Pure
Mathematics

Third Form Secondary

Student Book

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Introduction

بسم الله الرحمن الرحيم

We are pleased to introduce this book to show the philosophy on which the academic content has been prepared. This philosophy aims at:

1 Developing and integrating the knowledgeable unit in Math, combining the concepts and relating all the school mathematical curricula to each other.

2 Providing learners with the data, concepts, and plans to solve problems.

3 Consolidate the national criteria and the educational levels in Egypt through:

A) Determining what the learner should learn and why.

B) Determining the learning outcomes accurately. Outcomes have seriously focused on the following: learning Math remains an endless objective that the learners do their best to learn it all their lifetime. Learners should like to learn Math. Learners are to be able to work individually or in teamwork. Learners should be active, patient, assiduous and innovative. Learners should finally be able to communicate mathematically.

4 Suggesting new methodologies for teaching through (teacher guide).

5 Suggesting various activities that suit the content to help the learner choose the most proper activities for him/her.

6 Considering Math and the human contributions internationally and nationally and identifying the contributions of the achievements of Arab, Muslim and foreign scientists.

In the light of what previously mentioned, the following details have been considered:

★ This book contains: algebra and analytic solid geometry. The book has been divided into related and integrated units. Each unit has an introduction illustrating the learning outcomes, the unit planning guide, and the related key terms. In addition, the unit is divided into lessons where each lesson shows the objective of learning it through the title You will learn. Each lesson starts with the main idea of the lesson content. It is taken into consideration to introduce the content gradually from easy to hard. The lesson includes some activities, which relate Math to other school subjects and the practical life. These activities suit the students' different abilities, consider the individual differences throughout Discover the error to correct some common mistakes of the students, confirm the principle of working together and integrate with the topic. Furthermore, this book contains some issues related to the surrounding environment and how to deal with.

★ Each lesson contains examples starting gradually from easy to hard and containing various levels of thoughts accompanied with some exercises titled Try to solve. Each lesson ends in Exercises that contain various problems related to the concepts and skills that the students learned through the lesson.

Last but not least. We wish we had done our best to accomplish this work for the benefits of our dear youngsters and our dearest Egypt.

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Differential & Integral Calculus

Algebra

Unit One

Binomial theorem

Unit introduction

Nasser Eldin Eltosy was born (1201 - 1274) in Gahrood near tos in Iran in a family specialized in science and philosophy. He was a student of Kamal Eldin ElMowsly and Moeen Eldin Elmasry, so he studied wisedom, philosophy, astrology and mathematics. He had a great history in calculating the number of possibilities for different phenomena to happen. He also used permutations and combinations. Kardan (1901-1976) also had great interest in calculating the number of possibilities using the fundamental counting principle, which allowed him a great field in computer architecture which is about the design and structure of functional operations for computer. This unit is dealing with the principle of counting and the relation between permutations and combinations and their uses in solving some mathematical problems as well as the binomial theorem and solving some mathematical life applications.

Unit objectives

By the end of this unit and doing all the activities involved , the student should be able to:

- Identify the binomial theorem with positive integer power
- Deduce the general term in the expansion of the binomial theorem.
- Deduce the ratio between each term and the previous one in the expansion of the binomial theorem.
- Find the coefficient of any term in the expansion of the binomial theorem due to the order of this term.
- Find the coefficient of any power of x in the expansion of $(x + y)^n$
- Find the term free of x in the expansion of $(x + y)^n$
- Find the coefficient of the greatest term in the expansion of the binomial theorem
- Find the middle term in the expansion of the binomial theorem when n is an even number and the two middle terms when n is an odd number.
- Deduce relations between the combinations using the expansion of binomial theorem
- Deduce the relation between pascal's triangle and the coefficients of the expansion of binomial theorem and deduce some patterns using pascal's triangle .
- Solve mathematical life applications on the binomial theorem.

Key Terms

- *Binomial Theorem*

Materials

- *Scientific calculator*

Unit Lessons

Lesson (1 - 1): Bunomial theorem with integer positive power

Lesson (1 - 2): Finding the term containing x^K in the expansion of binomial

Lesson (1 - 3): The ratio between two consecutive terms in the binomial expansion

Unit Chart

Binomial theorem

Binomial theorem with integer positive power

The expansion of the binomial

The general term

Evaluation the term containing x^K

Unit One

1 - 1 Binomial theorem for positive integer power

You will learn

- Relating pascal's triangle and coefficient of the expansion of the binomial
- The general form of the expansion of $(x + a)^n$ $n \in Z^+$
- The general form of the general term T_{r+1} in the expansion of $(x + a)^n$
- The order and the value of the middle term and the two middle terms

Key terms

- The expansion
- Binomial
- The general term
- The middle term

Materials

- Scientific calculator
- Graphical programs

Think and discuss

We know that:

$(x + a)^1 = x + a$

$(x + a)^2 = x^2 + 2x\,a + a^2$

We can deduce that:

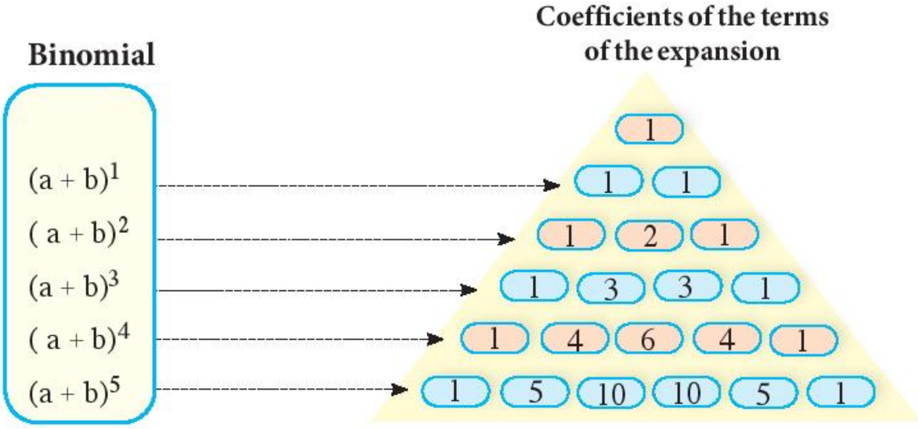
$(x + a)^3 = x^3 + 3x^2\,a + 3x\,a^2 + a^3$

$(x + a)^4 = x^4 + 4x^3\,a + 6x^2\,a^2 + 4x\,a^3 + a^4$

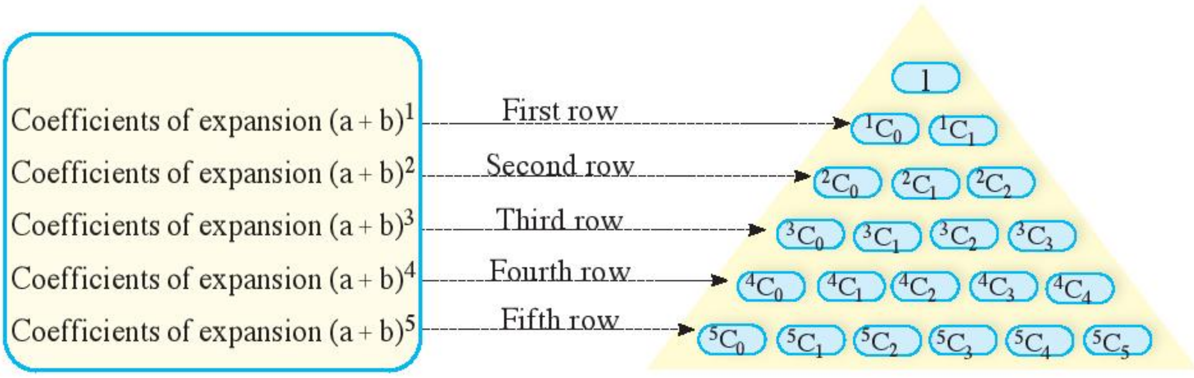
- What is the relation between the number of terms and the value of the power?
- What is the relation between the powers of the variables x, a in each term of the expansion ?
- What do you notice about the coefficients of terms in each term of the expansion?
- Can pascal's triangle be used to express the coefficients?
- Try to deduce a rule to expand $(a + b)^n$

Pascal's triangle

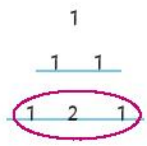
Notice that: **The coefficients of the expansion follow a pattern in pascal's triangle**



- Pascal's triangle can be written using the combinations as in the next figure:



By noticing the second row (say) from bascal's triangle, we notice that 1, 2, 1 represent $^2C_0$, $^2C_1$, $^2C_2$ respectively and the sum of these elements $^2C_0$, $^2C_1$, $^2C_2$ represent the number of subsets which can be formed from a set containing two elements where $^2C_0 + {}^2C_1 + {}^2C_2 = 4 = 2^2$

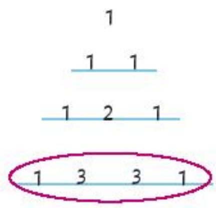


The set { x , y} all of its subsets are ϕ , {x} , {y},{x , y}

Similarly: the sum of elements of the third row

$^3C_0, {}^3C_1, {}^3C_2, {}^3C_3$ represent the number of subsets which we obtained from a set contains three elements and the number of these sets is $8 = 2^3$

That is $\quad {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 = 2^3$



In general , if we have a set of n elements , then the number of subsets which can be obtained from its elements = 2^n **i.e** $^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$

Verbal expression: using Bascal's triangle, find:

1) The coefficients of $(a+b)^6$ as combinations. **2)** The coefficients of $(a+b)^5$ as combinations.

Learn

The expansion of a binomial

If $a, x \in R$, $n \in Z^+$ then:

1- $(x+a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \dots + a^n$

2- $(x-a)^n = x^n - {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 - \dots + (-a^n)$

Remarks on the expansion of the binomial $(x+a)^n$

1) The number of terms of the expansion is $(n+1)$ term.

2) The expansion is arranged in a descending order according to the powers of x and arranged in an ascending order according to the powers of a.

3) The sum of powers of x and powers of a in any term equals n.

4) The value of r in ^nC_r of each term is always decreased the order of the term by one.

Example **Writing the expansion of a binomial**

1. Write the expansion of $(2x + 3y)^4$

Solution

$(2x + 3y)^4 = (2x)^4 + {}^4C_1(2x)^3(3y) + {}^4C_2(2x)^2(3y)^2 + {}^4C_3(2x)(3y)^3 + (3y)^4$

$= 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$

Try to solve

1. Write the expansion of:

a) $(3x + y)^5$ b) $(x^2 - 1)^6$

Special cases of the expansion of a binomial:

a) $(1 + x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + ... + x^n$

b) $(1 - x)^n = 1 - {}^nC_1 x + {}^nC_2 x^2 - ... + (-x)^n$

Example

2. Write the expansion of $(1 + x)^6$, then use it to find the numerical value of the expression:

$${}^6C_0 + {}^6C_1 + {}^6C_2 + ... + {}^6C_6$$

Solution

$(1 + x)^6 = 1 + {}^6C_1x + {}^6C_2x^2 + {}^6C_3x^3 + {}^6C_4x^4 + {}^6C_5x^5 + x^6$

Put $x = 1$ in both sides

$(1 + 1)^6 = 1 + {}^6C_1 + {}^6C_2 + {}^6C_3 + ... + 1$

$2^6 = {}^6C_0 + {}^6C_1 + {}^6C_2 + {}^6C_3 + ... + {}^6C_6$

The general term of the expansion of a binomia expansion

In the expansion of $(x + y)^n = x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + ... + y^n$

Notice that $T_2 = {}^nC_1 x^{n-1} y^1$, $T_3 = {}^nC_2 x^{n-2} y^2$

Similarly: $T_9 = {}^nC_8 x^{n-8} y^8$

Let the general term be T_{r+1} where $0 \leqslant r \leqslant n$ then T_{r+1}. It can be written as:

$$T_{r+1} = {}^nC_r (x)^{n-r} (y)^r$$

Example

3. In the expansion of $(x + \frac{2}{x})^8$ **according to the descending power of x, find the coefficient of the sixth term**

Solution

$T_6 = {}^8C_5 (x)^3 (\frac{2}{x})^5 = {}^8C_5 \times 2^5 x^{-2} = 1792 x^{-2}$

The coefficient of this term = 1792

Notice the coefficient of $T_{r+1} = {}^nC_r$ (the coefficient of first term)$^{n-r}$ (coefficient of second term)^r

Try to solve

(2) In the expansion of $(2x + \frac{1}{2})^7$ according to the descending powers of x, find each of T_3, T_7, and if $T_3 = T_7$, find the value of x.

