions of sum and difference of the measures of two angles.

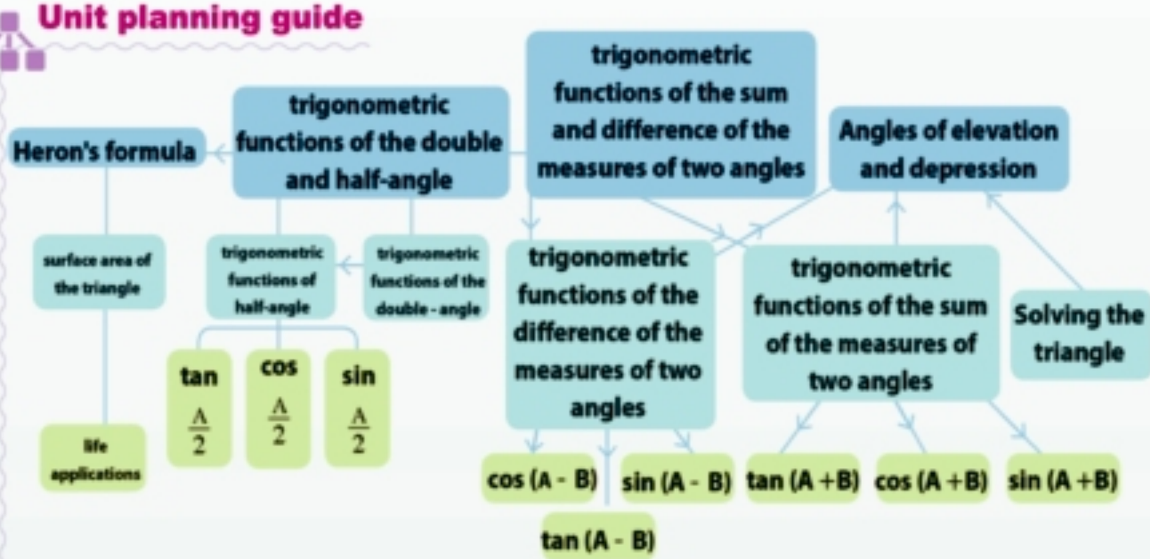
Lesson (4 - 3): The trigonometric functions of the double-angle.

Lesson (4 - 4): Heron's formula.

Materials

Scientific calculator- Internet

Unit planning guide



Unit Four

Angles of elevation and depression (applications on solving the triangle)

4 - 1

You will learn

- Concept of the angle of elevation.
- Concept of the angle of depression
- Solving applications on solving the triangle including angles of elevation and depression

Key terms

- Angle
- Angle of Elevation
- Angle of Depression

materials

- Scientific calculator

Activity

You have previously learned the angles of elevation and depression as an application to solve the right angled triangle. You can find the height of a minaret of the ground surface while you stand far from it for a known distance without measuring this minaret practically.

And now, after you learned the sine rule cosine rule and solving the triangle in general using those two rules, you can learn deeper applications on solving the triangle including the angles of elevation and depression in general .

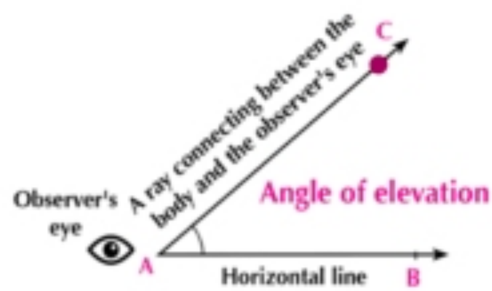
➤ Look up the mathematical refereces at your school library or log in **(Internet)** for various situations using the angles of elevation and depression to solve them as an application on solving the triangle in general, Now, we are going to review the concept of the angles of elevation and depression with you.

angles of elevation and depression

Learn

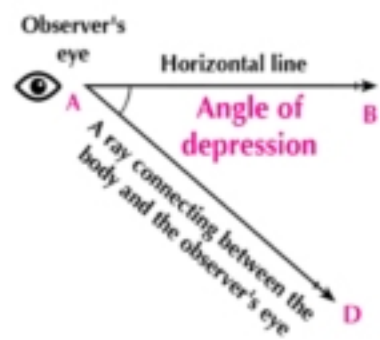
Angle of elevation

If person A observed point C higher than his horizontal eyesight level $\overrightarrow{AB}$, then the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is called the angle of elevation C of the horizontal eyesight of person A.



Angle of depression

If person A observed point D lower than his horizontal eyesight $\overrightarrow{AB}$, then the angle between $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is called the angle of depression D of the horizontal eyesight of person A.



Notes:

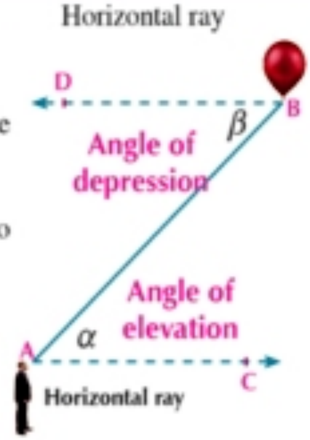
1- In the opposite figure:

$\angle CAB$ is the angle of elevation of the balloon with respect to to the the person at A.

$\angle DBA$ is the angle of depression of the person at A with respext to the balloon.

in this case $\alpha = \beta$

where α is the measurement of the angle of elevation and β is the measurement of the angle of depression.



2- to locate a point with respect to the cardinal points (main directions) from a known point, we find from the following figure that:

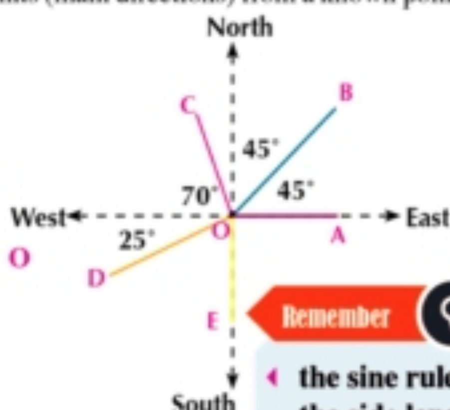
A lies East of O

B lies north of east of O

C lies in the direction of 70° north of west of O

D lies in the direction of 25° south of west of O

E lies South of O



> **Remember**
>
> - the sine rule: the side lengths of a triangle are propotional to the sines of the opposite angles.
> - the measurement of any exterior angle of a triangle equals the sum of non-adjacent interior angles.

Example

(1) From a point on the ground surface, a man observed the top of a tower of an angle of elevation of 25°, then he walked straight a head for 57 m at the horizontal level toward the tower base to find that the angle of elevation of the tower top is 52° 30′. Find the height of the tower to the nearest meter.

Solution

from the opposite figure :

$m(\angle CAD) = 52^\circ 30' - 25^\circ$

$= 27^\circ 30'$

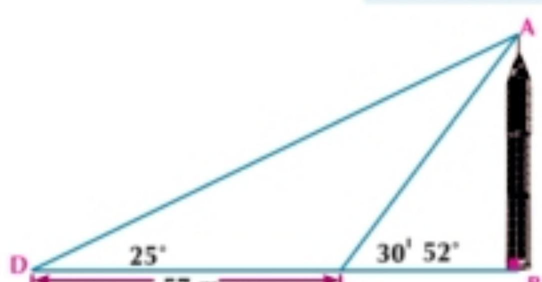
in $\triangle ACD$:

$$\frac{AC}{\sin 25^\circ} = \frac{57}{\sin 27^\circ 30'}$$

$$\therefore AC = \frac{57 \times \sin 25^\circ}{\sin 27^\circ 30'} \qquad (1)$$

in $\triangle ABC$:

$$\frac{AB}{\sin 52^\circ 30'} = \frac{AC}{\sin 90^\circ}$$



$\therefore\ AB = AC \times \sin 52^\circ\ 30'$ (2)

By substituting from (1) in (2), then:

$$AB = \frac{57 \times \sin 25^\circ}{\sin 27^\circ\ 30'} \times \sin 52^\circ\ 30'$$

$AB \simeq 41$m, i.e. the height of the tower is approximately 41 meters.

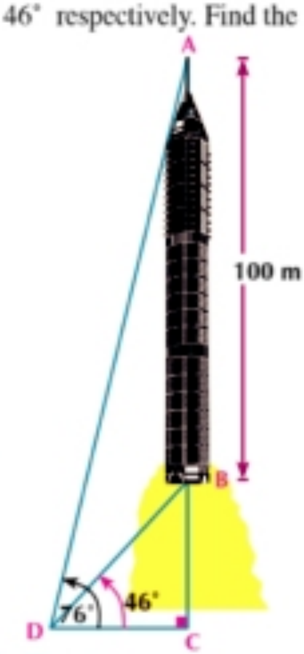
→ start 5 7 × sin 2 5 (× sin 5 2 °''' 3 0 °''') ÷
sin 2 7 °''' 3 0 °''' =

Try to solve

1. From a point on the ground surface a man observed the top of a tower at an angle of elevation of 20°, He walked on a horizontal way in the direction of the tower base for 50 meters, the measurement of the angle of elevation of the tower top is 42°. Find the height of the tower to the nearest meter .

Example

2. A tower of height 100 meters is constructed on a rock . From a point on the ground surface in the horizontal level passing through the rock base, the measure of the angles of elevation of the top and base of the tower were measured to give 76° and 46° respectively. Find the hight of the rock to the nearest meter.



Solution

$m(\angle ADB) = 76^\circ - 46^\circ = 30^\circ$

$m(\angle A) = 90^\circ - 76^\circ = 14^\circ$

In the triangle A B D: $\frac{BD}{\sin 14^\circ} = \frac{100}{\sin 30^\circ}$

$BD = \frac{100 \sin 14^\circ}{\sin 30^\circ}$ (1)

In the triangle B C D: $\frac{BC}{\sin 46^\circ} = \frac{BD}{\sin 90^\circ}$

$BC = BD \sin 46^\circ$ (2)

By substituting from (1) in (2) we deduce that

$$BC = \frac{100 \sin 14^\circ}{\sin 30^\circ} \times \sin 46^\circ \simeq 35 \text{ m}$$

Use the calculator respectively as follows:

start → 1 0 0 × sin 1 4) × sin 4 6)
÷ sin 3 0) =

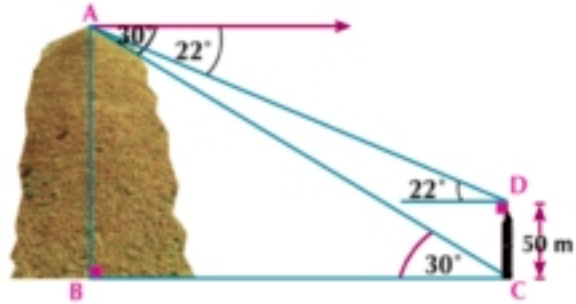
Try to solve

2. A tower of height 12 meters is constructed on a hill. If the measures of the two angles of elevation of the top and base of the tower from a point on the ground surface are 32° and 24° respectively, find the height of the hill to the nearest meter.