Example

(4) In the expansion of $(3x^2 - \frac{1}{2x})^{13}$ according to the descending power of x, find the tenth term from the end.

Solution

The tenth term from the end in the expansion of $(3x^2 - \frac{1}{2x})^{13}$ is the tenth term from the beginning of the expansion $(\frac{-1}{2x} + 3x^2)^{13}$

$$T_{10} = {}^{13}C_9(\frac{-1}{2x})^4 (3x^2)^9 = \frac{715 \times 3^9}{2^4} x^{14}$$

Another Solution

Notice that we can find the order of the tenth term from the end in the expansion $(3x^2 - \frac{-1}{2x})^{13}$, and its order equals $14 - 10 + 1 = 5$

T_{10} from the end is T_5 from the begining $T_5 = {}^{13}C_4(3x^2)^9 (\frac{-1}{2x})^4 = \frac{715 \times 3^9}{2^4} x^{14}$

Try to solve

(3) In the expansion of $(2x - \frac{1}{3x^2})^{11}$, find the fourth term from the end:

Rule

1) $(x + a)^n + (x - a)^n = 2 (T_1 + T_3 + T_5 + ...)$

2) $(x + a)^n - (x - a)^n = 2 (T_2 + T_4 + T_6 + ...)$, from expansion of $(x + a)^n$

Example

(5) Find in the simplest form $(x + 2)^6 + (x - 2)^6$

Solution

$$(x + 2)^6 + (x - 2)^6 = 2 (T_1 + T_3 + T_5 + T_7)$$
$$= 2 (x^6 + {}^6C_2\, x^4 \times 2^2 + {}^6C_4\, x^2 \times 2^4 + 2^6) = 2 (x^6 + 60x^4 + 240\, x^2 + 64)$$

Try to solve

(4) Find in the simplest form $(1 + \sqrt{x})^5 - (1 - \sqrt{x})^5$.

Example

(6) In the expansion $(3 + x)^{11} - {}^{11}C_1 (3 + x)^{10} (1 - 2x) + {}^{11}C_2 (3 + x)^9 (1 - 2x)^2 - \dots - (1 - 2x)^{11}$ find the fifth term, according to the ascending power of x .

Solution

The expression represents the expansion of $[(3 + x) - (1 - 2x)]^{11} = (2 + 3x)^{11}$

then:

$T_5 = {}^{11}C_4(2)^7 (3x)^4 = 330 \times 2^7 \times 3^4 x^4 = 3421440 x^4$

Try to solve

5) In the expansion $(1 - x)^8 + 24x(1 - x)^7 + 252x^2(1 - x)^6 + \ldots + 6561x^8$, find the numerical value of the sixth term when x = 2

Example

7) If $(1 + cx)^n = 1 + 20x + a_1x^2 + a_2x^3 + \ldots + a_{n-1}x^n$ and $16a_1 = 3a_2$, find the value of n, c where $c \neq 0$

Solution

$(1 + cx)^n = 1 + {}^nC_1 cx + {}^nC_2 c^2 x^2 + {}^nC_3 c^3 x^3 + \ldots$

$\therefore {}^nC_1 c = 20$ $\quad \therefore nc = 20$ $\quad \therefore c = \frac{20}{n}$ (1)

$\because 16 \times {}^nC_2 c^2 = 3 \times {}^nC_3 c^3$ $\quad \therefore 16 \times {}^nC_2 = 3 \times {}^nC_3 c$ (2)

Substituting from (1) in (2) $\quad \therefore 16 \times {}^nC_2 = 3 \times {}^nC_3 \times \frac{20}{n}$

$\therefore 16n = 3 \times \frac{{}^nC_3}{{}^nC_2} \times 20$ $\quad \therefore 16n = 3 \times \frac{n-2}{3} \times 20$ $\quad \therefore n = 10$

Substituting equation (1) $\quad \therefore c = \frac{20}{10} = 2$

Try to solve

6) In the expansion of $(1 + Kx)^{10}$, if the coefficient of the third term equals 180, and the coefficient of the fifth term equals 210, find the values of K and x where K is a positive integer.

Example

8) a) Prove that $= \frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n}{r}$

b) If the ratio between T_6 in the expansion of $(x + \frac{1}{x})^{15}$ and T_5 in the expansion of $(x - \frac{1}{x^2})^{14}$ equals $\frac{8}{9}$, find the value of x

Solution

$${}^nC_r \div {}^{n-1}C_{r-1} = \frac{\lfloor n}{\lfloor r \lfloor n-r} \times \frac{\lfloor r-1 \lfloor n-1-r+1}{\lfloor n-1} = \frac{n\lfloor n-1 \lfloor r-1}{r\lfloor r-1 \lfloor n-1} = \frac{n}{r}$$

$$= \frac{T_6 \text{ of } (x + \frac{1}{x})^{15}}{T_5 \text{ of } (x - \frac{1}{x^2})^{14}} = \frac{{}^{15}C_5 x^{10} (\frac{1}{x})^5}{{}^{14}C_4 x^{10} (\frac{-1}{x^2})^4} = \frac{8}{9}$$

$\therefore 3x^3 = \frac{8}{9}$ $\quad \therefore x^3 = \frac{8}{27}$ $\quad \therefore x = \frac{2}{3}$

The middle term in the expansion of $(x + a)^n$

In the expansion of $(x + a)^n$, we find that the number of terms of the expansion = $n + 1$

First: If n is an even number , then the number of terms of the expansion is an odd number and the expansion has one middle term of order $\frac{n+2}{2} = (\frac{n}{2} + 1)$

Second: If n is odd number , then the number of terms of the expansion is an even number, and the expansion has two middle terms of order $\frac{n+1}{2}$, $\frac{n+3}{2}$

Example

9) Find the middle term in the expansion of $(2x + \frac{1}{2x^2})^{12}$

Solution

The order of middle term = $\frac{12}{2} + 1 = 7$

$$T_7 = {}^{12}C_6\,(2x)^6\left(\frac{1}{2x^2}\right)^6 = {}^{12}C_6\,(2)^6\,(\tfrac{1}{2})^6\,x^{6-12} = {}^{12}C_6\,x^{-6}$$

Try to solve

7) Find the middle term in the expansion of $(x^2 + \frac{1}{2x})^{10}$. If the value of this term = $\frac{28}{27}$, find the value of x

Example

10) Find the two middle terms of the expansion of $(\frac{x^2}{3} + \frac{3}{x})^{15}$

Solution

The orders of the two middle terms are $\frac{15+1}{2}$, $\frac{15+3}{2}$ then the middle terms: T_8 , T_9

$$T_8 = {}^{15}C_7\left(\frac{x^2}{3}\right)^8\left(\frac{3}{x}\right)^7 = {}^{15}C_7 \times 3^{-8+7} \times x^{16-7} = {}^{15}C_7 \times \frac{1}{3}x^9 = 2145x^9$$

$$T_9 = {}^{15}C_8\left(\frac{x^2}{3}\right)^7\left(\frac{3}{x}\right)^8 = {}^{15}C_8 \times 3^{8-7} \times x^{14-8} = {}^{15}C_8 \times 3\,x^6 = 19305x^6$$

Try to solve

8) If the two middle terms of the expansion of $(3x + 2y)^{13}$ are equal, prove that $\frac{x}{y} = \frac{2}{3}$

Example

(11) Find the middle term in the expansion of $(3 + 2x)^8 + (3 - 2x)^8$

Solution

The expansion $= 2\,(T_1 + T_3 + T_5 + T_7 + T_9)$

$\therefore$ The middle term $= 2\,T_5$

$T_5 = {}^8C_4\,(3)^4\,(2x)^4 + {}^8C_4(3)^4\,(-2x)^4$

$= 2 \times {}^8C_4\,(3)^4\,(2x)^4 = 181440\,x^4$

Try to solve

(9) Find the middle term in the expansion of $\left(2\sqrt{x} + \frac{1}{2\sqrt{x}}\right)^{10} + \left(2\sqrt{x} - \frac{1}{2\sqrt{x}}\right)^{10}$

Exercises 1 - 1

Choose the correct answer:

(1) If the orders of the two middle terms of the expansion of $(x + y)^n$ are 7 and 8, then n equals:

a) 13 b) 15 c) 16 d) 56

(2) If $1 + 5x + \frac{5 \times 4}{2 \times 1}x^2 + \frac{5 \times 4 \times 3}{3 \times 2 \times 1}x^3 + \dots + x^5 = 1024$, then the value of x equals:

a) 1 b) 2 c) 10 d) 3

(3) Sum of the coefficients of terms of the expansion of $(x^2 - \frac{1}{x})^7$ equals:

a) 2^7 b) 2^5 c) 2^6 d) zero

(4) The coefficient of the fifth term in the expansion of $(1 + 2x)^{10}$ is

a) $16\ {}^{10}C_5$ b) $\frac{1}{16}\ {}^{10}C_5$ c) $16\ {}^{10}C_4$ d) $\frac{1}{16}\ {}^{10}C_4$

(5) In the expansion of a binomial theorem, if the general term is ${}^{12}C_r\,x^{24-4r}$, then the term containing x^{12} is:

a) T_3 b) T_4 c) T_5 d) Does not exist

(6) If the two middle terms of the expansion of $(a + 2b)^{2n+1}$ are equal, then:

a) $\frac{a}{b} = \frac{1}{2}$ b) $a = 4b$ c) $a = 8b$ d) $a = 2b$

(7) If the middle term of the expansion of $(\frac{2a}{3} + \frac{b}{a^2})^{8n}$ is the ninth term, then n equals:

a) 1 b) 2 c) 3 d) 4

8) In the expansion of $(1 + bx)^9$, the coefficient of the sixth term is:

a) ${}^9C_5$ b) ${}^9C_6$ c) ${}^9C_5 b^5$ d) ${}^9C_6 b^6$

9) In the expansion of a binomial , we have 7 positive terms and 6 negative terms , then the term is in the form of:

a) $(a - b)^{12}$ b) $(a - b)^{13}$ c) $(a + b)^{12}$ d) $(a - b)^{13}$

Second : Answer the following:

10) If $1 + 8x + {}^8C_2 x^2 + \ldots + x^8 = 256$, find the value of x

11) Find the value of x which satisfies $(1 + \sqrt{3})^6 - (1 - \sqrt{3})^6 = 480\sqrt{3}\,x$

12) Use the expansion of : $(1 + x)^{10} = 1 + {}^{10}C_1 x + {}^{10}C_2 x^2 + \ldots + x^{10}$ to prove that:

a) $1 + {}^{10}C_1 + {}^{10}C_2 + \ldots + {}^{10}C_{10} = 2^{10}$ b) $1 - {}^{10}C_1 + {}^{10}C_2 - \ldots + {}^{10}C_{10} = 0$

13) Write the expansion for each of :

a) $(\frac{2}{x} + \frac{x}{2})^4$ b) $(x - \frac{1}{x})5$

c) $(x + \sqrt{2})^4 + (x - \sqrt{2})^4$ d) $(\sqrt{3} + 2x)^5 - (\sqrt{3} - 2x)^5$

14) In the expansion of $(1 + x)^n$ according to the ascending powers of x, if $T_3 = 28x^2$, $T_5 = 1120$, find the value of n and x

15) In the expansion of $(1 + x)^n$, if the coefficient of the sixth term equals the coefficient of the tenth term, find the value of n.

16) In the expansion of $(ax + b)^{10}$ according to the descending powers of x, if the coefficient of $T_6 = \frac{63}{8}$, prove that $2ab = 1$

17) In the expansion of $(2x^2 + \frac{1}{2x})^{12}$, find the middle term.

18) In the expansion of $(\frac{x^2}{2} - \frac{2}{x})^{11}$,find the two middle terms.

19) In the expansion of $x^4 (x - \frac{1}{x})^9$ according to the descending powers of x, find the fourth term from the end.

20) If the middle term of the expansion of $(x^2 + \frac{1}{2x})^{10}$ equals $\frac{28}{27}$, find the value of x.

21) Find the ratio between the middle term and the fifth term in the expansion of $(\frac{2x}{3} + \frac{3}{2x})^{10}$, then find the numerical value of the ratio when $x = 3$

22) If the ratio between the fifth term in the expansion of $(x + \frac{1}{x})^{15}$ and the fourth term in the expansion of $(x - \frac{1}{x^2})^{14}$ equals -16 : 15, find the value of x.