Example

(3) From the top of a hill , a man observed the measures of the two angles of depression of the top and the base of a tower to give 22° and 30° respectively. If the height of the tower is 50 meters, find the height of the hill to the nearest meter known that the two bases of the tower and hill are in the same horizontal level .

Solution



From the opposite figure:

$m(\angle DAC) = 30° - 22° = 8°$

$m(\angle CDA) = 90° + 22° = 112°$

In the triangle A C D $\frac{AC}{\sin 112°} = \frac{50}{\sin 8°}$

$AC = \frac{50 \sin 112°}{\sin 8°}$ (1)

In the triangle A C B

$\frac{AB}{\sin 30°} = \frac{AC}{\sin 90°}$ $\therefore AB = AC \sin 30°$ (2)

By substituting from (1) in (2), then:

$AB = \frac{50 \sin 112° \times \sin 30°}{\sin 8°} \simeq 167$ meters.

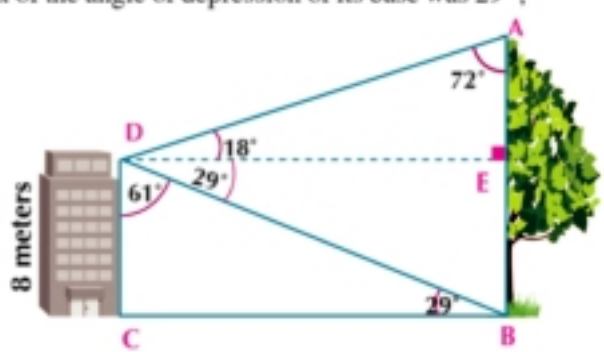
start → 5 0 × sin 1 1 2) × sin 3 0) ÷ sin 8) =

Try to solve

(3) From the top a rock of height 80 meters, the two angles of depression of the top and the base of a tower were measured to give 24° and 35° respectively. Find the height of the tower to the nearest meters known that the two bases of the rock and tower are in the same horizontal level.

Example

(4) From the top of a building of height 8 meters, the measurement of the angle of elevation of the top of a tree was 18° and the measurement of the angle of depression of its base was 29° , find to the nearest two decimals the distance between the bases of the building and the tree and the height of the tree known that the two bases of the building and tree are in the same horizontal level .



Solution

$m(\angle A) = 90° - 18° = 72°$

$m(\angle BDC) = 90° - 29° = 61°$

In $\triangle BCD$: $\because \frac{8}{\sin 29°} = \frac{BC}{\sin 61°}$

$\therefore BC = \dfrac{8\times \sin 61^\circ}{\sin 29^\circ} \simeq 14.43$ meters.

the distance between the two bases of the building and tree $\simeq$ 14.43 meters

In $\triangle AED$:

$\because \dfrac{DE}{\sin 18^\circ} = \dfrac{ED}{\sin 72^\circ}$ notice that $(ED = BC)$

$\therefore AE = \dfrac{\sin 18^\circ}{\sin 72^\circ} \times BC$ $\therefore AE = \dfrac{\sin 18^\circ}{\sin 72^\circ} \times 14.43 \simeq 4.69$ m

$\therefore$ Height of the tree $\simeq 8 + 4.69 \simeq 12.69$ m

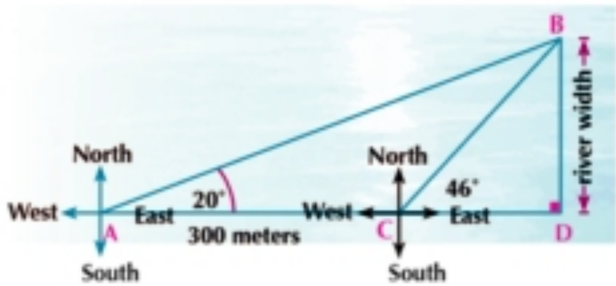
Try to solve

4) Two men stand on horizontal land to observe a flying balloon at the same time. The first man found the measurement of its angle of elevation is 63° while the other found the measurement of its angle of elevation is 39°. If the distance between the two men is 800 meters and the two men and the balloon are at the same vertical level and the projection of the balloon lying between the two men, find the height of the balloon from the ground surface to the nearest meter.

Example

5) From point A on a riverbank, a man observed the position of a home at point B on the other riverbank to find it in the direction of 20° North of the east. As he walks parallel to the riverbank in the direction of East for a distance of 300 meters to reach point C, he found point B in the direction of 46° North of the east. Find the width of the river to the nearest meter known that the two riverbanks are parallel and points A, B and C are at the same horizontal level.

Solution



Let the width of the river is $\overline{BD}$

In the triangle ABC

$m(\angle ABC) = 46^\circ - 20^\circ = 26^\circ$

$\dfrac{BC}{\sin 20} = \dfrac{300}{\sin 26^\circ}$

$BC = \dfrac{300 \sin 20^\circ}{\sin 26^\circ}$ (1)

In the triangle BDC

$\because \dfrac{BD}{\sin 46^\circ} = \dfrac{BC}{\sin 90^\circ}$ $\therefore BD = BC \times \sin 46^\circ$ $\therefore BD = \dfrac{300 \times \sin 20^\circ}{\sin 26^\circ} \times \sin 46^\circ \simeq 168$ meters

start → 3 0 0 × sin 2 0) × sin 4 6) ÷ sin 2 6) =

Try to solve

5) From point A on a riverbank, a man observed a tree at point B on the other bank. He found B is in the direction of 37° North of the east. As he walks parallel to the riverbank in the direction of west for a distance of 200 meters to reach point C, he found that point B is in the direction of 15° North of the east. Find the width of the river to the nearest meter .

Example

(6) From a point at the horizontal level passing through the base of a hill , a man observed the angle of elevation of the hill top to find its measure 27°. As he ascends the hill for a distance of 200 meters on a road inclined to the horizontal by an angle of measurement 14° to find the measure of the angle of elevation of the hill top is 38°. Find the height of the hill to the nearest meter.

Remember

the measurement of any exterior angle of a triangle equals the sum of non-adjacent interior angles..

Solution

In the triangle A C D

$m(\angle ACD) = 27° - 14° = 13°$

$\because$ Angle A E B is an exterior angle to the triangle A E C

$\therefore m(\angle EAC) = 38° - 27° = 11°$

$\therefore m(\angle ADC) = 180° - (11° + 13°) = 156°$

In the triangle A D C

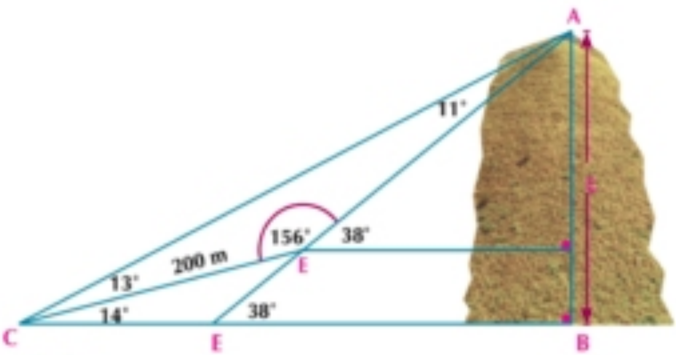
$\because \frac{AC}{\sin 156°} = \frac{200}{\sin 11°}$

$\therefore AC = \frac{200 \times \sin 156°}{\sin 11°}$ (1)

In the triangle A B C

$\because \frac{AB}{\sin 27°} = \frac{AC}{\sin 90°}$

$\therefore AB = AC \sin 27°$ (2)



By substituting from (1) in (2), we find:

$AB = \frac{200 \times \sin 156°}{\sin 11°} \times \sin 27° \simeq 194$ m.

Try to solve

(6) A man measured the angle of elevation of a hill top from a point on the ground surface to find it 22°. As he ascends the hill for 500 meters on a road inclined to the horizontal by an angle of measurement 7°, he found the measure of the angle of elevation of the hill top is 64°. Find the height of the hill to the nearest meter .

Example

(7) **Marine navigation:** A ship sailed from a certain point in the direction of 68° 25' south of the west at speed of 15 km/ h and at the same time , another ship sailed from the same point in the direction of 53° 48' North of the west at speed of 8 km/ h. Find the distance between the two ships after 3 hours to the nearest two decimals.

Solution

$\because$ Uniform velocity $= \dfrac{\text{distance traveled (covered)}}{\text{time taken to cover the distance}}$

Distance covered = uniform velocity × time taken to cover the distance

$\therefore AB = 15 \times 3 = 45$ km

$AC = 8 \times 3 = 24$ km

$\therefore m(\angle BAD) = 90° - 68°25' = 21°35'$

$\therefore m(\angle BAC) = 21°35' + 53°48'$

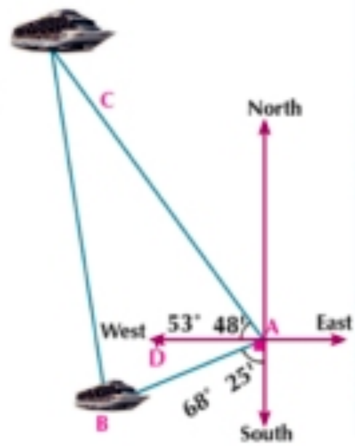
$= 75°23'$

In the triangle A B C

By applying the cosine rule

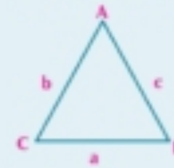
$(BC)^2 = (45)^2 + (24)^2 - 2 \times 45 \times 24$

$\cos 75°23'$ $\therefore BC \simeq 45.34$ km



Remember

Cosine rule



$a^2 = b^2 + c^2 - 2bc\cos A$

then $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

and so on with respect to the other side lengths and measurements of angles.

start → 4 5 x^2 + 2 4 x^2 - 2 × 4 5 × 2 4 × Cos 7 5 °''' 2 3 °''') = √ Ans =

Try to solve

(7) **Marine navigation:** A ship sailed from a certain point in the direction of 70° East of the south at speed of 12 km/h and at the same time, another ship sailed from the same point in the direction of 55° North of the east at speed of 5km/h. Find the distance between the two ships after 2 hours to the nearest two decimals .

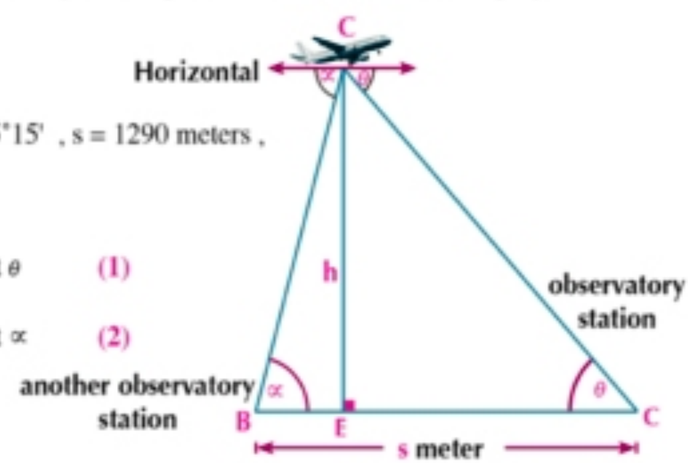
Example

(8) **(Theoritical proving):** A pilot observed two observatory stations A and B on horizontal land where A B = s meter to find the measurement of their two angles of depression θ and $\propto$ respectively. If the plane and the two observatory stations are at the same vertical level, the height of the plane on that time above the ground equals h meter and the vertical projection of the plane $\in \overline{AB}$, prove that:

$h = \dfrac{s}{\cot\theta + \cot\propto}$

if $m(\angle\theta) = 48°31'$, $m(\angle\propto) = 75°15'$, s = 1290 meters ,

Find the value of h.



Solution

$\because \cot\theta = \dfrac{AD}{h}$ $\therefore AD = h\cot\theta$ (1)

$\because \cot\propto = \dfrac{BD}{h}$ $\therefore BD = h\cot\propto$ (2)

By adding (1) and (2)

$\therefore$ AD + BD $= h \cot \theta + z \cot \propto = h (\cot \theta + \cot \propto)$

$\therefore$ s $= h (\cot \theta + \cot \propto)$

i.e. $h = \dfrac{S}{\cot \theta + \cot \propto}$

when $m(\angle \theta) = 48^\circ\ 31'$, $m(\angle \propto) = 75^\circ\ 15'$, s = 1290 meter

$\therefore\ h = \dfrac{1290}{\cot 48^\circ\ 31' + \cot 75^\circ\ 15'} \simeq 1124.2$ meter

1 2 9 0 ÷ (1 ÷ tan 4 8 °'" 3 1 °'") + 1 ÷
tan 7 5 °'" 1 5 °'")) =

Try to solve

(8) **Theoretical proving:** A ladder of length L is placed by one of its ends on a vertical wall and the other end on horizontal land, the ladder inclined to the horizontal by an angle of measurement $\propto$ and the lower end of the ladder moved for a distance s meters far from the wall and it inclined to the horizontal by an angle of measurement θ. Prove that $L = \dfrac{s}{\cos \theta - \cos \propto}$
if s = 40cm , $m(\angle \theta) = 30^\circ$, $m(\angle \propto) = 40^\circ$, find the ladder length.

Exercises 4 - 1

(1) A man observed the angle of elevation of a tower top from a point on the ground surface to find its measurement is $20^\circ\ 35'$. He walked on a horizontal road in the direction of the tower base for a distance of 50m to find the measurement of the angle of elevation of the tower top is 42°. Find the height of the tower to the nearest meter .

(2) From a top of a house of height 15m , the measurement of the angle of elevation of a tower top is 67° and the measurement of the angle of depression of a tower base is 35° . Find the height of the tower to the nearest meter known that the tower base and the house base are at the same horizontal level.

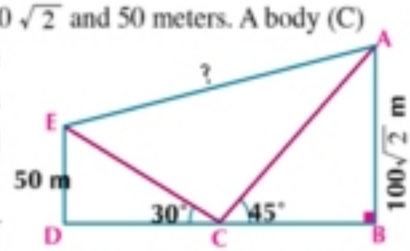
(3) From a tower top of height 65 meters, the two angles of depression of points A and B at the horizontal level are measured to give 32° and $21^\circ\ 12'$ respectively. If D represents the tower base and $A \in \overline{BD}$, find the length of $\overline{AB}$ to the nearest meter.

(4) From a hill top, an observer found that the measurement of the two angles of depression of a tower top and its base are 15° and 26° respectively. If the height of the tower is 50 meters, calculate the height of the hill to the nearest meter known that the two bases of the hill and tower are at the same horizontal level.

(5) A ship on the sea observed a lighthouse to find that it is located at a distance of 50 km towards the East, then the ship sailed in the direction of North of East and after two hours the lighthouse lied in the direction of 50° South of East. Calculate the velocity of the ship.

(6) A lighthouse of height 60 meters is constructed on a hill close to the seashore. The two angles of elevation of the top and base of the lighthouse were measured from a boat on the sea to give 70° and 45° respectively. Find the height of the hill from the sea level to the nearest meter.

(7) **In the opposite figure:** two balloons A and E of height $100\sqrt{2}$ and 50 meters. A body (C) on the ground lying at the vertical level passing through the balloons is observed . If the measurement of the two angles of depression of the body are 45° and 30°respectively, find the distance between the two balloons to the nearest meter.



(8) From the top of a rock 80 meter heigh, the two angles of depression of the top and base of a tower were measured to give 24°and 35° respectively. Find the height of the tower known that the bases of the rock and tower are at the same horizontal level.

(9) From a top of a house, A man observed a car at the rest at the same horizontal level of the house to find the measure of the angle of depression is 70°, when this man descends vertically down for a distance of 12 meters, he observed the measure of the angle of depression of the car was 30°. Find the height of the house to the nearest meter and the distance between the car and house.

(10) Three villages A , B and C: the village A lies west of village B where A B = 20km and village C lies in the direction of 48° East of North of village A and 60° North of west of village B. Find the distance between the two villages B and C to the nearest km.

(11) From a point on the ground surface , it is found that the measurement of the angle of elevation of a top of a tree is 50° and from another point distant 45 meters from the previous point and above it exactly, it is found that the measurement of the angle of depression of the top of the tree is 30°. Find the height of the tree to the nearest meter .

(12) From the peak of a mountain of height 100 meters above sea level, a man observed the angle of depression of a rock top to find its measurement is 42° 37'.Find the height of the rock above sea level if it is distant from the mountain 22 meters known that both of them are at the same horizontal land .

(13) Two people moved from the same point and at the same time. The first moved in the direction of 40° west of north at speed of 32 meter/ min and the second moved in the direction of 70° south of west at speed of 38 m/min . Find the distance between them after 5 minutes to the nearest meter .

(14) From a point at the horizontal level passing through a hill base a man observed the angle of elevation of a hill peak to find its measurement is 24°. As he ascends the hill for a distance of 400 meters on a level inclined to horizontal by an angle of measurement 15°, he found the measurement of the angle of elevation of the hill peak is 36° . Find the height of the hill to the nearest meter .

(15) A plane C is observed from two stations A and B at the time it passes through the vertical level passing through the straight line $\overleftrightarrow{AB}$ where A B = 3000 meters to find the measurement of its angle of elevation from A is 53° 21', the measurement of its angle of elevation from B is 34° 26' and the vertical projection of the plane $\in \overline{AB}$. Find the height of the plane from the ground surface to the nearest meter.

(16) **Marine navigation:** A ship sails Northeast at a uniform velocity of a magnitude of 28 km/h. A passenger on it watched two constant points in the direction of 37° West of North and after 3 hours, the passenger found that one of those points was in the direction of 24° south of west with respect to him and the other point was in the direction of 19° North of West with respect to him. Find the distance between the two points to the nearest Km known that the two points and the passenger are at the same horizontal level .

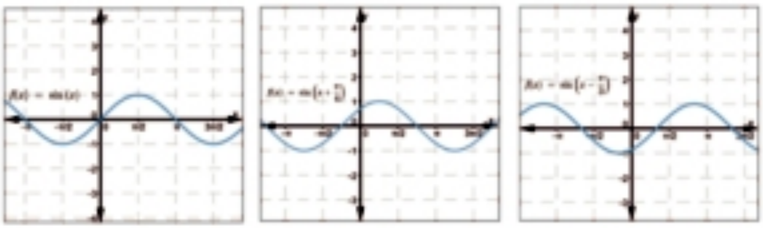
Trigonometric functions of sum and difference of the measures of two angles

4 - 2

Think and discuss

Use geogebra to graph the following functions :

$f(x) = \sin x$, $y = \sin(x + \frac{\pi}{4})$, $m(x) = \sin(x - \frac{\pi}{3})$



What do you notice from your learning to the concept of the geometric transformation?

Notice that the second function shown in the second graph includes the adding of the two angles x and $\frac{\pi}{4}$

and the third function shown in the third graph includes the subtration of the two angles x and $\frac{\pi}{3}$.

As a result, it was necessary to use the rules of the trigonometric ratios of the sum or difference of two angles in order to find the trigonometric functions of a certain angle. for example, to find the value of $\sin 75^\circ$, we can put it in the form of $\sin(30^\circ + 45^\circ)$, and also $\cos 15^\circ$ can be put in the form of $\cos(60^\circ - 45^\circ)$ or $\cos(45^\circ - 30^\circ)$ and so on....

Learn

Trigonometric functions of sum and difference of the measures of two angles

From the opposite figure : (proof is not required)

Notice that $m(\angle A) = m(\angle ZLM)$. Why?

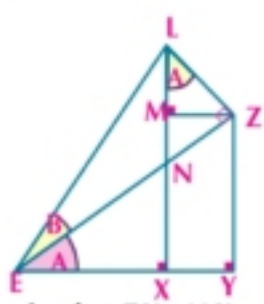
$$\sin(A + B) = \frac{LX}{LE} = \frac{MX}{LE} + \frac{LM}{LE}$$

$$= \frac{ZY}{LE} + \frac{LM}{LE}$$

$$= \frac{ZY}{LE} \times \frac{ZE}{ZE} + \frac{LM}{LE} \times \frac{LZ}{LZ}$$

$$= \frac{ZY}{ZE} \times \frac{ZE}{LE} + \frac{LM}{LZ} \times \frac{LZ}{LE}$$

$$= \sin A \times \cos B + \cos A \times \sin B$$



(notice that: $ZY = MX$ - why?)

You will learn

- Deduce the trigonometric functions of sum and difference of the measures of two angles.
- Solve various applications on the trigonometric functions of sum and difference of the measures of two angles.
- Use the identities of the sum and difference to prove the correctness of some other identities.

Key terms

- Trigonometric function
- Sine function
- Cosine function
- Tangent function
- Trigonometric functions of sum of the measures of two angles
- Trigonometric functions of difference of the measures of two angles

Materials

- Scientific calculator

Then $\sin(A + B) = \sin A \cos B + \cos A \sin B$

By putting (- B) instead of B, we get:

$\sin [A + (- B)] = \sin A \cos (- B) + \cos A \sin (- B)$

$\sin (A - B) = \sin A \cos B - \cos A \sin B$

Remember

$\sin (-A) = - \sin A$

$\cos (-A) = \cos A$

$\tan (-A) = - \tan A$

Use the same figures to prove:

$\cos (A + B) = \cos A \cos B - \sin A \sin B$

then , we deduce that: $\cos (A - B) = \cos A \cos B + \sin A \sin B$

Example

(1) Find:

(a) $\sin 75^\circ$ (b) $\cos 15^\circ$ what do you notice?

Solution

(a) $\sin 75^\circ = \sin (30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{1 + \sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2} + \sqrt{6}}{4}$

(b) $\cos 15^\circ = \cos (45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{1 + \sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2} + \sqrt{6}}{4}$

we notice that: $\sin 75^\circ = \cos 15^\circ$

Try to solve

(1) Find.