Unit One

1 - 2 Finding the term containing x^k in the expansion of binomial

You will learn

- *Using the general term to find the term containing x^k and term free of x.*
- *Finding the coefficient of the term containing x^k.*
- *Finding the highest coefficient of the expansion.*

Key terms

- *General term*
- *Term free of x*
- *Heightest power*
- *Coefficient*

Materials

- *Scientific calculator*

Think and discuss

We studied in the previous lesson that :

$$(x^2 - \frac{1}{2x})^{20} = (x^2)^{20} - {}^{20}C_1(x^2)^{19}(\frac{1}{2x}) + {}^{20}C_2(x^2)^{18}(\frac{1}{2x})^2 - {}^{20}C_3(x^2)^{17}(\frac{1}{2x})^3 + \dots + (\frac{-1}{2x})^{20}$$

Is it easy to find the term containing x^{16} or x^{24} or term free of x without writing the terms of the expansion?

We notice that it is a difficult method to find the term containing x^k by writing all the terms of the expansion , so we follow the next :

1- Suppose that this term is the general term T_{r+1} , we find this term in terms of r.

2- Find the sum of the powers of x in the general term in terms of r and place this sum to be equal to the required power k then find r which satisfies the inclusion of this term on the required power k and we have

- **a** If $r \in N$, then $r + 1$ is the required term.
- **b** If $r \notin N$ then there is no term containing the required power in this expansion.

To find the term free of x, we put the sum of powers of x in the general term = 0

Example

1 In the expansion of $(\frac{3x}{2} + \frac{2}{3x})^{11}$, find the coefficient of x in this expansion.

Solution

$$T_{r+1} = {}^{11}C_r (\frac{3x}{2})^{11-r} (\frac{2}{3x})^r$$

Compairing the powers $x^{11-r-r} = x^1$

$$11 - 2r = 1 \qquad r = 5$$

Required term is the sixth term coefficient of $T_6 = {}^{11}C_5 (\frac{3}{2})^6 (\frac{2}{3})^5 = 693$

Try to solve

1. Find the coefficient of x^8 in the expansion of $(\frac{2x}{3} - \frac{3}{x})^{12}$

Example

2. In the expansion of $(2x - \frac{1}{2x^2})^9$, find :

   a. The coefficient of x^3
   b. The term free of x
   c. Prove that the term does not contain x^2

Solution

$T_{r+1} = {}^9C_r\,(2x)^{9-r}\left(\frac{-1}{2x^2}\right)^r = {}^9C_r\,(-1)^r\,(2)^{9-2r}\,x^{9-3r}$

Comparing the powers

a. To find the coefficient of x^3

$x^{9-3r} = x^3$ $\quad 9 - 3r = 3$ $\quad r = 2$

the third term contains x^3

the coefficient of $T_3 = {}^9C_2 \times 2^5 = 36 \times 32 = 1152$

b. To find the term free of x $\quad 9 - 3r = 0$ $\quad r = 3$

Required term is $T_4 = {}^9C_3\,(-1)^3\,(2)^3 = -672$

c. Put $9 - 3r = 2$ $\quad \therefore 7 = 3r$ $\quad \therefore r = \frac{7}{3} \notin N$

$\therefore$ This expansion does not contain x^2

Try to solve

2. a. Find the term free of x in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{12}$

   b. Find the coefficient of x^{-10} in the expansion of $(\frac{x^2}{3} - \frac{2}{x^2})^{15}$

   c. In the expansion of $\left(ax + \frac{1}{bx}\right)^{10}$ according to the descending power of x , if the term free of x equals the coefficient of the seventh term, Prove that $6\,ab = 5$

Example

3. If n is a positive integer, prove that there is no term free of x in the expansion of $\left(x^5 + \frac{1}{x^2}\right)^n$ except if n is a multiple of 7 , then find this term in the case of $n = 7$

Solution

$T_{r+1} = {}^{n}C_r (x^5)^{n-r} \left(\frac{1}{x^2}\right)^r = {}^{n}C_r\ x^{5n-7r}$

$x^{5n-7r} = x^0 \qquad 5n - 7r = 0 \qquad r = \frac{5n}{7}$

$\frac{5n}{7} \in \mathbb{N}$ when n is a multiple of 7 $\qquad$ when $n = 7$

$r = 5 \qquad$ the term free of x is T_6

$T_6 = {}^{7}C_5 = 21$

Try to solve

3) In the expansion of $\left(x^2 + \frac{1}{x}\right)^{3n}$ find :

a) The coefficient of the term which contains x^{3n}

b) If $n = 6$, find the ratio between the coefficient of the term containing x^{3n} and the coefficient of the middle term

Example

4) In the expansion of $\left(2 + \frac{x}{3}\right)^9$, find the value of x which makes the two middle terms equal.

Solution

The order of the two middle terms $\frac{9+1}{2}$ and the consecutive i.e. T_5 , T_6

$\because T_5 = T_6 \qquad \therefore {}^{9}C_4 (2)^5 \left(\frac{x}{3}\right)^4 = {}^{9}C_5 (2)^4 \left(\frac{x}{3}\right)^5$

$2 = \frac{x}{3} \qquad \therefore x = 6$

Exercises 1 - 2

Choose the correct answer:

1) The term containing x^4 in the expansion of $(1 + 2x)^{10}$ equals:

a) ${}^{15}C_4$ b) $\frac{1}{16}\ {}^{10}C_4$ c) $16\ {}^{10}C_4$ d) $32\ {}^{10}C_5$

2) In the expansion of $\left(x + \frac{1}{x}\right)^{10}$, the term free of x is:

a) T_4 b) T_5

c) T_6 d) not found

3) In the expansion of $x^3 (1 + x)^7$, the coefficient of the term containing x^4 is:

a) ${}^{7}C_4$ b) ${}^{7}C_3$ c) ${}^{7}C_1$ d) 21

4. In the expansion of $(x^2 + \frac{2}{x})^6$, the term free of x is

   a) Third. b) Fourth.
   c) Fifth. d) not found

5. In the expansion of $(ax^2 + \frac{1}{ax})^{11}$, if the coefficients of x^4 and x^7 are equal , then a =

   a) 1 b) -1 c) ± 1 d) ± 2

6. If the term free of x in the expansion of $(x + \frac{1}{x})^n$ is T_7 , then n = ...

   a) 6 b) 10 c) 12 d) 8

7. In the expansion of $(x^2 + \frac{1}{ax})^8$, if the coefficient of the middle term equals the coefficient of x^7 , then a = ...

   a) $\frac{4}{5}$ b) $\frac{-4}{5}$ c) $\frac{-5}{4}$ d) $\frac{5}{4}$

8. In the expansion of $(ax + \frac{1}{bx})^{10}$ according to the descending powers of x, if the term free of x equals the coefficient of the seventh term, then:

   a) $ab = \frac{6}{5}$ b) $ab = \frac{5}{6}$ c) $ab = \frac{36}{25}$ d) $ab = \frac{25}{36}$

9. The term free of x in the expansion of $(2x + \frac{1}{2x})^8$

   a) 35 b) 140 c) 70 d) 56

10. In the expansion of $(1 + ax)^7$ according to the ascending powers of x, if the coefficient of $T_5 = 560$, then a = ...

    a) 2 b) 4 c) ± 2 d) ± 4

Answer the following questions :

11. In the expansion of $(4x^2 + \frac{1}{2x})^{12}$, find the term free of x

12. Find the coefficient of x^{12} in the expansion of $x^2 (\frac{x^2}{2} + \frac{2}{x^3})^{15}$

13. If the sixth term of the expansion $(2x - \frac{1}{x^3})^n$ according to the descending powers of x is free of x, find the value of n. Then investigate if any of these terms of this expansion contains x^{-6} or not?

14. In the expansion of $(2x - \frac{1}{x^2})^9$:

    first: find the coefficient of x^3 **second :** find the term free of x

    third: Prove that this expansion does not contain a term containing x^2

(15) Prove that $^{n}C_r : {}^{n-1}C_{r-1} = \frac{n}{r}$ and if the ratio between the coefficient of T_{11} in the expansion of $(1 + x^2)^n$ and the coefficient of T_{10} in the expansion of $(1 - y)^{n-1}$ equals -3 : 2, find the value of n.

(16) Find the coefficient of $(\frac{x}{y})^4$ in the expansion of $(\frac{2x}{y} + \frac{y}{2x})^{10}$

(17) Find the coefficient of x^n in the expansion of $(1 + x)^{2n}$, then prove that it is equal to twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$

(18) In the expansion of $(x + \frac{1}{x})^{2n}$, prove that the term free of x is the middle term, then find the value of this term when $n = 8$

(19) In the expansion of $(x^k + \frac{1}{x})^6$ where k is a positive integer , find :

First: The value of k which makes the expansion have a term free of x

Second: The ratio between the term free of x and the coefficient of the middle term at the highest value of k you have got from First.

(20) In the expansion of $(x^2 + \frac{1}{ax})^{12}$, if the ratio between the term free of x and the coefficient of x^3 in this expansion equals 5 : 16 , find the value of a , then find the value of the middle term when $x = 2$.

(21) In the expansion of $(2x^2 + \frac{a}{x^3})^{10}$, if the coefficient of x^5 equals the coefficient of x^{15} , find the value of a.

(22) In the expansion of $(x^2 + \frac{1}{8x})^{13}$ according to the descending power of x :

First: Prove that there is no term free of x **Second:** If $T_4 = T_{11}$, find the value of x

(23) In the expansion of $(x + \frac{1}{x^2})^9$ find:

First: The order and the value of the term free of x

Second: The value of x which makes the sum of the two middle terms in the expansion equal to zero.

(24) Find the value of the term free of x in the expansion of $(9x^2 + \frac{1}{3x})^9$, then find the value of x which makes the two middle terms equal.

(25) In the expansion of $(x^2 + \frac{1}{x})^{3n}$, prove that the term free of x equals the coefficient of the term containing x^{3n} , if n = 6, find the ratio between the term free of x and the coefficient of the middle term.

Unit One

1 - 3

Ratio between two consecutive terms in a binomial expansion

In the expansion of $(x + a)^n$ and the two consecutive terms are T_{r+1}, T_r

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r\,(x)^{n-r}\,(a)^r}{{}^nC_{r-1}\,(x)^{n-r+1}(a)^{r-1}}$$

$$= \frac{{}^nC_r}{{}^nC_{r-1}} \times \frac{a}{x} = \frac{\lfloor n}{\lfloor r\ \lfloor n-r} \times \frac{\lfloor r-1\ \lfloor n-r+1}{\lfloor n} \times \frac{a}{x}$$

$$= \frac{\lfloor r-1\ (n-r+1)\lfloor n-r}{r\lfloor r-1\ \lfloor n-r} \times \frac{a}{x}$$

$$\boxed{\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \times \frac{a}{x}}$$

and : $\dfrac{\text{coefficient of } T_{r+1}}{\text{coefficient of } T_r} = \dfrac{n-r+1}{r} \times \dfrac{\text{coefficient of } a}{\text{coefficient of } x}$

Example

1. In the expansion of $(3 + 2y)^{12}$ find each of :

a) $\dfrac{T_3}{T_2}$ b) $\dfrac{\text{coefficient of } T_7}{\text{coefficient of } T_8}$

c) $\dfrac{T_6}{T_4}$ d) $\dfrac{\text{coefficient of } T_8}{\text{coefficient of } T_6}$

Solution

a) $\dfrac{T_3}{T_2} = \dfrac{12-2+1}{2} \times (\dfrac{2y}{x}) = \dfrac{11}{2} \times \dfrac{2y}{x} = \dfrac{11y}{x}$

b) $\dfrac{\text{coefficient of } T_7}{\text{coefficient of } T_8} = \dfrac{7}{12-7+1} \times \dfrac{1}{2} = \dfrac{7}{12}$

c) $\dfrac{T_6}{T_4} = \dfrac{T_6}{T_5} \times \dfrac{T_5}{T_4}$

$= \dfrac{12-5+1}{5} \times (\dfrac{2y}{x}) \times \dfrac{12-4+1}{4}(\dfrac{2y}{x})$

$= \dfrac{8}{5} \times \dfrac{2y}{x} \times \dfrac{9}{4} \times \dfrac{2y}{x} = \dfrac{72\,y^2}{5\,x^2}$

d) $\dfrac{\text{coefficient of } T_8}{\text{coefficient of } T_6} = \dfrac{\text{coefficient of } T_8}{\text{coefficient of } T_7} \times \dfrac{\text{coefficient of } T_7}{\text{coefficient of } T_6}$

$= \dfrac{12-7+1}{7} \times \dfrac{2}{1} \times \dfrac{12-6+1}{6} \times \dfrac{2}{1}$

$= \dfrac{6}{7} \times \dfrac{2}{1} \times \dfrac{7}{6} \times \dfrac{2}{1} = 4$

You will learn

- *Find the ratio between two consecutive terms.*
- *Find the ratio between the two coefficients of two consecutive terms*

Key terms

- *Consecutive terms*

Materials

- *Scientific calculator*

Try to solve

1) In the expansion of $(x^2 + \frac{2}{x})^8$

First: Find the ratio between the fifth and the sixth terms. If this ratio equals 8 : 25, find the value of x

Second: Prove that this expansion does not have a term free of x

Example

2) In the expansion of $(x + y)^8$ if $2T_5 = T_4 + T_6$, find $\frac{x}{y}$ numerically.

Solution

$T_4 + T_6 = 2T_5$ by dividing by T_5 $\qquad \frac{T_4}{T_5} + \frac{T_6}{T_5} = 2$

$\frac{4}{8-4+1}(\frac{x}{y}) + \frac{8-5+1}{5}(\frac{y}{x}) = 2$ $\qquad \frac{4x}{5y} + \frac{4y}{5x} = \frac{2}{1}$ **by multiplying by** $5xy$

$4x^2 + 4y^2 = 10xy$ $\qquad 4x^2 - 10xy + 4y^2 = 0 \div 2$

$2x^2 - 5xy + 2y^2 = 0$ $\qquad (2x - y)(x - 2y) = 0$

$2x = y$ $\qquad x = 2y$

$\frac{x}{y} = \frac{1}{2}$ or $\frac{x}{y} = \frac{2}{1}$

Try to solve

2) In the expansion of $(\sqrt{x} + \frac{1}{x})^8$, If $T_4, T_5, 25T_7$ and T_6 are proportional, find the value of x.

Example

3) If the coefficients of three consecutive terms of the expansion of $(1 + x)^n$ are 35, 21, 7 according to the ascending power of x, find the value for each of n and the orders of these three terms.

Solution

Let the terms T_r, T_{r+1}, T_{r+2}

$\frac{\text{coefficient of } T_{r+1}}{\text{coefficient of } T_r} = \frac{n-r+1}{r} = \frac{21}{35}$ $\qquad \frac{n-r+1}{r} = \frac{3}{5}$

$5n - 5r + 5 = 3r$ $\qquad \boxed{5n - 8r = -5}$ $\qquad (1)$

$\frac{\text{coefficient of } T_{r+2}}{\text{coefficient of } T_{r+1}} = \frac{n-(r+1)+1}{r+1} = \frac{7}{21}$ $\qquad \frac{n-r}{r+1} = \frac{1}{3}$

$3n - 3r = r + 1$ $\qquad \boxed{3n - 4r = 1}$ $\qquad (2)$

By solving the two equations: (1) and (2) $\quad \therefore n = 7, r = 5$

Try to solve

3) If the third, fourth and fifth terms of the expansion $(x + y)^n$ are 112, 448, 1120 respectively, find the values of each of n, y, x

Example **Finding the greatest term**

(4) Find the greatest term in the expansion of $(x + y)^{10}$ when $x = 2$, $y = 3$

Solution

$\because \dfrac{T_{r+1}}{T_r} = \dfrac{10 + 1 - r}{r} \times \dfrac{3}{2}$ $\therefore \dfrac{T_{r+1}}{T_r} = \dfrac{11 - r}{r} \times \dfrac{3}{2} = \dfrac{33 - 3r}{2r}$

First: $\dfrac{33 - 3r}{2r} \geqslant 1$ $\therefore 33 - 3r \geqslant 2r$ $\therefore 5r \leqslant 33$ $\therefore r \leqslant 6.6$

From this, we deduce that $T_7 > T_6 > T_5 > ..... > T_1$

Second: $\dfrac{33 - 3r}{2r} \leqslant 1$ $\therefore 33 - 3r \leqslant 2r$ $\therefore 5r \geqslant 33$ $\therefore r \geqslant 6,6$

From this, we deduce that $T_{11} < T_{10} < T_9 < T_8 < T_7$

$\therefore T_7$ is the greatest term in the expansion of $(x + y)^{10}$

$\therefore T_7 = {}^{10}C_6 \times 2^4 \times 3^6$ $\therefore T_7 = 2449440$

Exercises 1 - 3

Choose the correct answer from the given answers:

1) In the expansion of $(x + y)^{10}$, the ninth term : the eighth term equals ....

a) $\frac{3y}{8x}$

b) $\frac{3x}{8y}$

c) $\frac{8y}{3x}$

d) $\frac{8x}{3y}$

2) In the expansion of $(1 - x)^{12}$ the coefficient of sixth term : the coefficient of fifth term ....

a) $\frac{8}{5}$

b) $\frac{5}{8}$

c) $\frac{-8}{5}$

d) $\frac{-5}{8}$

3) In the expansion of $(x + y)^{8}$, then the ratio $\frac{T_6}{T_4} =$ ....

a) $\frac{25y^2}{16x^2}$

b) $\frac{25x^2}{16y^2}$

c) 1

d) $\frac{y^2}{x^2}$

4) In the expansion of $(3a - 2b)^{11}$, if the ratio between the two middle terms respectively equals $\frac{-3}{2}$, then a : b = ...

a) 9 : 4

b) 4 : 9

c) 1

d) -1

Second : Answer the following questions:

5) In the expansion of $(2 x^2 + \frac{3}{x^2})^{11}$, find each of:

a) $\frac{T_3}{T_2}$ b) $\frac{T_4}{T_5}$

c) $\frac{T_6}{T_8}$ d) $\frac{\text{coefficient of } T_4}{\text{coefficient of } T_6}$

6) In the expansion of $(1 + x)^{12}$, if $T_3 = 2T_2$, find the value of x ....

7) In the expansion of $(a + b)^n$ if $T_2 = 240$, $T_3 = 720$, $T_4 = 1080$, find the value of each of a, b, n

8) If $T_2 : T_3$ in the expansion of $(a + b)^n$ equals the ratio $T_3 : T_4$ in the expansion of $(a + b)^{n+3}$, find the value of n

9) In the expansion of $(1 + mx)^n$, if $4T_6 = 7T_8$, $\frac{T_4}{T_6} = \frac{1}{4}$ when $x = 1$, find the value of m, n

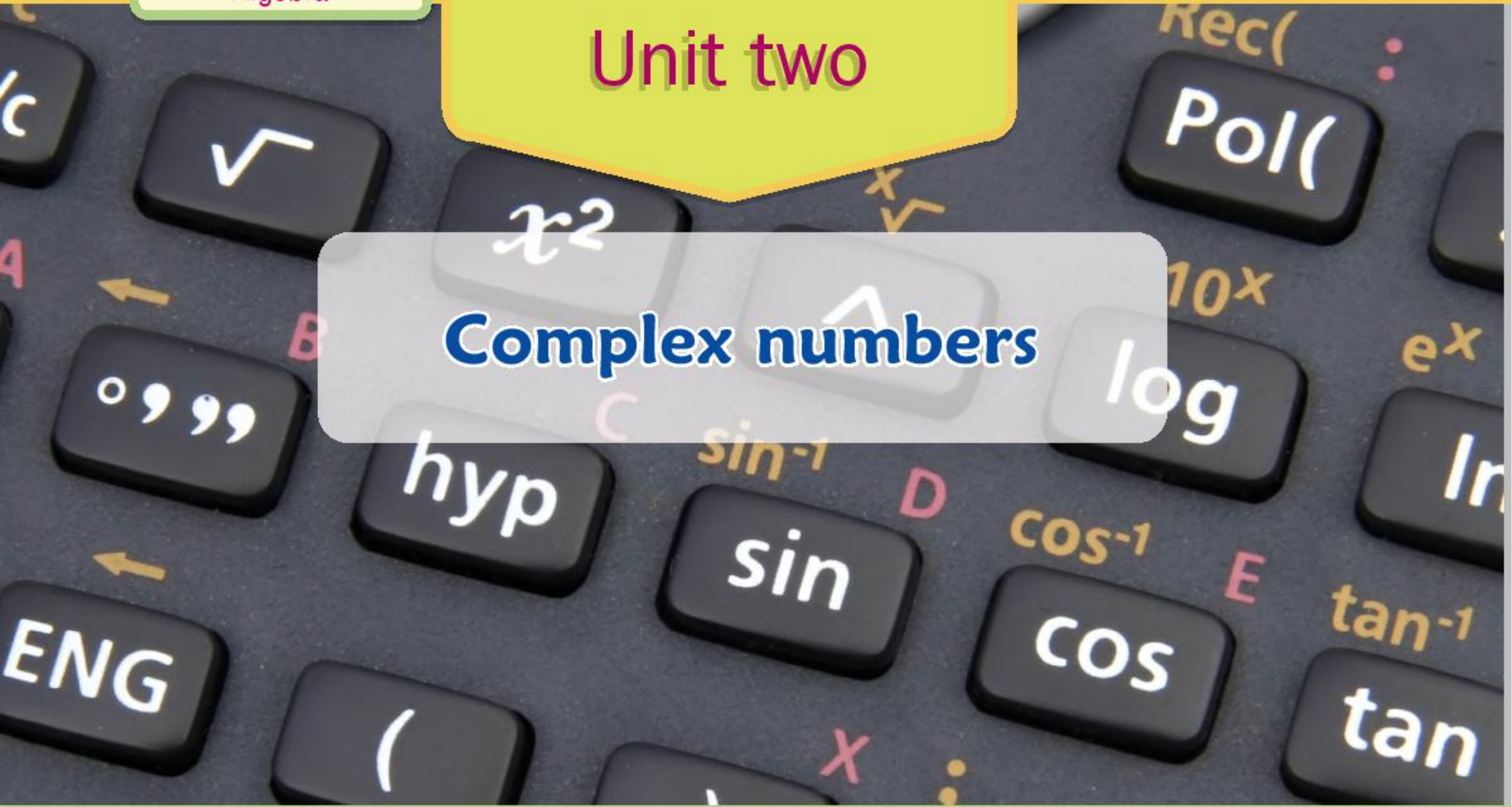
10) Find numerically the value of the greatest term in the expansion of $(3 - 5x)^{15}$ when $x = \frac{1}{5}$

11) In the expansion of $(x + y)^n$ according to the descending power of x, if the second term is the arithmetic mean between the first and the third terms when $x = 2y$, find the value of n.

Algebra

Unit two

Complex numbers



Unit introduction

Jean-Robert Argand is one of the most popular mathematicians. He had been the first to study the complex numbers in details and had used them to prove that all the algebraic equations have roots whether these roots are true or imaginary. The complex numbers are represented by the Argand's Diagram to honor the French Scientist Argand either by point (x, y) where x is a real number on x-axis and y represents the imaginary number on y-axis or vector of amagintude $\sqrt{x^2+y^2}$ and direction $\tan^{-1}\frac{y}{x}$. in this unit, you are going to identify the cubic roots of unity and solve applications on the complex numbers such as electricity, dynamics, theorem of relativity and the physical different fields. These numbers are flexiable to help to get a final result satisfactorily

Unit objectives

BY THE END OF THIS UNIT AND DOING ALL THE ACTIVITIES INCLUDED, THE STUDENT SHOULD BE ABLE TO:

- Represent the complex number and its conjugate graphically by points (ordered pairs) in the cartesian plane.
- Determine the modulus and the amplitude of the complex number.
- Identify the principle amplitude of a complex number
- Identify the trigonometric from of a complex number
- Identify De Moivre's theorem and its applications
- Deduce the n^{th} roots of any complex number
- Express sin ni , cos ni interms of the powers of sini , cosi
- Identify the expansion of sini , cosi, as series
- Deduce the Eular's rule from the series
- Identify and apply the methods to convert the different forms of the complex number.
- Identify the cubic roots of unity.
- Identify the modulus and the amplitude of the product and quotient of two complex numbers.
- Perform the basic operations on the complex numbers in the trigonometric form
- Solve Application problems on the cubic roots of unity
- Use the complex numbers in solving Mathematical problems.
- Use some computer programs in solving mathematical problems including complex numbers
- Deduce the properties of addition and multiplication operations on the complex numbers.
- Deduce the properties of two conjugate numbers
- Deduce the properties of the cubic roots of unity.