(a) $\cos 105^\circ$ (b) $\sin 75^\circ \cos 15^\circ + \cos 75^\circ \sin 15^\circ$

(c) $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$

Remember

$\sin (180-A) = \sin A$

$\cos (180-A) = - \cos A$

$\sin (180+A) = - \sin A$

$\cos (180+A) = - \cos A$

Example

(2) If $\sin A = \frac{3}{5}$ where $90^\circ < A < 180^\circ$, $\cos B = \frac{-5}{13}$

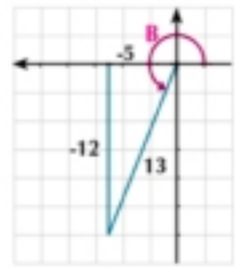
where $180^\circ < B < 270^\circ$

find $\cos (A - B)$, $\sin (A + B)$

Solution

$\because A \in$ second quadrant $\quad \therefore \sin A = \frac{3}{5} \quad \cos A = \frac{-4}{5}$

$\because B \in$ third quadrant $\quad \therefore \sin B = \frac{-12}{13} \quad \cos B = \frac{-5}{13}$



then:

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$= (\frac{-4}{5})(\frac{-5}{13}) + (\frac{3}{5})(\frac{-12}{13}) = \frac{-16}{65}$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$= (\frac{3}{5})(\frac{-5}{13}) + (\frac{-4}{5})(\frac{-12}{13}) = \frac{33}{65}$

Try to solve

(2) If $\cos A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$ where A and B are the measurement of two acute angles , find without using the calculator:

a) $\sin(A + B)$ b) $\sec(A - B)$

(3) In the triangle A B C , $\cos A = \frac{-3}{5}$ and $\sin B = \frac{5}{13}$, Find sin C without using the calculator.

Example

Electricity: If the electric current intensity is given by the relation : $C = \frac{5}{2} \sin 165^\circ t$

a) Rewrite the previous relation using the sum of measures of two angles.

b) Find the electric current intensity after one second (without using the calculator)

Solution

$C = \frac{5}{2} \sin 165^\circ t$ — **given relation**

$C = \frac{5}{2} \sin(45^\circ t + 120^\circ t)$ — **because** $165^\circ = 45^\circ + 120^\circ$

$= \frac{5}{2} \sin(45^\circ + 120^\circ)$ — **by substituting** n =1 second

$= \frac{5}{2} [\sin 45^\circ \cos 120^\circ + \cos 45^\circ \sin 120^\circ]$ — **using the expansion of the sum of measures of two angles**

$= \frac{5}{2} [\frac{1}{\sqrt{2}} (-\frac{1}{2}) + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}]$ — **by substituting the values of trigonometric ratios**

$= \frac{5}{2} \times \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ — **by multiplying both** numerator and denominator $\times \sqrt{2}$

$= \frac{(\sqrt{6} - \sqrt{2})}{8}$ by simplifying

Try to solve

(4) If the electric current intensity is given by the relation $C = \frac{3}{2} \cos 285^\circ t$

a) Rewrite the previous relation using the difference of measures of two angles.

b) Find the electric current intensity after one second (without using the calculator)

Tangent function of sum and difference of the measures of two angles

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

By dividing the nominator and denominator by $\cos A \cos B \neq 0$, then:

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$ when putting (-B) instead of B, then:

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$ where A and $B \neq \frac{\pi}{2}(2n + 1)$ and $n \in Z$

Example

(3) Without using the calculator , prove that:

a) $\tan 50^\circ = \frac{1 + \tan 5^\circ}{1 - \tan 5^\circ}$

b) $\tan (45^\circ - A) = \frac{\cos A - \sin A}{\cos A + \sin A}$

Solution

a) The left side = $\tan 50^\circ$

$= \tan(45^\circ + 5^\circ) = \frac{\tan 45^\circ + \tan 5^\circ}{1 - \tan 45^\circ \tan 5^\circ} = \frac{1 + \tan 5^\circ}{1 - \tan 5^\circ}$ = the right side

b) The left side = $\tan(45^\circ - A)$

$= \frac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ \tan A} = \frac{1 - \tan A}{1 + \tan A}$

$= \frac{1 - \frac{\sin A}{\cos A}}{1 + \frac{\sin A}{\cos A}} \times \frac{\cos A}{\cos A} = \frac{\cos A - \sin A}{\cos A + \sin A}$ = the right side

Remember

$\tan A = \frac{\sin A}{\cos A}$

Try to solve

(5) If each of A and B are the measures of two positive acute angles where $\sin A = \frac{12}{13}$, and $\sin B = \frac{4}{5}$, find the value of $\tan(A - B)$

(6) If A, B and C are the measures of the angles of a triangle where $\tan B = \frac{4}{3}$, $\tan C = 7$, prove that $A = 45^\circ$

(7) If $\tan A = \frac{5}{6}$ and $\tan B = \frac{1}{11}$, prove that $(A + B) = 45^\circ$

Example

(4) Find the solution set for each of the following equations where $0^\circ < x < 360^\circ$

a $\tan x + \tan 20^\circ + \tan x \tan 20^\circ = 1$ **b** $\sin (x + 30^\circ) = 2 \cos x$

Solution

a $\because \tan x + \tan 20^\circ + \tan x \tan 20^\circ = 1$

$\therefore \tan x + \tan 20^\circ = 1 - \tan x \tan 20^\circ$

$\therefore \frac{\tan x + \tan 20^\circ}{1 - \tan x \tan 20^\circ} = 1$ (**By dividing the two sides of the equation by:** $1 - \tan x \tan 20^\circ$)

i.e.: $\tan (x + 20^\circ) = 1$

$\therefore x + 20$ **lies in the first quadrant**

$\therefore \tan (x + 20^\circ) = \tan 45^\circ$ $x + 20^\circ = 45^\circ$ $x = 25^\circ$

or $x + 20^\circ$ **lies in the third quadrant**

$\tan (x + 20^\circ) = \tan 225^\circ$ $x + 20^\circ = 225^\circ$ $x = 205^\circ$

Solution set is $\{25^\circ, 205^\circ\}$

b $\because \sin (x + 30^\circ) = 2 \cos x$

$\therefore \sin x \cos 30^\circ + \cos x \sin 30^\circ = 2 \cos x$

$\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x = 2 \cos x$

$\frac{\sqrt{3}}{2} \sin x = \frac{3}{2} \cos x$

By dividing the two sides by $\cos x$, **where:** $\cos x \neq 0$

$\therefore \frac{\sin x}{\cos x} = \frac{3}{2} \times \frac{2}{\sqrt{3}} = \sqrt{3}$ **i.e.** $\tan x = \sqrt{3} > 0$

$\therefore x$ lies in the first or third quadrant and

$x = 60^\circ$ **or** $x = 60^\circ + 180^\circ = 240^\circ$

Solution set is $\{60^\circ, 240^\circ\}$

Try to solve

(8) Find the solution set for each of the following equations where $0^\circ < x < 360^\circ$

a $\sin x \cos 20^\circ - \cos x \sin 20^\circ = \frac{1}{2}$ **b** $\cos 2x \cos x + \sin 2x \sin x = \frac{\sqrt{3}}{2}$

Exercises 4 - 2

Complete:

1. $\sin 40^\circ \cos 10^\circ - \cos 40^\circ \sin 10^\circ =$ ....
2. $\dfrac{\tan 75^\circ - \tan 15^\circ}{1 + \tan 75^\circ \tan 15^\circ} =$ ....
3. $\cos 70^\circ \cos 20^\circ - \sin 70^\circ \sin 20^\circ =$ ....
4. $\sin(X + Y)\cos Y - \cos(X + Y)\sin Y =$ ....
5. If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, then $\tan(A + B) =$ ....

Choose the correct answer:

6. $\sin(\theta + \frac{\pi}{6}) =$

   a) $\frac{1}{2}(\cos\theta + \sqrt{3}\sin\theta)$ b) $\frac{1}{2}(\cos\theta + \sin\theta)$

   c) $\frac{1}{2}(\sqrt{3}\cos\theta + \sin\theta)$ d) $\frac{1}{2}(\sqrt{3}\cos\theta + \sqrt{2}\sin\theta)$

7. $\sin 75^\circ \cos 15^\circ + \cos 75^\circ \sin 15^\circ$ equals

   a) zero b) $\frac{1}{2}$ c) 1 d) $\frac{\sqrt{3}}{2}$

8. $\sin 5x \sin 3x + \cos 5x \cos 3x$ equals

   a) $\cos 2x$ b) $\cos 8x$ c) $\sin 8x$ d) $\sin 2x$

9. $\tan 15^\circ$ equals

   a) $2 + \sqrt{3}$ b) $-\sqrt{3} - 2$ c) $\sqrt{3} - 2$ d) $2 - \sqrt{3}$

10. If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, then $\tan(A + B)$ equals

    a) 1 b) $-\frac{5}{6}$ c) $\frac{1}{6}$ d) $\frac{5}{6}$

11. $\cot(2\pi - \frac{5\pi}{12}) =$

    a) $2 + \sqrt{3}$ b) $2 - \sqrt{3}$ c) $-2 - \sqrt{3}$ d) $\sqrt{3} - 2$

Put in the simplest form:

(12) $\cos(A - B) - \cos(A + B)$

(13) $\sin(A + B) - \sin(A - B)$

(14) $\cos A \cos(A + \frac{\pi}{2}) + \sin A \sin(A + \frac{\pi}{2})$

(15) $\cos 40^\circ \cos x - \sin 40^\circ \sin x$

(16) If you know that $\sin A = \frac{-8}{17}$ where $180^\circ < A < 270^\circ$, $\cos B = \frac{-5}{13}$ where $90^\circ < B < 180^\circ$

Without using the calculator find the values of:

a $\cos(A - B)$ b $\sin(A + B)$ c $\tan(A - B)$

(17) If $\tan(\theta + 45^\circ) = \frac{3}{2}$, find the value of $\tan\theta$.

(18) If you know that $\frac{\cos(A + B)}{\cos(A - B)} = \frac{1}{3}$, prove that $2\sin A \sin B = \cos A \cos B$

then prove that: $2\tan A = \cot B$ and if you know that $\tan A = \frac{2}{5}$, find tan B, and then find $\tan(A - B)$

(19) If A and B are two acute angles where $\tan A = \frac{4}{5}$, $\tan B = \frac{1}{9}$, prove that $A + B = 45^\circ$.

(20) If $\cos A = 0.6$, $\cos B = 0.8$, where A and B are the measures of two acute angles,

Find without using the calculator the value for each of the following:

a $\sin(A + B)$ b $\cos(A - B)$ c $\tan(A - B)$

(21) If A and B are two acute angles where $\cos A = \frac{3}{5}$ and $\tan B = \frac{8}{15}$, Find without using the calculator the value for each of the following:

a $\sin(A + B)$ b $\tan(A - B)$ c $\sec(A - B)$ d $\cot(A + B)$

(22) If $\sin A \sin B = \frac{1}{2}$ and $\cos A \cos B = \frac{1}{3}$, where A and B are the measures of the acute angles,

find the value of: $\cos(A + B)$ and $\cos(A - B)$

(23) If $\sin A = \frac{3}{5}$ where $0^\circ < A < 90^\circ$ and $\tan B = -7$ where $90^\circ < B < 180^\circ$, prove that $A + B = 135^\circ$

(24) If $x + y + z = 90^\circ$, prove that $\tan x \tan y + \tan y \tan z + \tan z \tan x = 1$

(25) **Creative thinking:** If $\tan A = \frac{2}{3}$ where $\pi < A < \frac{3\pi}{2}$, $\tan B = \frac{1}{5}$ where $0 < A < \frac{\pi}{2}$,

find the value for each of: $\tan(A + B)$, $\cos(A + B)$, then prove that $A + B = \frac{5\pi}{4}$

4 - 3

You will learn
- Deduce the trigonometric functions of the double -angle.
- Deduce the trigonometric functions of the half-angle.
- Solve various applications on the trigonometric functions of the double -angle
- Use the identities of the double-angle to prove the correctness of other identities

Key terms
- Trigonometic function
- Double - angle
- Half - angle
- Sine function
- Cosine function
- Tangent function

materials
- Scientific calculator

Think and discuss

You have already learned the trigonometric functions of the sum and difference of the measures of two angles. Now, we are going to raise a question:

From what you learned in the previous lesson can you deduce the trigonometric functions of the double-angle A if A is the measurement of a given angle?

Discuss with your instructors the answers you reached.

Learn

The trigonornetric functions of the double - angle

You know that:

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

(by putting B = A)

$\therefore \sin(A + A) = \sin A \cos A + \cos A \sin A$

$\therefore \sin 2A = 2\sin A \cos A$ for each $A \in R$ **(1)**

Similarly:

$\cos 2A = \cos^2 A - \sin^2 A$ for each $A \in R$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$ where tan A is defined, $\tan^2 A \neq 1$

Remember

The basic relations among the trigonometric functions:

$\sin^2 C + \cos^2 C = 1$

$\tan^2 C + 1 = \sec^2 C$

$\cot^2 C + 1 = \csc^2 C$

$\csc C = \dfrac{1}{\sin C}$

$\sec C = \dfrac{1}{\cos C}$

$\tan C = \dfrac{\sin C}{\cos C}$

$\cot C = \dfrac{1}{\tan C}$

$\tan C = \dfrac{1}{\cot C}$

Verbal expression:

1- Write the form of the previous rules if we double the angle 2A to be 4A .

Example **Trigonometric identities of the double -angle**

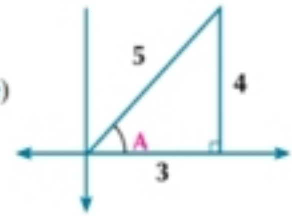
(1) If you know $\sin A = \frac{4}{5}$ where $0° < A < 90°$, find the value for each of the following without using the calculator:

(a) sin 2A (b) cos 2A (c) tan 2A

Solution

$\because \sin A = \frac{4}{5}$ $\because$ A lies in the first quadrant

$\therefore \cos A = \frac{3}{5}$ **(positive because A is an acute angle)**



a $\sin 2A$ $= 2\sin A \cos A = 2 \times \frac{4}{5} \times \frac{3}{5} = \frac{24}{25}$

b $\cos 2C$ $= 1 - 2\sin^2 A = 1 - 2 \times \frac{16}{25} = \frac{-7}{25}$

(you can use the other forms of the cosine rule of the double-angle)

c $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{24}{25} \div (\frac{-7}{25}) = \frac{-24}{7}$

Try to solve

(1) If $\cos A = \frac{4}{5}$, $0° < A < 90°$, find the values for each of the following without using the calculator:

a $\sin 2A$ **b** $\cos 2A$ **c** $\tan 2A$

Example **Trigonometric identities of the double-angle**

(2) Find the value for each of the following, without using the calculator , :

a $2\sin 15° \cos 15$ **b** $2\cos^2 22° 30' - 1$

Solution

a $2\sin 15° \cos 15° = \sin 2 \times 15° = \sin 30° = \frac{1}{2}$

b $2\cos^2 22° 30' - 1 = \cos(2 \times 22° 30') = \cos 45° = \frac{1}{\sqrt{2}}$

Try to solve

(2) Find the value for each of the following, without using the calculator:

a $2\sin 22° 30' \cos 22° 30'$ **b** $2\cos^2 75° - 1$ **c** $\cos^2 67.5° - \sin^2 67.5°$

d $\frac{2\tan 22° 30'}{1 - \tan^2 22° 30'}$ **e** $\frac{2\cos^2 165° - 1}{\sin 75° \cos 75°}$

The trigonornetric functions of the half-angle

Learn

You have previously learned that: $\cos A = 1 - 2\sin^2 \frac{A}{2}$ **(identity of double-angle)**

I.e.: $2\sin^2 \frac{A}{2} = 1 - \cos A$ **(from the properties of algebraic expressions)**

$\sin^2 \frac{A}{2} = \frac{1 - \cos A}{2}$ **(By dividing the two sides by 2)**

$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$ and similarly, the trigonometric functions can be found for each of: $\cos\frac{A}{2}$ and $\tan\frac{A}{2}$

$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$, $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$, $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$ **where** $\cos A \neq -1$
the sign is determined according to the quadrant at which the angle $\frac{A}{2}$ lies in

Example **Trigonometric identities of half-angle**

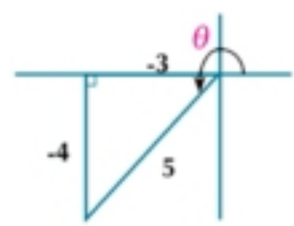
(3) Find the value for each of the following Without using the calculator :

a $\sin\frac{\theta}{2}$ known that, $\sin\theta = -\frac{4}{5}$, $180^\circ < \theta < 270^\circ$ **b** $\cos 75^\circ$

c $\tan 22^\circ\ 30'$

Solution

a $\because \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$



$\therefore \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-(-\frac{3}{5})}{2}} = \pm\sqrt{\frac{8}{10}}$

$\therefore \sin\frac{\theta}{2} = \pm\frac{2}{\sqrt{5}} = \pm\frac{2}{5}\sqrt{5}$

$\because 180^\circ < \theta < 270^\circ$ by dividing by 2, then $90^\circ < \frac{\theta}{2} < 135^\circ$

$\because \sin\frac{\theta}{2} = \frac{2}{5}\sqrt{5}$ (the sign is positive because it lies in the second quadrant)

b $\cos 75^\circ = \cos\frac{150^\circ}{2}$ (because $75^\circ = \frac{150^\circ}{2}$)

$= \sqrt{\frac{1+\cos 150^\circ}{2}}$ $(0 < 75^\circ < 90^\circ)$ **lies in the first quadrant so the value is positive**

$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}}$ **by multiplying numenator and denominator** $\times 2$

$\frac{\sqrt{2-\sqrt{3}}}{2}$ **by simplifying**

$\frac{\sqrt{2-\sqrt{3}}}{2} = \frac{\sqrt{\frac{4-2\sqrt{3}}{2}}}{2} = \frac{\sqrt{\frac{(\sqrt{3}-1)^2}{2}}}{2}$

$\frac{(\sqrt{3}-1)}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

c $\tan 22^\circ\ 30' = \tan\frac{45^\circ}{2}$ (because $22.5^\circ = \frac{45^\circ}{2}$)

$= \sqrt{\frac{1-\cos 45^\circ}{1+\cos 45^\circ}}$ $0 < (22.5^\circ < 90^\circ)$ **it lies in the first quadrant so the value is positive**

$= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}} = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$ **by multiplying by the conjugate of denominator** $(2-\sqrt{2})$

$$= \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}} \times \frac{2-\sqrt{2}}{2-\sqrt{2}}} = \sqrt{\frac{(4+2)-4\sqrt{2}}{4-2}}$$

$$= \sqrt{3-2\sqrt{2}} = \sqrt{2-2\sqrt{2}-1} = \sqrt{(\sqrt{2}-1)^2} = \sqrt{2}-1$$

Try to solve

(3) Find the value for each of the following without using the calculator:

a) $\cos\frac{\theta}{2}$ known that $\sin\theta = -\frac{4}{5}$, $180^\circ < \theta < 270^\circ$

b) $\cos 22^\circ\ 30'$

c) $\tan 15^\circ$.

Example **Proving the correctness of a trigonometric identity**

(4) Prove the correctness of the identity: $\csc 2x + \cot 2x = \cot x$, then use the previous identity to find the value of $\cot 15^\circ$.

Solution

The left side $= \frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x} = \frac{1+\cos 2x}{\sin 2x}$

$= \frac{1+(2\cos^2 x-1)}{2\sin x\cos x} = \frac{2\cos^2 x}{2\sin x\cos x} = \frac{\cos x}{\sin x} = \cot x$ (the right side)

By putting $x = 15^\circ$ in the identity: $\csc 2x + \cot 2x = \cot x$

$\therefore \cot 15^\circ = \csc 30^\circ + \cot 30^\circ = 2 + \sqrt{3}$

Try to solve

(4) Prove that: $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$ then find the value of $\frac{1-\tan^2 15^\circ}{1+\tan^2 15^\circ}$

Example **Finding the value of a trigonometric ratio**

(5) If $4\cos 2C + 3\sin 2C = 0$, find without using the calculator the value of $\tan C$, where C is the measurement of a positive acute angle.

Solution

$\because 4\cos 2C + 3\sin 2C = 0$ $\qquad \therefore 4\cos 2C = -3\sin 2C$

$\therefore \frac{\sin 2C}{\cos 2C} = \frac{4}{-3}$ $\qquad \therefore \tan 2C = \frac{4}{-3}$

$\therefore \frac{2\tan C}{1-\tan^2 C} = \frac{-4}{3}$ $\qquad \therefore 6\tan C = -4 + 4\tan^2 C$

$\therefore 2\tan^2 C - 3\tan^2 C - 2 = 0$

$(2\tan C + 1)(\tan C - 2) = 0$ $\qquad \therefore \tan C = -\frac{1}{2}$ (refused) or $\tan C = 2$

Think: Use the trigonometric functions of half-angle to find the value of tan C.

Try to solve

(5) If $4 \sin E + 3 \cos E = 3$, prove without using the calculator that $\tan \frac{E}{2} = \frac{4}{3}$, Where E is the measurement of a positive acute angle.