Key terms

- Argand plane
- conjugate
- Modulus
- principle amplitude
- Trigonmetric
- De Moivre's thearem
- root
- square root
- cubic root
- Unit circle
- Polar

Materials

- Scientific calculator

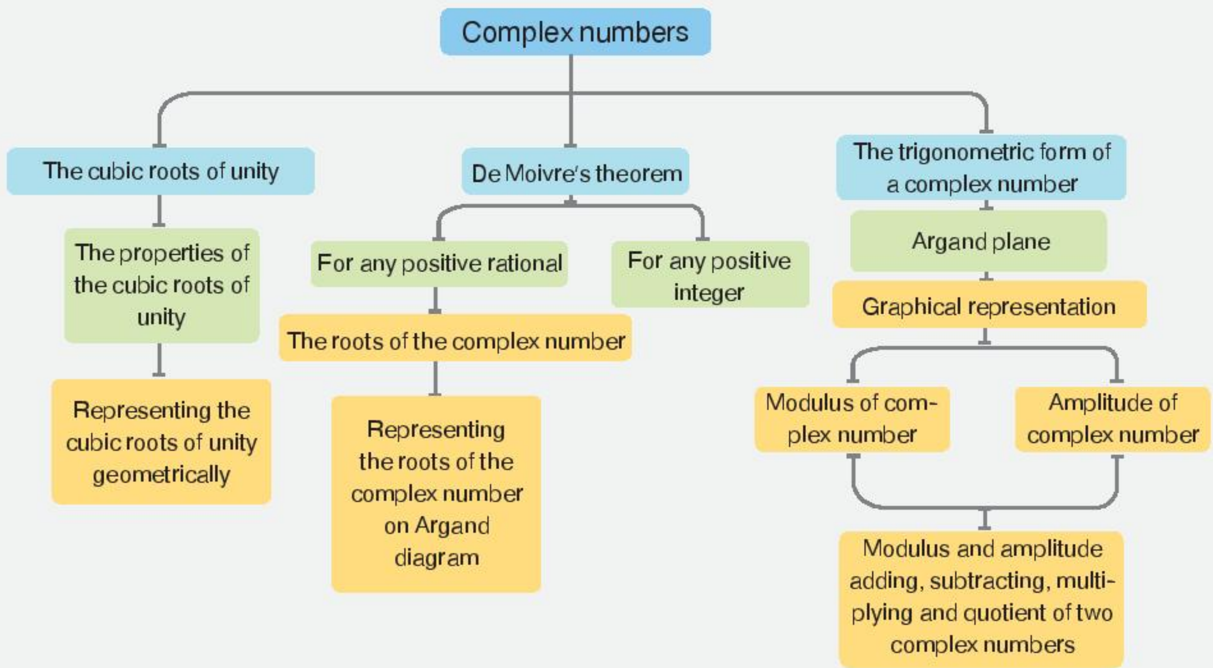
Unit Lessons

Lesson (2-1): The trigonometric form of a complex number

Lesson (2-2): De Moivre's theorem

Lesson (2-3): The cubic roots of unity

Unit chart



Unit two

2 - 1 Trigonometric form of a complex number

You will learn

- The graphical representation of the complex number and its conjugate in Argand's plane
- Graphical representation of the sum of two complex numbers
- Modulus of the complex number
- Amplitude of the complex number
- Principle amplitude of the complex number
- The trigonometric form of the complex number
- The modulus and the amplitude of the product and quotient of two complex numbers.

Key terms

- Argand's plane
- Conjugate
- Modulus
- Principle amplitude
- Trigonmetric form

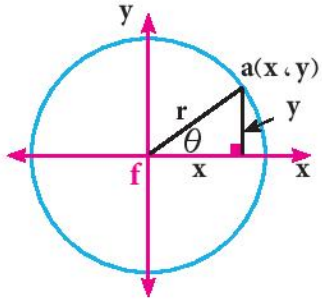
Materials

- Scientific calculator

You have studied the complex numbers and known that a complex number can be written in the form $z = x + y\,i$ **(Algebraic form),** where x , y are real numbers, $i^2 = -1$. In this lesson, we will identify another form to write the complex number and how to represent it graphically .

Polar and cartesian coordinates:

The opposite figure represents a circle with radius length r, A (x , y) lies on the circle and opposite to an angle of measure θ .



$\cos\theta = \frac{x}{r}$ $\sin\theta = \frac{y}{r}$

$x = r\cos\theta$ $y = r\sin\theta$

where $r = \sqrt{x^2 + y^2}$, $\tan\theta = \frac{y}{x}$

i.e. : $\theta = \tan^{-1}\frac{y}{x}$ if we meditate the cartesian plane as a polar plane where the polar axis coincides on the positive part of x-axis, then we can change polar coordinates to Cartesian and vise versa.

Converting polar co-ordinates to cartisian co-ordinates

If the point A in polar coordinates is A(r, θ), then the Cartesian coordinate of A is (x, y) where :

$x = r\cos\theta$, $y = r\sin\theta$

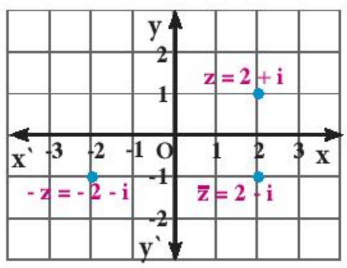
Then: $(x, y) = (r\cos\theta , r\sin\theta)$

Argand's plane

The mathematician **Argand** had represented the complex number z graphically on the orthogonal Cartesian coordinates such that the axis $\overleftrightarrow{XX`}$ represents the real part of the complex number and the axis $\overleftrightarrow{yy`}$ represents the imaginary part of the complex number. so the point (x , y)represents the complex number x + i y

Example

1 In the opposite Argand's diagram, we notice that the two points representing z , - z are symmetric about the point of origin (O).



We also notice that the two points representing the two conjugate number z, $\bar{z}$ are symmetric about $\overleftrightarrow{xx^-}$ axis

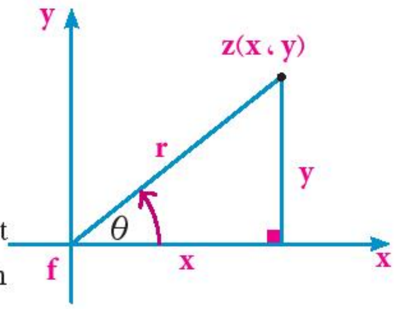
Try to solve

1. Represent on Argend's diagram each of the numbers:
$z = 3 + 4i$, $\bar{z}$, $-z$, $1 + z$

Critical thinking: What do all complex number with real part equals 2 represent on Argand's diagram?

Learn

The modulus and the amplitude (argument) of a complex number



If $z = x + y\,i$ is a complex number represented by the point z(x , y) in Argand's plane. Then the modulus is its distance from the origin and denoted by $|z|$ or r and θ is called amplitude of the complex number:

$r = \sqrt{x^2 + y^2}$, $\tan\theta = \frac{y}{x}$ thus $\theta = \tan^{-1}\frac{y}{x}$

Where $\frac{-\pi}{2} < \tan^{-1}(\frac{y}{x}) < \frac{\pi}{2}$

The trigonometric (polar) form of a complex number

If $z = x + yi$ is a complex number with modulus r and principle amplitude θ where $\theta \in]-\pi , \pi]$, then it is written as $z = r(\cos\theta + i\sin\theta)$ and the measure of θ is determined according to the following:

a) $x > 0 , y > 0$ θ lies in the first quadrant $\theta = \tan^{-1}(\frac{y}{x})$

b) $x < 0 , y > 0$ θ lies in the second quadrant $\theta = \pi + \tan^{-1}(\frac{y}{x})$

c) $x < 0 , y < 0$ θ lies in the third quadrant $\theta = -\pi + \tan^{-1}(\frac{y}{x})$

d) $x > 0 , y < 0$ θ lies in the fourth quadrant $\theta = \tan^{-1}(\frac{y}{x})$

Notic that

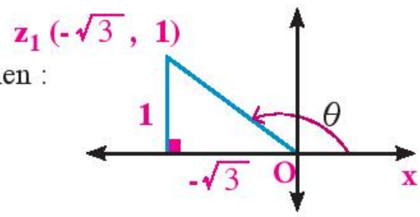
$x > 0, y = 0$, then $\theta = 0$
$x < 0, y = 0$, then $\theta = \pi$
$x = 0 , y > 0$, then $\theta = \frac{\pi}{2}$
$x = 0 , y < 0$ then $\theta = \frac{-\pi}{2}$

Example

2. Find the modulus and the principle amplitude of each of the following complex numbers:

a) $z_1 = -\sqrt{3} + i$ b) $z_2 = -1 - i$

Solution



$\because$ The form of the complex number is : $z = x + i\,y$, then :

a) $x = -\sqrt{3}$, $y = 1$

$\therefore$ z_1 lies in the second quadrant

$r = \sqrt{x^2 + y^2} = \sqrt{(-\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2$

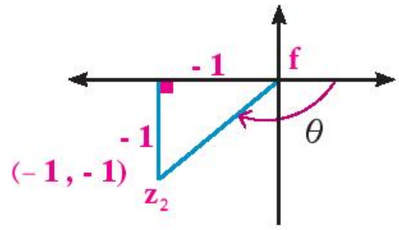
$\theta = \pi + \tan^{-1}(\frac{-1}{\sqrt{3}}) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

b $x = -1 \quad , \quad y = -1$

$\therefore$ z_2 lies in third quadrant

$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\theta = -\pi + \tan^{-1}(\frac{-1}{-1}) = -\pi + \frac{\pi}{4} = \frac{-3\pi}{4}$



Try to solve

2) Find the modulus and the principle amplitude of each of the following:

a $z_1 = \sqrt{2} + \sqrt{2}\, i$ **b** $z_2 = 1 - \sqrt{3}\, i$

c $z_3 = -\sqrt{3}\, i$ **d** $z_4 = 5$

Remember

$\frac{\theta^\circ}{180^\circ} = \frac{\theta^{rad}}{\pi}$

This rule is used to convert from degree measure into radian measure and vise versa

The properties of the modulus and the amplitude of a complex number

For each complex number $z = x + y\,i$ with argument θ :

1) $|z| > 0$

2) The amplitude of the complex number takes an infinite number of values by adding the multiple of 2π. Thus the amplitude of the complex number is $\theta + 2\pi n$ where $n \in Z$.

3) $|z| = |\bar{z}| = |-z| = |\overline{-z}|$ where $\bar{z}$ is the conjugate of z.

4) $z\bar{z} = |z|^2$

Critical thinking: If the principle amplitude of Z is θ, find the principle amplitude of $-Z$, $\bar{z}$, $\frac{1}{Z}$

Example

3) Write each of the following complex numbers in a trigonometric form:

a $Z_1 = 2 - 2\sqrt{3}\, i$ **b** $Z_2 = -4i$ **c** $Z_3 = \frac{-4}{\sqrt{3} + i}$ **d** $Z_4 = -2$

Solution

$\because$ The form of the complex number is : $Z = x + i\,y$, then:

a $x = 2 \quad , \quad y = -2\sqrt{3}$

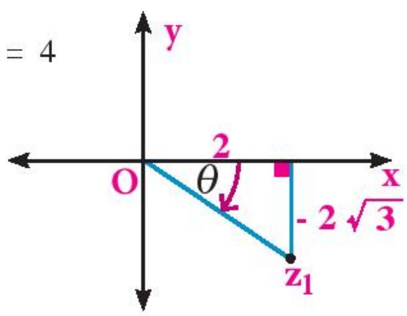
$\therefore Z_1$ lies in the fourth quadrant

$r = |Z_1| = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (-2\sqrt{3})^2} = \sqrt{16} = 4$

$\theta = \tan^{-1}(\frac{-2\sqrt{3}}{2}) = \frac{-\pi}{3}$

$\therefore Z_1 = r(\cos\theta + i\sin\theta)$

$= 4(\cos(\frac{-\pi}{3}) + i\sin(\frac{-\pi}{3}))$

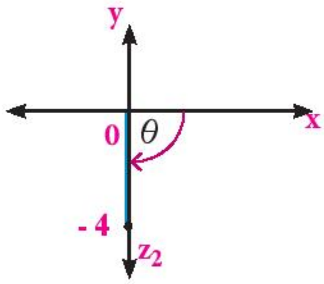


b $\because$ $x = 0$, $y = -4$

$\therefore$ Z_2 lies on y- axis

$r = |Z_2| = \sqrt{x^2 + y^2} = \sqrt{(0)^2 + (-4)^2} = 4$

$\theta = \frac{-\pi}{2}$ $\quad Z_2 = 4(\cos(\frac{-\pi}{2}) + i\sin(\frac{-\pi}{2}))$



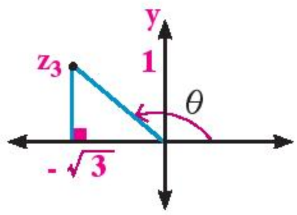
c $Z_3 = \frac{-4}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i} = -\sqrt{3} + i$

$x = -\sqrt{3}$, $y = 1$ $\quad\therefore$ Z_3 lies in the second quadrant

$r = |Z_3| = \sqrt{x^2 + y^2} = \sqrt{3+1} = 2$

$\theta = \pi + \tan^{-1}(\frac{-1}{\sqrt{3}}) = \frac{5\pi}{6}$

$Z_3 = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$



d $\because$ $x = -3$, $y = 0$ $\quad\therefore$ Z_4 lies on x-axis

$r = |Z| = \sqrt{(-3)^2 + (0)^2} = 3$

$\theta = \pi$ $\quad\therefore$ $Z_4 = 3(\cos\pi + i\sin\pi)$

Remember

- $1 = \cos 0 + i\sin 0$
- $-1 = \cos\pi + i\sin\pi$
- $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
- $-i = \cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}$

Try to solve

3 Write each of the following in a trigonometric form:

a $Z_1 = 8$ **b** $Z_2 = 5i$ **c** $Z_3 = -3-3i$

Example

4 Find the modulus and the principle amplitude for each of the following numbers:

a $Z_1 = -8(\cos 45^\circ + i\sin 45^\circ)$

b $Z_2 = 2(\sin\frac{4}{3}\pi - i\cos\frac{4}{3}\pi)$

Remember

$\cos\theta - i\sin\theta$
$= \cos(-\theta) + i\sin(-\theta)$

Solution

a $Z_1 = -8(\cos 45^\circ + i\sin 45^\circ)$

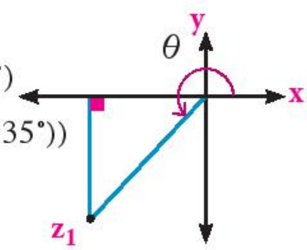
$= 8(-\cos 45^\circ - i\sin 45^\circ)$

$\because$ $x < 0$, $y < 0$ $\quad\therefore$ Z_1 lies in the third quadrant

$\because$ $-\cos 45^\circ = \cos(180^\circ + 45^\circ)$, $-\sin 45^\circ = \sin(180^\circ + 45^\circ)$

$\therefore$ $Z_1 = 8(\cos 225^\circ + i\sin 225^\circ) = 8(\cos(-135^\circ) + i\sin(-135^\circ))$

$\therefore$ $Z_1 = 8$, principle amplitude $\theta = -135^\circ = -3\frac{\pi}{4}$



(b) $Z_2 = 2\,(\sin \frac{4}{3}\pi - i \cos \frac{4}{3}\pi)$

$\because\ x > 0\ ,\ \ y < 0$

$\because\ \sin \frac{4}{3}\pi = \cos(\frac{3\pi}{2} + \frac{4}{3}\pi) = \cos \frac{17}{6}\pi = \cos(\frac{5\pi}{6})$

$-\cos \frac{4}{3}\pi = \sin(\frac{3\pi}{2} + \frac{4}{3}\pi) = \sin(\frac{17}{6}\pi) = \sin(\frac{5\pi}{6})$

$\therefore\ z_2 = 2(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}))$

$\therefore\ |z_2| = 2$, principle amplitude $\frac{5\pi}{6}$

Remember

$\sin(\frac{\pi}{2} + \theta) = \cos\theta$

$\cos(\frac{\pi}{2} + \theta) = -\sin\theta$

$\sin(\frac{3\pi}{2} + \theta) = -\cos\theta$

$\cos(\frac{3\pi}{2} + \theta) = \sin\theta$

Try to solve

(4) Find the modulus and the principle argument for each of the following complex numbers:

(a) $Z_1 = 2\,(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3})$

(b) $Z_2 = \frac{-1}{\sqrt{2}}(\sin 45° - i \sin 45°)$

Learn

multiplying and dividing complex numbers using the trigonometric form

If $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then

1) $Z_1Z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$ **(1)**

Thus. $|Z_1Z_2| = r_1r_2 = |Z_1||Z_2|$

$\text{Arg}(Z_1Z_2) = \theta_1 + \theta_2$

2) $\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times \frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$ **(2)**

then $|\frac{Z_1}{Z_2}| = \frac{r_1}{r_2} = \frac{|Z_1|}{|Z_2|}$, $\text{Arg}(\frac{Z_1}{Z_2}) = \theta_1 - \theta_2$

Ask your teacher to prove the relations (1) and (2)

Example

(5) Express $3\,(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}) \times 4\,(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$ in the form of $x + y\,i$

Solution

$3\,(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}) \times 4(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$

$= 3 \times 4((\cos\frac{5\pi}{12} + \frac{\pi}{12}) + i\sin(\frac{5\pi}{12} + \frac{\pi}{12}))$

$= 12\,(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 12(0 + i(1)) = 12\,i$

Try to solve

5) Express $2(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15}) \times 3(\cos\frac{-2\pi}{5} + i\sin\frac{-2\pi}{5}))$ in the form of $x + i\,y$

Remember

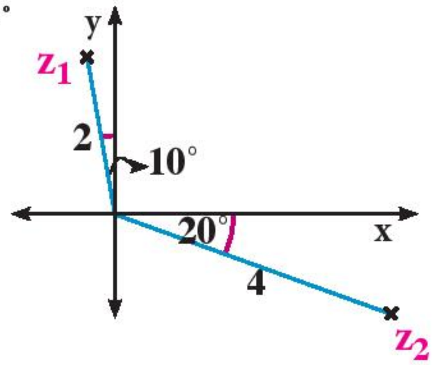
- $1 = \cos 0 + i\sin 0$
- $-1 = \cos\pi + i\sin\pi$
- $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
- $-i = \cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}$

Example

6) If Z_1 , Z_2 are two complex numbers represented on Argend's plane as in the opposite figure, find in the form of $x + y\,i$ the number $\frac{Z_2}{Z_1}$

Solution

From the graph $|Z_1| = 2$, arg of $Z_1 = 90° + 10° = 100°$



$\therefore\ Z_1 = 2(\cos 100° + i\sin 100°)$

$|Z_2| = 4$, $\arg Z_2 = -20°$

$\therefore\ z_2 = 4(\cos(-20°) + i\sin(-20°))$

$\therefore\ \frac{Z_2}{Z_1} = \frac{4}{2} \times \frac{\cos(-20°) + i\sin(-20°)}{(\cos 100° + i\sin 100°)}$

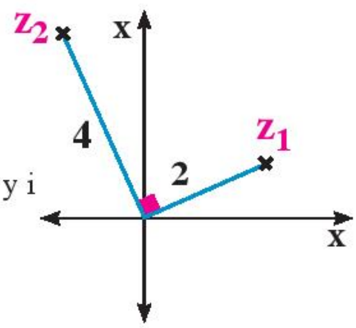
$= 2[\cos(-20° - 100°) + i\sin(-20° - 100°)]$

$= 2(\cos(-120°) + i\sin(-120°))$

$= 2(-\frac{1}{2} - i \times \frac{\sqrt{3}}{2}) = -1 - \sqrt{3}\,i$

Try to solve

6) Use the opposite Argand's diagram to find $\frac{Z_2}{Z_1}$ in the form $x + y\,i$



Results:

1) If $Z = r(\cos\theta + r\sin\theta)$ then

(a) $\frac{1}{Z} = \frac{1}{r}(\cos(-\theta) + r\sin(-\theta))$ **(b)** $Z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

2) We can generalize the product of finite number of complex numbers if $Z_1, Z_2, \ldots Z_n$ are complex numbers and if:

$Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2), \ldots\ldots, Z_n = r_n(\cos\theta_n + i\sin\theta_n)$

then: $Z_1 Z_2 \ldots Z_n = r_1 r_2 \ldots r_n(\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n))$

in a special case when $Z_1 = Z_2 = \ldots = Z_n = r(\cos\theta + i\sin\theta)$ then:

$$Z^n = r^n(\cos n\theta + i\sin n\theta)$$

Example

7) Put the number $1 - i$ in the trigonometric form, then find $(1 - i)^8$

Solution

$\because\ x = 1 \quad , \quad y = -1 \qquad r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\because\ x > 0 \ , \ y < 0 \qquad \therefore$ Z lies in the fourth quadrant

$\theta = \tan^{-1}(\frac{-1}{1}) = \frac{-\pi}{4} \qquad \therefore\ 1 - i = \sqrt{2}\,(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4})$

$\therefore\ (1 - i)^8 = (\sqrt{2})^8\ (\cos(\frac{-\pi}{4}) + i\sin(\frac{-\pi}{4}))^8$

$= 16\,(\cos(-2\pi) + i\sin(-2\pi))$

$= 16\,(\cos 0 + i\sin 0) = 16$

Try to solve

7) If $Z_1 = 2(\cos 10° + i\sin 10°)$, $Z_2 = 3(\cos 40° + i\sin 40°)$

Find $Z_1^4\ Z_2^2$ in the form x + y i

Exponential form of a complex number (Euler form)

Any function of x can be expressed as a series from the power of x called (Taylor series) Here are Taylor expansion for some functions that we are going to study in this unit.

1) The sine function $y = \sin x$

$$\sin x = \frac{x}{\lfloor 1} - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \dots + (-1)^n \times \frac{x^{2n+1}}{\lfloor 2n+1} + \dots$$

(Sine function is an odd function because $\sin(-x) = -\sin x$ so the expansion contains the odd power of x)

2) The cosine function $y = \cos x$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \dots + (-1)^n \times \frac{x^{2n}}{\lfloor 2n} + \dots$$

(cosine function is an even function because $\cos(-x) = \cos x$ so the expansion contains even powers of x)

3) exponential function $y = e^x$

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} \dots + \frac{x^n}{\lfloor n} + \dots$$

Notice that

$$e^{ix} = 1 + \frac{ix}{\lfloor 1} + \frac{i^2x^2}{\lfloor 2} + \frac{i^3x^3}{\lfloor 3} + \dots + \frac{i^nx^n}{\lfloor n} + \dots$$

$$= 1 + \frac{ix}{\lfloor 1} - \frac{x^2}{\lfloor 2} - \frac{ix^3}{\lfloor 3} + \dots$$

$$= (1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \dots) + i\,(\frac{x}{\lfloor 1} - \frac{x^3}{\lfloor 3} + \dots)$$

$$e^{ix} = \cos x + i\sin x$$

i.e. The complex number $Z = x + iy = r(\cos\theta + i\sin\theta)$ can be written in the form :

$Z = r\,e^{\theta i}$ it called Euler form where θ is in radian measure.

Tip

Euler equation
$e^{\pi i} + 1 = 0$
it conects the most 5 famous constants in Euler form
θ **should be in radian measure**

Example

8 Write each of the following complex numbers in the exponential form (Euler's form):

a $Z_1 = 1 + i$ **b** $Z_2 = -1+\sqrt{3}\ i$ **c** $Z_3 = e^{3+\frac{\pi}{6}i}$ **d** $Z_4 = -2i$

Solution

a $Z_1 = 1 + i$ $\therefore x = 1 \quad , \quad y = 1$

$r = |Z_1| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

$\because x > 0 \ , \ y > 0$ $\therefore Z_1$ lies in the first quadrant

$\theta = \tan^{-1}(\frac{1}{1}) = \frac{\pi}{4}$ $\therefore Z_1 = r e^{\theta i} = \sqrt{2}\ e^{\frac{\pi}{4}i}$

b $Z_2 = -1+\sqrt{3}\ i$ $\therefore x = -1 \quad , \quad y = \sqrt{3}$

$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\because x < 0 \ , \ y > 0$ $\therefore Z_2$ lies in the second quadrant

$\therefore \theta = \pi + \tan^{-1}(-\sqrt{3}) = \frac{2}{3}\pi$ $\therefore Z_2 = r e^{\theta i} = 2\ e^{\frac{2}{3}\pi i}$

c $Z_3 = e^{3+\frac{\pi}{6}i} = e^3 \times e^{\frac{\pi}{6}i}$ $r = |Z_3| = e^3 \quad , \quad \text{amp } Z_3 = \frac{\pi}{6}$

d $\because Z_4 = -2i$ $\therefore x = 0 \quad , \quad y = -2$ $\therefore r = \sqrt{(0)^2 + (-2)^2} = \sqrt{4} = 2$

$\because x = 0 \quad , \quad y = -2$ $\therefore Z_4$ lies on y-axis

$\therefore \theta = \frac{-\pi}{2}$ $\therefore Z_4 = 2\ e^{\frac{-\pi}{2}i}$

Try to solve

8 If $Z = \frac{\sqrt{2}\ i}{1+i}$,write Z in the exponential form.

Multiplying and dividing the complex numbers using the exponential form.