Example **Solving the trigonometric equations**

(6) Find the values of x included between 0 and 2π which satisfy the following equations:

a $\sin 2x = \sin x$ **b** $\cos^2 x - \sin^2 x = -\frac{1}{2}$ **c** $\tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} = 1$

Solution

a $\because \sin 2x = \sin x$

$2 \sin x \cos x = \sin x$

(**because:** $\sin 2x = 2 \sin x \cos x$)

$\sin x (2\cos x - 1) = 0$

$\sin x = 0$

$x = \pi$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$

the values of x which satisfy the equation are:

$\frac{\pi}{3}$, π or $\frac{5\pi}{3}$

think: Do you have other solutions? Find one of them.

b $\cos^2 x - \sin^2 x = -\frac{1}{2}$

$\cos 2x = -\frac{1}{2}$

(**because:** $\cos 2x = \cos^2 x - \sin^2 x$)

either $\cos 2x = \cos (\frac{2\pi}{3} + 2 n\pi)$

or $\cos x = \cos (\frac{4\pi}{3} + 2n\pi)$

where $n \in y$

$2x = \frac{2\pi}{3} + 2 n\pi$

$2x = \frac{4\pi}{3} + 2 n\pi$

By dividing by (2)

$x = \frac{\pi}{3} + \pi n$ $x = \frac{2\pi}{3} + n\pi$

by putting n= 0.1 ,

$x = \frac{\pi}{3}$, $x = \frac{4\pi}{3}$

$x = \frac{2\pi}{3}$, $x = \frac{5\pi}{3}$

the values of x which satisfy the equation are:

$\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$

c $\because \tan^2 \frac{\tan}{2} + 2 \tan \frac{X}{2} = 1$

$\therefore 2 \tan \frac{X}{2} = 1 - \tan^2 \frac{X}{2}$

$\therefore \frac{2\tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} = 1$ $\therefore \tan x = 1$

(positive tangent in both the first and third quadrants)

in the first quadrant: $x = \frac{\pi}{4}$

in the third quadrant: $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$\therefore$ **the values of x which satisfy the equation are** $\frac{\pi}{4}$ or $\frac{5\pi}{4}$

Try to solve

(6) Find the values of x included between 0 and 2π which satisfy the equation $\cos x + \cos 2x = 0$

(7) **Geometry:** X Y Z is a triangle in which x = 12cm, y = 18cm, z = 15cm, prove that m(∠Y) = 2 m(∠X).

Exercises 4 - 3

Choose the correct answer:

1. If $\cos C = \frac{1}{3}$, then cos 2 C equals

 a) 0 b) $\frac{-2}{3}$ c) $\frac{-7}{9}$ d) $\frac{2}{3}$

2. sinA cos A equals

 a) $\frac{1}{2}\sin 2A$ b) $\sin 2A$ c) $\cos 2A$ d) $\frac{1}{2}\cos 2A$

3. $\cos^2 C - \cos 2C$ equals

 a) sin C b) cos C c) $\sin^2 C$ d) tan C

4. $1 - 2\sin^2 50$ equals

 a) sin 100° b) cos 50° c) cos 100° d) sin 50°

5. 1 + cos 4A equals

 a) $2\cos^2 4A$ b) $\cos^2 2A$ c) cos4 ²ıA d) $2\cos^2 2A$

6. Find without using the calculator the values of : $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$ if:

 a) $\sin\theta = \frac{4}{5}, 0° < \theta < 90°$ b) $\cos\theta = \frac{1}{3}, 0° < \theta < \frac{\pi}{2}$

 c) $\csc\theta = \frac{5}{3}, \pi < \theta < \frac{3\pi}{2}$ d) $\cot\theta = \frac{3}{2}, 180° < \theta < 270°$

7. Find without using the calculator the values of: $\sin 2\theta$, $\cos 2\theta$, $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$:

 a) $\cos\theta = \frac{1}{4}, 0° < \theta < 90°$ b) $\sin\theta = \frac{3}{5}, 90° < \theta < 180°$

 c) $\tan\theta = \frac{4}{3}, \pi < \theta < \frac{3\pi}{2}$ d) $\sin\theta = \frac{15}{17}, \frac{3\pi}{2} < \theta < 2\pi$

8. If you know that $\cos A = \frac{5}{13}$, where A is measurement of a positive acute angle, find without using the calculator the value of:

 $\sin 2A$, $\cos 2A$, $\tan 2A$, $\cos\frac{A}{2}$

9. If A is the measurement of an acute angle and $\cos 2A = \frac{119}{169}$,without using the calculator, find the value of: sinA , cosA , tan2A

10. If $\sin\frac{A}{2} = \frac{1}{3}$, without using the calculator, find the value of: $\cos\frac{A}{2}$, sin A , cosA

(11) If $\cos B = \frac{-4}{5}$, $\pi < B < \frac{3\pi}{2}$, without using the calculator, find the value of $\frac{\sin 2B}{2\cos 2B}$

(12) Express each of the following in the form of an only trigonometric ratio:

a $\sin 35^\circ \cos 35^\circ$ b $\frac{\tan 40^\circ}{1 - \tan^2 40}$ c $\sin 25^\circ \cos 35^\circ - \cos 25^\circ \sin 35^\circ$

d $\frac{\tan 50^\circ - \tan 40^\circ}{1 + \tan 50^\circ \tan 40^\circ}$ e $\cos^2 25 - \sin^2 25$ f $\frac{1 - \cos\theta + \sin\theta}{1 + \cos\theta + \sin\theta}$

(13) Find the values of X included between 0° and 2π which satisfy each of the following equation:

a $\sin 2x = \sin x$ b $\sin^2 2x + 2\sin 2x + 1 = 0$ c $\cos x - 2\sin^2 \frac{1}{2}x = 0$

(14) Prove the correctness for each of the following identities:

a $\frac{1 - \cos 2A}{3 + \cos 2A} = \frac{\sin^2 A}{1 + \cos^2 A}$ b $\frac{\cot\theta - 1}{\cot\theta + 1} = \frac{\cos 2\theta}{1 + \sin 2\theta}$

(15) **Mechanics:** A football player kicked the ball by an angle of measurement 30° with the ground surface with initial velocity of a magnitude of 14.7 m/sec. If the horizontal distance S which the ball covers is given by the relation : $S = \frac{2v_0^2 \sin\theta \cos\theta}{g}$ where g is the free fall acceleration and equals 9.8 m/sec2 and V_0 represents the initial velocity.

First: put the previous relation in the simplest form

Second: Find the horizontal distance D which the ball covers in meters.

Activity **Using the graphical calculator**

(16) Use the graphical calculator or a graphical program of a computer or a tablet to graph the curve of the function f where :

$f(x) = \sin\theta \cos\frac{\pi}{4} - \cos\theta \sin\frac{\pi}{4}$ where: $-2\pi \leqslant \theta \leqslant 2\pi$

a Can you deduce the function F from its graphical representation ? Explain.

b Prove the correctness of the deduced function from the graph using the rules.

Heron's Formula

4 - 4

Historical introduction:

Heron lived in Alexandria B.C. He credited the heron's formula to find the surface area of the triangle in terms of its side lengths. There is a thought that this formula is known before Heron but it is credited for him because it is found in a part of his factory called Metrica. This part included several geometric knowledge in plane geometry, solids, areas and so on.

You will learn

- Finding the surface area of the triangle in terms of its side lengths.
- Finding the radius length of the inscribed circle of a triangle touching its sides

Key terms

- Heron formula

Materials

- Scientific calculator
- Internet

Finding the surface area of the triangle in terms of its side lengths

Let a , b and c be the side lengths of the triangle A B C where:

$a + b + c = 2P$ (where P is half the triangle perimeter)

From the cosine rule, we know that:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \qquad (1)$$

and from pythagoras' identity: $\sin C = \sqrt{1 - \cos^2 C}$

(2)

(**Notice that:** $0^\circ < C < 180^\circ$ then $\sin C > 0$)

By substituting from (1) and (2), then:

$$\sin C = \sqrt{\frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2b^2}}$$ by unifying the denominators

$$\sin C = \frac{1}{2ab}\sqrt{(2ab + (a^2 + b^2 - c^2))\ (2ab - (a^2 + b^2 - c^2))}$$

factorizing the difference between two squares

$$= \frac{1}{2ab}\sqrt{[(a+b)^2 - c^2]\,[c^2 - (a - b)^2]}$$

putting the expression in the form of a perfect square

$$= \frac{1}{2ab}\sqrt{(a + b + c)(b + c - a)(a + c - b)(a + b - c)}$$

By factorizing as a difference between two squares

$$= \frac{1}{2ab}\sqrt{2P(a + b + c - 2a)(a + b + c - 2b)(a + b + c - 2c)}$$

$$= \frac{1}{2\,a\,b}\sqrt{2p\,(2p-2a)\,(2p-2b)\,(2p-2c)}$$

$$= \frac{1}{2\,a\,b}\sqrt{16p\,(p-a)\,(p-b)\,(p-c)}$$

$$= \frac{\cancel{4}\;2}{2\,a\,b\;1}\sqrt{p(p-a)\,(p-b)\,(p-c)} \qquad (3)$$

But: $a\,(\triangle ABC) = \frac{1}{2}\,a\,b\,\sin C \qquad (4)$

By substituting from (3) in (4) $a\,(\triangle ABC) = \sqrt{p\,(p-a)(p-b)\,(b-c)}$

I.e: the surface area of the triangle whose side lengths are a, b and c is:

> $\Delta = \sqrt{p(p-a)\,(p-b)\,(p-c)}$ **where P is half of the triangle perimeter**

Important note : (proof is not required)

Activity (1)

Log in internet to prove the correctness of Heron's formula.

Example

1) Find the surface area of the triangle whose side lengths are 6, 8 and 10 centimetres using Heron's formula

Solution

$\because\ 2P = 6 + 8 + 10 = 24$ cm $\qquad P = 12$ cm

$P - a = 12 - 6 = 6$cm , $p - b = 12 - 8 = 4$ cm , $p - c = 12 - 10 = 2$ cm

$\therefore$ Area of $\Delta = \sqrt{p\,(p-a)(p-a)\,(p-c)}$

$= \sqrt{12 \times 6 \times 4 \times 2} = 24\ cm^2$

Critical thinking: Can you find another way to find the surface area of the triangle in the example above? Explain.

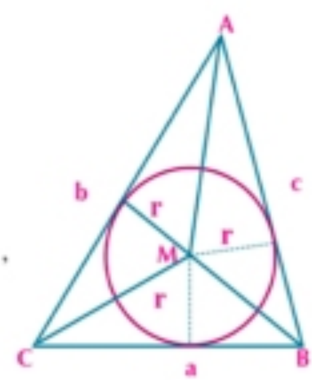
Try to solve

1) Find the surface area of the triangle A B C in which:
a= 5cm , b = 12 cm , c = 13cm using Heron's formula.

Activity (2)

Finding the radius length of the inscribed circle of a triangle touching its sides.

1) Tell the relation between $\triangle$A B C and the triangles M AB , M B C and M AC.



(2) Can you find a relation between the radius length r and the surface area of triangle A B C ?

We can deduce that: $a(\triangle ABC) = \frac{1}{2}ar + \frac{1}{2}b \times r + \frac{1}{2}cr$

$$\sqrt{p(p-a)(p-b)(p-c)} = r \times P$$

i.e. $$r = \frac{\sqrt{p(p-a)(p-b)(p-c)}}{P} = \frac{a(\triangle ABC)}{P}$$

Example

(2) Use the relation above to find the radius length of the circle interiorly touching the sides of the triangle AB C whose side lengths are 7, 9 and 14 centimeters to the nearest decimal.