If $z_1 = r_1 e^{\theta_1 i}$, $z_2 = r_2 e^{\theta_2 i}$

Then $z_1 z_2 = r_1 r_2 e^{\theta_1 i} \times e^{\theta_2 i} = r_1 r_2 e^{(\theta_1 + \theta_2)i}$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \times \frac{e^{\theta_1 i}}{e^{\theta_2 i}} = \frac{r_1}{r_2}\ e^{(\theta_1 - \theta_2)i}$$

Example

9 Find the result for each of the following in the exponential form :

a $2(\cos 25° + i \sin 25°) \times 2(\sin 158° - i \sin 158°)$ **b** $(\frac{1 + i}{1 - i})^7$

Solution

a Convert Z_2 to the standard trigonometric form as follows:

$\because$ (sin 158° - i cos 158°) = sin (90° + 68°) - i cos (90° + 68°) = cos 68° + i sin 68°

$\therefore$ 3 (cos 25° + i sin 25°) × 2 (cos 68° + i sin 68°)

= 6 (cos (25° + 68°) + i sin (25° + 68°))... sin 25°) ×2 (cos 68°.....

= 6 (cos 93° + i sin 93°) = 6 $e^{1,62i}$

b $\because 1 + i = \sqrt{2}$ (cos 45° + i sin 45°)

$1 - i = \sqrt{2}$ (cos (-45°) + i sin (-45°))

$\therefore \left(\frac{1+i}{1-i}\right)$ = cos (45° + 45°) + i sin (45° + 45°)

= cos 90° + i sin 90° = $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$\therefore \left(\frac{1+i}{1-i}\right)^7 = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^7 = \cos\frac{7\pi}{2} + i\sin\frac{7\pi}{2}$

$= \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right) = e^{\frac{-\pi}{2}i}$

Note

$\frac{93°}{180°} = \frac{\theta^E}{\pi}$

i.e: $\theta^d = \frac{93°}{180°} \times \pi$

$\theta^d \simeq 1,62$

Try to solve

9. If $Z_1 = 1 - \sqrt{3}\, i$, $Z_2 = 1 + i$, find each of the following in the trigonometrical form:

a $Z_1 Z_2$ **b** $\frac{z_2}{z_1}$ **c** $(Z_2)^6$

Example

10. Express $Z = \sqrt{2}\, e^{\frac{3\pi}{4}i}$ in the algebraic form x + y i where x , y ∈ R

Solution

$\because Z = \sqrt{2}\, e^{\frac{3\pi}{4}i}$ $\therefore r = |Z| = \sqrt{2}$, $\theta = \frac{3\pi}{4}$

$\therefore Z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$= \sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= -1 + i$

Note

$\cos\frac{3\pi}{4} = \cos 135° = \frac{-1}{\sqrt{2}}$

$\sin\frac{3\pi}{4} = \sin 135° = \frac{-1}{\sqrt{2}}$

Try to solve

10. Express $Z = 8\, e^{\frac{\pi}{6}i}$ in the algebraic form x + y i where x , y ∈ R

Exercises 2 - 1

Complete the following

1. The number $Z = 3 - 4i$ is represented on Argand's diagram by the point A where A = (............ , ...)

2. If the point A represents the complex number Z on Argand's plane and point B represents the number $\overline{Z}$ on Argand's plane, then B is the image of A by reflection in ....

3. The modulus of the complex number $Z = -5i$ equals ....

4. If $Z = \frac{2-i}{2+i}$ then $|Z| =$ ....

5. If θ is the principle argument of the complex number Z, then the principle arg of $\overline{Z}$ is ....

6. If $Z = \frac{1}{\overline{Z}}$, then $|Z| =$ ....

7. The exponential form of the number $-1 + i$ is ....

8. If $Z = 1 + \sqrt{3}\,i$, then the principle amplitude of $(1 + \sqrt{3}\,i)^8$ is ....

9. The trigonometric form of the number $Z = 2 - 2\sqrt{3}\,i$ is ....

10. If Z is a complex number and $\arg(z) = \theta$, then $\arg 2Z$ is ....

Choose the correct answer from the given:

11. If $Z = \sqrt{2}(\sin 30^\circ + i\cos 30^\circ)$, then the principle amplitude of Z is ....
    - a) 30°
    - b) 60°
    - c) 90°
    - d) 120°

12. If $Z = (1 + \sqrt{3}\,i)^n$ and $|Z| = 8$ then the principle amplitude of Z is ....
    - a) $\frac{\pi}{2}$
    - b) $\frac{\pi}{3}$
    - c) $\frac{\pi}{6}$
    - d) π

13. If $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ and if $\theta_1 + \theta_2 = \pi$, then $Z_1 Z_2 =$ ....
    - a) $r_1 r_2$
    - b) $-r_1 r_2$
    - c) $r_1 r_2 i$
    - d) $-r_1 r_2 i$

14. The amplitude of the complex number $Z = -3$ is ....
    - a) 0°
    - b) 90°
    - c) 180°
    - d) 270°

15. If $Z = -1 + \sqrt{3}$, then $|\overline{Z}| =$ ....
    - a) $-1 - \sqrt{3}\,i$
    - b) $\sqrt{2}$
    - c) 2
    - d) -2

16 If $Z_1 = -1 - i$, then the exponential form of Z is ....

a $e^{\frac{3\pi}{4} i}$ b $e^{\frac{5\pi}{4} i}$ c $\sqrt{2}\, e^{\frac{-3\pi}{4} i}$ d $\sqrt{2}\, e^{225 i}$

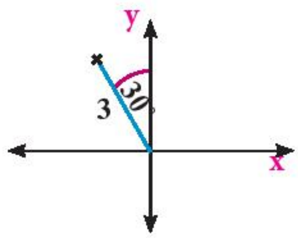
17 If $Z_1 = 2 + 2\sqrt{3}\, i$, $Z_2 = -3 - 3\sqrt{3}$ then $\arg Z_1 + Z_2 =$ ....

a 60° b 240° c 180° d 300°

18 If $x + y\,i = \frac{a + b\,i}{a - b\,i}$, then $x^2 + y^2$ ....

a $a^2 + b^2$ b $a^2 - b^2$ c $2a\ b$ d 1

19 The opposite figure represents the complex number ....



a 3(cos 30°+ i sin 30°)

b 3(cos 60°+ i sin 60°)

c 3(cos 120°+ i sin 120°)

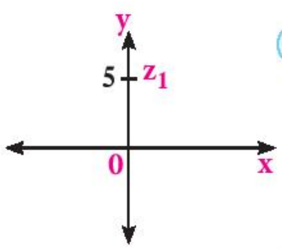
d 3(cos 150°+ i sin 150°)

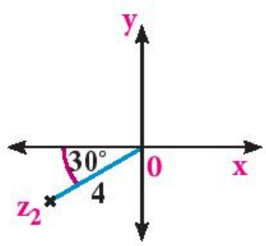
20 If Z is a complex number whose principle argument is θ, then the arg of $\frac{1}{Z}$ ....

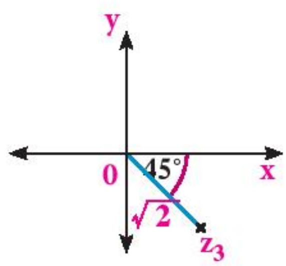
a θ b $-\theta$ c $\pi - \theta$ d $-\pi + \theta$

Answer the following:

21 Write each of the following complex numbers in the trigonometric form:

a 

b 

c 

d $Z_4 = -3 + 4i$

e $Z_5 = 4 (\cos 40° - i \sin 40°)$

22 Find the modulus and the principle argument for each of the following complex numbers:

a $Z_1 = -1 + i$

b $Z_2 = \frac{4}{\sqrt{3} - i}$

c $Z_3 = -2(\cos 45° + i \sin 45°)$

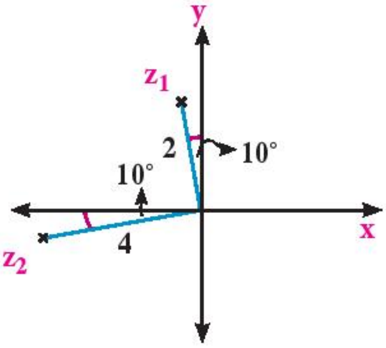
d $Z_4 = 1 + i \tan 20°$

23) If $Z_1 = \cos 114° + i \sin 66°$, $Z_2 = \cos 42° + i \sin 138°$

$Z_3 = \sin 24° + i \sin 114°$, find the algebraic form of the number: $\frac{Z_1 Z_2}{Z_3}$

24) If $Z_1 = 2 (\cos 75° + i \sin 75°)$, $Z_2 = 4 (\cos 15° + i \sin 15°)$, find the exponential form of: $Z_1 Z_2$, $\frac{Z_1}{Z_2}$

25) **In the opposite figure,** find the exponential form of: $\frac{Z_1}{Z_2}$



26) Write each of the following numbers in the algebraic form:

a) $Z_1 = e^{\frac{\pi}{3} i}$

b) $Z_2 = 2 e^{\frac{3\pi}{4} i}$

c) $3 e^{\frac{-\pi}{6} i}$

27) If $Z = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$, prove that $\frac{1}{z} = \frac{1}{2} e^{\frac{5\pi}{3} i}$

28) If $Z = \sqrt{3} + i$, find the algebraic form of Z^6

29) If $Z = \frac{(a + b) + i (a - b)}{(a - b) - i (a + b)}$, find Z in the simplest form then find $|Z|$ where a, b $\in R$

30) **Creative thinking:** If $Z_1 = \cos 75 + i \sin 75$, $Z_2 = \cos 15 + i \sin 15$, find the trigonometric form of the number: $Z_1 + Z_2$

31) If $\arg Z_1 = \frac{\pi}{3}$, $\arg Z_2 = \frac{3\pi}{4}$, $\arg Z_3 = \frac{\pi}{6}$, find:

a) $\arg (Z_1^3 \; Z_2^2)$ b) $\arg (2Z_1 \cdot Z_2)$ c) $\arg (\frac{Z_1 Z_2}{Z_3})$ d) $\arg (Z_3^6)$

32) **Creative thinking:** Prove that $\cos \theta = \frac{1}{2} (e^{\theta i} + e^{-\theta i})$, $\sin \theta = \frac{-i}{2} (e^{\theta i} - e^{-\theta i})$

Unit two

2 - 2

Demoivre's Theorem

You will learn

- Demoivre's theorem with positive integer power
- De Moivre's theorem with positive rational power
- The roots of complex number
- Representing the roots of the complex number in Argend's plane

Key terms

- De Moivre's thearem
- root

Think and discuss

a If the complex number Z whose modulus r and argument θ, then find:

1) The modulus of the number Z^3

2) The argument of the number Z^3

b If Z is complex number and the principle amplitude of Z^3 is θ, then the principle amplitude of Z is ....

Learn

De moivre's theorem with positive integer power

If n is a positive integer number

Then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Example

1. Express $\cos 3\theta$ in terms of $\cos\theta$

Solution

$\because (\cos\theta + i\sin\theta)^3 = \cos^3\theta + i\sin^3\theta$ **(1) De Moivre's theorem.**

also $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + {}^3C_1\cos^2\theta\,(i\sin\theta)$.

$+ {}^3C_2\cos\theta\,(i\sin\theta)^2 + (i\sin\theta)^3$ **(binomial theorem).**

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$ **(2)**

From (1) , (2) equating the real parts.

$\because \sin^2\theta + \cos^2\theta = 1$ $\therefore \sin^2\theta = 1 - \cos^2\theta$

$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta\,(1 - \cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$.

Try to solve

1. Express $\sin 3\theta$ in terms of $\sin\theta$

De moivre's theorem with positive rational power

We know that $\cos\theta + i\sin\theta = \cos(\theta + 2r\pi) + i\sin(\theta + 2r\pi)$: where r is an integer.

If K is a positive number , then $(\cos\theta + i\sin\theta)^{\frac{1}{k}} = \cos\dfrac{\theta + 2r\pi}{k} + i\sin\dfrac{\theta + 2r\pi}{k}$

i.e $(\cos\theta + i\sin\theta)^{\frac{1}{k}}$ takes several values according to values of r, the number of this values equals K by putting r = ... , - 2 , - 1 , 0 , 1 , 2 .... which makes the amplitude:

$\dfrac{\theta + 2r\pi}{k}$ included between $-\pi$,π

Example

(2) Find in C the trigonometric and exponential forms the roots of the equation $Z^4 = 8(1-\sqrt{3}\,i)$, then write the solution set.