Solution

$\because 2P = 7 + 9 + 14 = 30$ cm $P = 15$ cm , $p - a = 8$cm , $p - b = 6$cm , $p - c = 1$ cm

By substituting in the relation above:

$$\therefore r = \frac{\sqrt{p(p-a)(p-b)(p-c)}}{P} = \frac{\sqrt{15 \times 8 \times 6 \times 1}}{15} = \frac{4}{5}\sqrt{5} \text{ cm}$$

Critical thinking: What do you expect if: $p - a$ **or** $p - b$ **or** $p - c$ is negative or equals zero? Explain .

Example

(3) Find the area of the following triangles.

a) A triangle of side lengths 7 , 9 and 12 centimeters.

b) A triangle of two-side lengths 24 and 40cm and the included angle of measurement 30°.

c) Is the triangle whose side lengths are 12 , 14 and 30 existed? find its area if possible.

Solution

a) $\because 2P = 28$ cm $\therefore p = 14$ cm , $p - a = 7$cm , $p - b = 5$cm , $p - c = 2$cm

By applying Heron's formula:

Area of the triangle = $\sqrt{14 \times 7 \times 5 \times 2} = 14\sqrt{5}$ cm²

b) Area of the triangle $= \frac{1}{2} \times 24 \times 40 \times \sin 30$

$= \frac{1}{2} \times 24 \times 40 \times \frac{1}{2} = 240$ cm²

c) $2P = 56$ cm $\therefore P = 28$ cm , p<a side

$\therefore$ there is no triangle to find its area

Remember

The surface area of a triangle in terms of two side lengths and the included angle.

Area of triangle = $\frac{1}{2}$ product of its two sides × cosine of the included angle

Think: Can you use another way to prove that there is no triangle to be drawn ? Explain.

Try to solve

(3) Find the area of the following triangles:

a) A triangle of side lengths 6 , 6 and 8 centimeters.

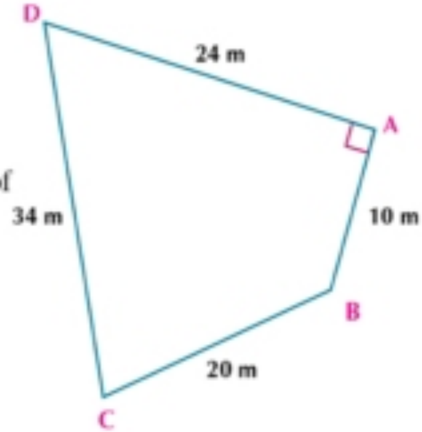
b) The triangle whose side lengths are 24, 36 and 60 centimeters. Find its surface area if possible.

c) A triangle of side lengths 10 , 24 and 26 centimeters.

Example

Geometry

(4) The opposite figure illustrates a quadrilateral piece of land , find its area.



Solution

Draw $\overline{BD}$

in triangle A B D which is right angled at A

$(BD)^2 = (AB)^2 + (AD)^2$ **Pythagoras theorem**

$= 100 + 576 = 676$ $\therefore BD = 26$ m

$a(\triangle ABD) = \frac{1}{2} AB \times AD = \frac{1}{2} \times 10 \times 24 = 120 \text{ m}^2$

in triangle B C D:

$\because 2P = 20 + 34 + 26 = 80$ $\therefore P = 40$ m

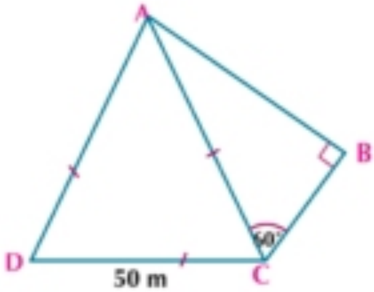
$a(\triangle BCD) = \sqrt{40 \times 20 \times 6 \times 14} = 40\sqrt{42} \text{ m}^2$

Area of the piece of land = area of $\triangle$ A B D + area of $\triangle$ B C D

$= 120 + 40\sqrt{42} = 40(3 + \sqrt{42}) \text{ m}^2$

Try to solve

(4) The opposite figure illustrate a piece of land of dimensions shown in the figure.
Find its area.



Exercises 4 - 4

Complete:

1. The surface area of the equilateral triangle whose side length is 6cm equals...

2. The surface area of the isosceles triangle which one of its side length is 10cm and the measurement of one of its base angles is 45° equals...

3. The surface area of the triangle whose side lengths are 3, 4 and 5 centimeters equals...

4. The surface area of the triangle whose two side lengths are 6 and 8 centimeters and the measurement of the angle included between them is 30° equals...

Find the surface area of the triangle A B C in each of the following cases:

5. $a = 15$cm , $b = 12$cm , $c = 9$ cm
6. $b = 16$cm , $c = 20$ cm, $m(\angle A) = 60^\circ$
7. $a = 16$cm , $b = 18$cm , $c = 24$ cm
8. $a = 32$cm , $b = 36$, $c = 30$ cm

Choose the correct answer:

9. The surface area of the triangle whose side lengths are 6cm , 8cm and 10cm in cm^2 is :

   a) 24 b) 30 c) 40 d) 48

10. In the opposite figure : surface area of $\triangle$ A B C equals cm^2

   a) 20 b) $4\sqrt{5}$

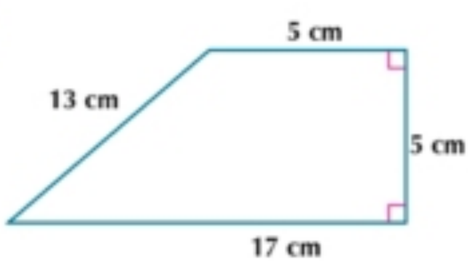
   c) $2\sqrt{5}$ d) 10

11. If the perimeter of a triangle is 60 cm and the length of one side is 26cm , then the lengths of the other two sides in centimeter can be:

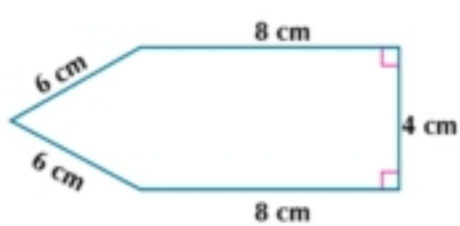
   a) 4 , 30 b) 31 , 3 c) 20 , 14 d) 2 , 32

Find the surface area for each of the following figures using the given data shown:

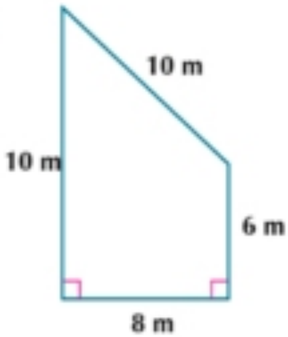
12.



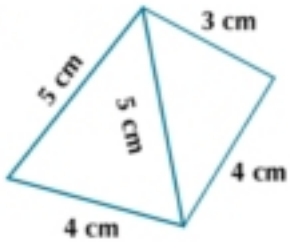
13.



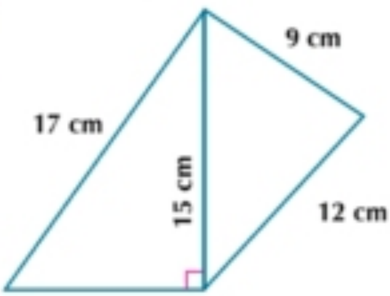
(14)



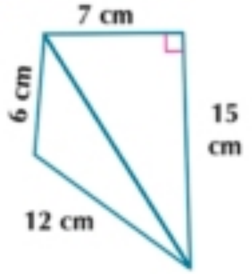
(15)



(16)



(17)



(18) Find the surface area of the quadrilateral A B C D in which m(∠B) = 90°, A B = 5cm , B C = 12cm , A D = C D = 13cm.

(19) Find the surface area of the triangle A B C in which:

a) a = 8cm , b = 11cm , c = 13 cm

b) a = 12cm , b = 17cm , c = 25 cm

c) a = 40 cm , b = 24cm , c = 32 cm

d) a = 11cm , b = 8cm , c = 6 cm

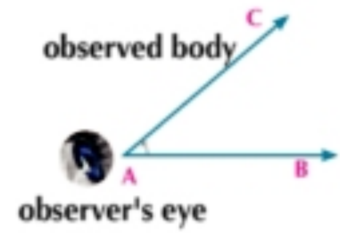
(20) **Yards:** A triangular yard at which the ratio among its side lengths is 7 : 5 : 3. If the perimeter of the yard is 300 meters , find its surface area .

Activity

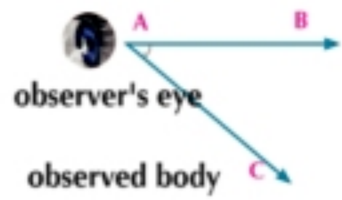
(21) **Environment:** The following diagram illustrates a triangular yard, find its surface area

Unit summary

1 **Angle of elevation:** if it is supposed that there is an observer at point A and looks at a body at point C above the eye sight level, then the angle included between the horizontal ray and the ray $\overrightarrow{AC}$ connecting between the observer's eye and the observed body is called the angle of elevation of the observed body C with respect to point A

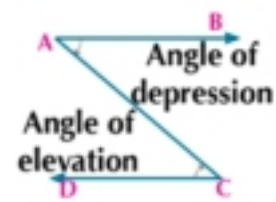


2 **Angle of depression:** if it is supposed that there is an observer at point A and looks at a body at point C down the eye sight level, then the angle included between the horizontal ray $\overrightarrow{AB}$ and the ray $\overrightarrow{AC}$ connecting between the observer's eye and the observed body is called the angle of depression of the observed body C with respect to point A.



3 The measurement of the angle of depression C with respect to A equals the measurement of the angle of elevation A with respect to C

4 The trigonometric functions of the sum and difference of the measures of two angles:



- If A and B are the measures of two angles, then:

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(C \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$, where $A \neq \frac{\pi}{2}(2K+1)$, $B \neq \frac{\pi}{2}(2K+1)$, $\tan A \tan B \neq \pm 1 \quad K \in z$

5 The trigonometric functions of double -angle:

$\sin 2A = 2 \sin A \cos A$ For each $A \in R$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1 = 1 - 2\sin^2 A$ For each $A \in R$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$, where tan A is defined and $\tan^2 A \neq 0$

we can deduce that:

$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$

$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} = 2\cos^2 \frac{A}{2} - 1 = 1 - 2\sin^2 \frac{A}{2}$

$\tan A = \frac{2\tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$

6 The surface area of a triangle in terms of two side lengths and the included angle = $\frac{1}{2}$ product of its two sides × cosine of the included angle .

7 If a, b and c are the side lengths of a triangle whose perimeter is 2P then:

Surface area of the triangle = $\sqrt{p(p-a)(p-b)(p-c)}$

Accumulative test

First: Multi - choice questions

(1) Choose the correct answer:

$\sin 3x \cos 2x + \cos 3x \sin 2x$ equals

(1) $\cos 5x$ (2) $\sin 5x$ (3) $\sin x$ (4) $\cos x$

(2) In the triangle A B C if a= 5cm, then b equals

(1) $\frac{5\sin A}{\sin B}$ (2) $\frac{5\sin B}{\sin A}$ (3) $\frac{5\sin A}{\sin C}$ (4) $\frac{5\sin C}{\sin B}$

(3) The triangle A B C in which a: b: c = 3: 2: 2, then cos a equals

(1) $-\frac{1}{4}$ (2) $-\frac{1}{8}$ (3) $\frac{1}{8}$ (4) $-\frac{3}{4}$

(4) In the triangle A B C if a = 15cm , b = 25cm , c = 35cm, then the measurement of the largest angle in the triangle equals:

(1) 90° (2) 40° (3) 120° (4) 150°

Second : short answered questions:

(5) Find the radius length of the circumcircle of the triangle A B C in which $m(\angle A) = 30^\circ$, a = 10cm.

(6) In the triangle A B C if $2 \sin A = 3 \sin B = 4 \sin C$, find a : b: c

(7) Find the measurement of the largest angle in the triangle A B C in which a = 6cm , b = 14cm, c = 10cm

(8) If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$ find $\tan (A + B)$

(9) Find the value of $\tan 15^\circ$ without using the calculator.

(10) If $\sin A = \frac{1}{3}$, find the value of $\cos 2A$

Third : long - answered questions

(11) Solve the triangle in which $m(\angle A) = 2\ m(\angle B) = 86^\circ$, c = 9cm

(12) The triangle X Y Z in which x= 9.4cm, y = 15.6cm, $m(\angle Z) = 54^\circ$

find Z to the nearest decimal and the area of the circumcircle of the triangle.

(13) Find the measurement of the smallest angle in the triangle A B C in which a= 13cm , b = 14cm , c = 15cm, then find its surface area.

14. A pilot observed a ground target to find the measure of its angle of depression is 45° and when he flew down for a distance of 2km vertically, he found the measure of the angle of depression of the target was 30°. Find the height of the plane at the beginning of observation to the nearest kilometer .

15. Find the surface area of the isosceles triangle whose perimeter is 30cm and the length of one of its sides is 12cm.

16. Find the surface area of a triangle-like piece of land whose perimeter is 180m and the ratio between its side lengths is 2: 3: 4 to the nearest meter.

General Exercises

For more exercises, please visit the website of Ministry of Education.

Enrichment Information

Please visit the following links.

General tests

First : tests of algebra

Exam 1 Algebra

Answer the following questions

Question 1: Choose the correct answer :

(1) If $\lfloor n+1 = 30 \lfloor n-1$, then the value n is

(a) 5 (b) 6 (c) 29 (d) 30

(2) The value of the series $\sum_{r=1}^{15} (r^2 + r + 1)$ is:

(a) 1375 (b) 3720 (c) 14400 (d) 2232000

(3) The number of the terms of the arithmetic sequence (7, 11, 15,..., 271) is :

(a) 34 (b) 67 (c) 169 (d) 9313

(4) If $x > 0$, then the common ratio of the geometric sequence $(4, x - 3, 2x + 6, ...)$ is :

(a) 1 (b) 5 (c) 3 (d) 24

Question 2:

(1) If ${}^5P_r = 2 \times {}^6P_{r-1}$ find the value of r.

(2) Find the order of the first negative term of the terms of the sequence (152 - 9n), then find the greatest sum can be got from the terms of this sequence .

Question 3:

(1) How many different three-digit even numbers can be formed from the set of the numbers {2, 3, 4, 5, 7}?

(2) Find the geometric sequence whose terms are positive, the sum of the first three terms equals 14 and its first term is greater than its second term by 4, then find the sum of infinite number of its terms starting from its first term.

Question 4:

(1) Find the geometric sequence whose sum of its first three terms equals $\frac{171}{32}$ and its second term equals $\frac{27}{16}$, then find its tenth term.

(2) A 25 - row theatre; the first row contains 20 seats, the second row contains 22 seats and the third row contains 24 seats and so on.... Find the number of seats in all the theatre rows.

Question 5:

1. If ${}^{2n+m}C_2 = 190$, ${}^{n-2m}P_3 = 60$, find the value of n and m.

2. An arithmetic sequence whose sum of its first and last terms is 26 and the sum of its terms is 468, find the number of its terms. If its tenth term equals 47, find this sequence.

Exam 2 Algebra

Answer the following questions،

Question 1: Choose the correct answer:

1. The number of ordered pairs (a , b) which can be formed from the elements of the set $\{1, 2, 3\}$ where $a \neq b$

   a) 2 b) 3 c) 6 d) 9

2. The n^{th} term of the sequence $(2, 2, \frac{8}{3}, 4, ...)$is

   a) $(n - 1)$ b) $2^n - 1$ c) 2^{n-1} d) $\frac{2^n}{n}$

3. The sum of the first 25 terms of the terms of the sequence $(3 - 2n)$ is

   a) 650 b) 600 c) -575 d) -600

4. If (x , y, z,...) are geometrically sequenct , then:

   a) $2y < x + z$ b) $y^2 > x z$ c) $y = x z$ d) $\sqrt{y} = x z$

Question 2:

1. If ${}^{25}C_{2r+1} = {}^{25}C_{3r-1}$, find the value of r

2. Find the number of terms that can be taken from the terms of the sequence (-43 , -36 , -29 ,...) starting from its first term to get a sum of 221.

Question 3:

1. A university student learns different eight subjects and he cannot join the next grade till he succeeds in six subjects at least. How many ways can the student join the next grade?

2. A geometric sequence in which the sum of an infinite number of its terms starting from its first term equals 108 and its first term is greater than its second term by 12. Find the sequence and the sum of its first seven terms.

Question 4:

1. Find the sum of the odd ordered terms of the arithmetic sequence (2 , 5 , 8 ,... , 110)

2. An agricultural crops storing company has seven warehouses to store the wheat so that the first warehouse holds 270 tons and each warehouse after that can hold two third of the amount of the directly previous warehouse. Can the company store 800 tons of wheat? What is the greatest amount of wheat the company can store in its warehouses to the nearest ton?

Question 5:

1. If $\underline{|3n-7}$ = 120 find the value of ${}^{n}C_{n-1}$.

2. Insert 28 arithmetic means between 4 and 91, then find the sum of the terms of the arithmetic sequence resulted.

Exam 3 — calculus & trigonometry

Answer the following questions:

Question 1: choose the correct answer:

1. If $y = \sin 2x$ then $\frac{dy}{dx}$ when $x = \frac{\pi}{6}$ equals.
   - a) 2
   - b) 1
   - c) $\frac{1}{2}$
   - d) $\sqrt{3}$

2. If $\cos\theta = \frac{2}{3}$, then $\cos 2\theta =$
   - a) $\frac{4}{9}$
   - b) $\frac{3}{2}$
   - c) $\frac{-1}{9}$
   - d) $\sqrt{3}$

3. $\int (2x+3)^4 \, dx =$.
   - a) $\frac{1}{5}(2x+3)^5 + C$
   - b) $\frac{1}{10}(2x+3)^5 + C$
   - c) $\frac{1}{10}(2x+3)^3 + C$
   - d) $10(2x+3)^3 + C$

4. The average rate of change of the function f where $f(x) = x^2$ when x varies from 3 to 3.1 equals.
   - a) 0.61
   - b) 6.1
   - c) 9
   - d) 9.61

Question 2:

1. Find the first derivative if: $y = x^2 \sin 2x$

2. Without using the calculator, prove that $\frac{\sin 2x}{1 + \cos 2x} = \tan x$

Question 3:

1. Find the slope of the tangent to the curve of the function f wher $f(x) = \frac{x^2+3}{x-2}$ when $x = 1$

(2) Find:

(a) $\int (x^2 + 2x)\, dx$ (b) $\int (\sin x - \cos x)^2\, dx$

Question 4:

(1) Find the point (s) lying on the curve of the function: $y = \frac{1}{x - 3}$ at which the tangent is parallel to the straight line x + y = zero.

(2) From a house top of 25 meters high, the measurement of the angle of elevation of a tower top was 70° and the measurement of the angle of depression of the tower base was 30°. Find the height of the tower known that the bases of the house and tower are at the same horizontal level.

Question 5:

(1) If the function f where
$$f(x) = \begin{cases} x^2 - 2 & \text{for each } x \leqslant 2 \\ 2a\,x - 3b & \text{for each } x > 2 \end{cases}$$

(2) Find $\frac{dy}{dx}$ if $y = (z^3 - z^2)$, $z = 2x + 1$ when $x = -1$

Exam 4 calculus & trigonometry

Answer the following questions:

Question 1: choose the correct answer:

(1) The slope of the tangent to the curve of the function f where $f(x) = 3x^2 + 2x - 1$ when $x = 2$ equlas

(a) 4 (b) 8 (c) 17 (d) 14

(2) $\sin A \cos B - \cos A \sin B =$

(a) $\sin(A + B)$ (b) $\cos(A + B)$ (c) $\sin(A - B)$ (d) $\cos(A - B)$

(3) $\int \frac{x^2 + 3x}{x}\, dx =$

(a) $x + 3$ (b) $\frac{1}{2} x^2 + 3x + C$ (c) $x^2 + 3x + C$ (d) $\frac{x^3 + 3x^2}{x^2}$

(4) $\frac{d}{dx} (\sin x \cos x) =$

(a) $\sin x$ (b) $\cos x$ (c) $\frac{1}{2} \cos 2x$ (d) $\cos 2x$

Question 2:

(1) If $y = f(x)$ where $y = x^2 - a x$, find the slope of the tangent to the curve of the function f at point (3, 0) lying on it.

(2) If $\sec A = \frac{5}{4}$ and $\csc B = \frac{13}{5}$ where A and B are the measurements of two acute angles , find $\sec(A - B)$.

Question 3:

(1) Discuss the differentiability of the function f where

$$f(x) = \begin{cases} x^2 & , \quad x > 2 \\ 4x - 1 & , \quad x \leqslant 2 \end{cases} \qquad \text{when } x = 2$$

(2) Find $\int (1 - \cos x)^2 \, dx$

Question 4:

(1) A ship sailed from a certain point in the direction of 60° north of the west at velocity 26 km/h and at the same time and place, another ship sailed in the direction of the East at velocity 15 km/h. Find the distance between the two ships after 3 hours.

(2) If $y = z^5 + 3$ and $z = (x - 1)^3$, find the value of $\frac{dy}{dx}$ when $x = 2$

Question 5:

(1) If $y = \left(\frac{x^2 + 1}{x - 3}\right)^5$, find $\frac{dy}{dx}$ when $x = 1$

(2) Find the tangent equation to the curve of $y = 2x \sin x \cos x$ when $x = \pi$

المواصفات الفنيـة:

رقـم الكتـــاب:	٢٧/٢/٢٢/١٥/١٠/١٥٣٦
مقـاس الكتـــاب:	$\frac{1}{8}$ (٥٧ × ٨٢) سم
طبـــع المتـــن:	٤ ألوان
طبـــع الغـلاف:	٤ ألوان
ورق المتـــــن:	٨٠ جم أبيض
ورق الغـــلاف:	٢٠٠ جم كوشيه
عدد الصفحات بالغلاف:	١٦٠ صفحة

الأشــراف برنتنج هاوس