Solution

$\because Z^4 = 8 - 8\sqrt{3}\,i \qquad \therefore x = 8 , y = -8\sqrt{3}$

$r = \sqrt{(8)^2+(-8\sqrt{3})^2} = 16 \quad , \tan\theta = \dfrac{-8\sqrt{3}}{8} = -\sqrt{3}$

$\because x > 0 \quad , \quad y < 0 \qquad \therefore Z^4$ lies in the fourth quadrant

$\theta = \tan^{-1}(-\sqrt{3}) = -\dfrac{\pi}{3}$

$\therefore Z^4 = 16\left(\cos\dfrac{-\pi}{3}\right) + i\sin\left(\dfrac{-\pi}{3}\right))$

$\therefore Z = 2\left(\cos\frac{1}{4}\left(-\frac{\pi}{3} + 2\pi r\right) + i\sin\frac{1}{4}\left(-\frac{\pi}{3} + 2\pi r\right)\right)$

when r = 0 , **then** $Z_1 = 2\left(\cos\left(\frac{-\pi}{12}\right) + i\sin\left(\frac{-\pi}{12}\right)\right) = 2e^{\frac{-\pi}{12}i}$

when r = 1, **then** $z_2 = 2\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right) = 2e^{\frac{5\pi}{12}i}$

when r = - 1, **then** $z_3 = 2\left(\cos\left(\frac{-7\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12}\right)\right) = 2e^{\frac{-7\pi}{12}i}$

when r = 2, **then** $z_4 = 2\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right) = 2e^{\frac{11\pi}{12}i}$

$\therefore$ S.S. = $\{2e^{\frac{-\pi}{12}i} ,\ 2e^{\frac{5\pi}{12}i} ,\ 2e^{\frac{-7\pi}{12}i} ,\ 2e^{\frac{11\pi}{12}i}\}$

Try to solve

(2) Find in C the solution set of $Z^4 = 2 + 2\sqrt{3}\,i$

Example

(3) Find the roots of the equation $Z^3 = 1$, then represent the roots on Argend's plane.

Solution

$Z^3 = 1$

$= \cos 0^\circ + i \sin 0^\circ$

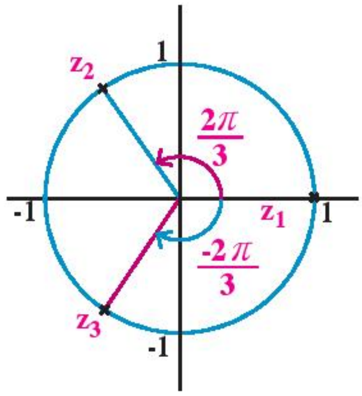
$\therefore\ Z = (\cos 0^\circ + i \sin 0^\circ)^{\frac{1}{3}}$

$= \cos \frac{1}{3}\ (2r\pi) + i \sin \frac{1}{3}\ (2r\pi)$

when r = 0, then $Z_1 = \cos 0^\circ + i \sin 0^\circ = 1$

when r = 1, then $Z_2 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$

when r = - 1, then $Z_3 = \cos \frac{-2\pi}{3} + i \sin \frac{-2\pi}{3}$



We notice that the roots divide the circle whose center is the origin , and radius length is the unity into 3 equal arcs and the measure of each is 120°(the coordinates of the points form the vertices of an equilateral triangle).

Try to solve

3) Find the roots of the equation $Z^4 = 1$, then represent the roots on the Argand's plane.

The n^{th} roots

The equation $x^n = a$ where a is complex number has n roots in the form of $x = a^{\frac{1}{n}}$.
We can calculate them by finding the trigonometric form of a, then apply De Movre's theorem.
All roots lie on Argend's plane on a circle whose center is the origin and its radius length is $|a|^{\frac{1}{n}}$ to form a regular polygon of n vertices.

Example (The fifth roots of - 32)

4) Represent in Argand's diagram the fifth roots of - 32

Solution

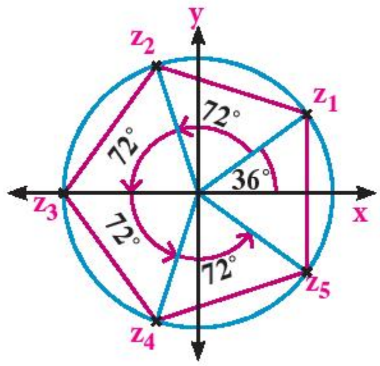
The fifth roots of - 32 are the roots of the equation $Z^5 = -32$

Convert the number - 32 into a trigonometric form.

$\therefore Z^5 = 32\ (\cos \pi + i \sin \pi)$

$\therefore Z = 32^{\frac{1}{5}}\ (\cos \pi + i \sin \pi)^{\frac{1}{5}}$

$= 2\ (\cos \frac{1}{5}\ (\pi + 2r\pi) + i \sin \frac{1}{5}\ (\pi + 2r\pi)).$



The first root when r = 0

$\therefore\ Z = 2\ (\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}) = 2\ (\cos 36^\circ + i \sin 36^\circ)$

Thus, the measure of the angle between each two successive roots is $\frac{360^\circ}{5} = 72^\circ$

$Z_2 = 2\ (\ \cos (36^\circ + 72^\circ) + i \sin (36^\circ + 72^\circ)) = 2\ (\cos 108^\circ + i \sin 108^\circ)$

$Z_3 = 2\ (\cos(\ 36^\circ + 2 \times 72^\circ) + i \sin (36^\circ + 2 \times 72^\circ)) = 2\ (\cos 180^\circ + i \sin 180^\circ)$

$Z_4 = 2\ (\ \cos (36^\circ + 3 \times 72^\circ) + i \sin (36^\circ + 3 \times 72^\circ))\ = 2\ (\cos 252^\circ + i \sin 252^\circ)$

$= 2\,(\cos(-108^\circ) + i\sin(-108^\circ))$

$Z_5 = 2\,(\cos(36^\circ + 4 \times 72^\circ) + i\sin(36^\circ + 4 \times 72^\circ)) \quad = 2\,(\cos 324^\circ + i\sin 324^\circ)$

$= 2\,(\cos -36^\circ + i\sin -36^\circ)$

Try to solve

4) Represent on Argand's diagram the sixth roots of 1

Example

5) Find the square roots of $3 + 4i$

Solution

Let $(3+4i)^{\frac{1}{2}} = x + yi$ by squaring both sides

$\therefore\ 3 + 4i = x^2 - y^2 + 2xyi$

by equating the true part with the true part and the imaginary part with the imaginary part

$\therefore\ x^2 - y^2 = 3 \longrightarrow$ **(1)** , $\quad 2xy = 4 \longrightarrow$ **(2)** by squaring **(1)** , **(2)** and adding

$\therefore\ x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = 9 + 16 \qquad \therefore\ x^4 + 2x^2y^2 + y^4 = 25$

$\therefore\ (x^2 + y^2)^2 = 25 \qquad \therefore\ (x^2 + y^2) = 5$ **(3)**

By adding **(1)**, **(3)** $\quad 2x^2 = 8 \quad \therefore\ x = \pm 2$

when $x = 2$ by substituting in (2) $\quad y = 1$

when $x = -2 \quad y = -1$

$\therefore$ the first root $= 2 + i \qquad \therefore$ the second root $= -2 - i$

Try to solve

5) Find the square roots of the number $7 - 24i$

Example

6) Find in C the solution set of the equation $(1 - i)x^2 - (6 - 4i)x + 9 - 7i = 0$

Solution

The equation can be in the form of:

$$x^2 - \frac{6 - 4i}{1 - i}x + \frac{9 - 7i}{1 - i} = 0 \qquad x^2 - (5 + i)x + 8 + i = 0$$

Use the general rule to solve the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{(5+i) \pm \sqrt{(5+i)^2 - 4 \times 1(8+i)}}{2}$$

$$x = \frac{(5+i) \pm \sqrt{24 + 10i - 32 - 4i}}{2}$$

$$x = \frac{(5+i) \pm \sqrt{-8 + 6i}}{2}$$

let $a + b\,i = \sqrt{-8 + 6i}$ **by squaring both sides**

$a^2 - b^2 + 2ab\,i = -8 + 6i$

$a^2 - b^2 = -8$ **(1)** $\qquad 2ab = 6$ **(2)**

$a^2 + b^2 = 10$ **(3)** $\qquad$ **from (1) . (3)** $a^2 = 1 \qquad \therefore a = \pm 1 \qquad \therefore b = \pm 3$

$\therefore\ a + b\,i = \pm(1 + 3\,i)$

$\therefore\ x = \dfrac{5 + i \pm (1 + 3i)}{2} \qquad x = 3 + 2\,i \quad$ or $\quad x = 2 - i$

Try to solve

6. Find in C the solution set of the equation $x^2 + (1 + i)\,x - 6 + 3i = 0$

Exercises 2 - 2

1. Use De moivre's theorem to prove each of the following identities:
   a) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ $\qquad$ b) $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

2. Find in C the solution set in each of the following equations; write the roots in the form $x + y\,i$:
   a) $Z^4 = 16$ $\qquad$ b) $Z^3 + 8 = 0$ $\qquad$ c) $Z^3 + 8\,i = 0$

3. Find the solution set of the equation $Z^5 + 243 = 0$ where $Z \in C$

4. Find the solution set of the equation of $Z^4 = 2 + 2\sqrt{3}\,i$ in the exponintial form .

5. Find the square roots for each of the following:
   a) $2 - 2\sqrt{3}\,i$ $\qquad$ b) $1 - i$ $\qquad$ c) $8\,i$ $\qquad$ d) $3 + 4\,i$ $\qquad$ e) $5 - 12\,i$

6. Find the cubic roots of 8, then represent them on Argand's plane.

7. Find the fourth roots of -1, then represent them on Argand's plane.

8. If $\dfrac{7 - 11\,i}{4 + i} = a + b\,i$, find the value of $(\sqrt{-b} + a\,i)^{\frac{3}{2}}$

9. Put the number $2\sqrt{2}\,(1 + i)$ in the trigonometric form, then find its square roots in the exponintial form.

10. If $Z = 8 - 6\,i$, find $Z^{\frac{3}{2}}$ in the algebraic form.

11. **Creative thinking :** prove that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos(2\theta) + 3)$

Cubic roots of unity

Unit two

2 - 3

You will learn

- The cubic roots of unity
- Properties of cubic roots of unity
- Representing the cubic roots of one geometrically
- Congugate of $a + \omega$, $a i + \omega$

Key terms

- Root
- Square root
- Cubic root
- Unit circle
- Conjugate

Materials

- Scientific calculator
- Graphical programs

Co-operation work :

Use De Moivre's theorem to find the solution set of the equation $Z^3 = 1$

Find the previous roots in algebraic form.

Find the sum of the 3 roots. What do you notice?

Learn

Cubic roots of unity

Use De Moivre's theorem we to find that the solution set of $Z^3 = 1$ is :

$$1 \,,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\, i \,,\ \frac{-1}{2} - \frac{\sqrt{3}}{2}\, i.$$

Also we notice that the square of one of the complex roots equals the other complex root, thus we can assume the cubic roots in the form of: $1 , \omega, \omega^2$

where $\omega = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, $\omega^2 = -\frac{1}{2} \mp \frac{\sqrt{3}}{2} i$

Critical thinking :

Can you find the cubic roots of one using the algebraic form of the complex number ?

Properties of cubic roots of one:

If 1 , ω , ω^2 are the cubic roots of one, then

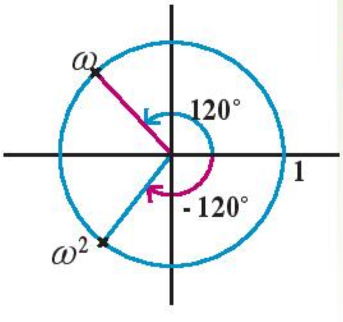
1- $1 + \omega + \omega^2 = 0$ (the sum of the roots = 0)

$(1 + \omega = -\omega^2 \,,\ 1 + \omega^2 = -\omega \,,\ \omega + \omega^2 = -1)$

2- $\omega^3 = 1$

$\left(\frac{1}{\omega} = \omega^2 , \frac{1}{\omega^2} = \omega \right)$

3- The cubic roots of one lies on a circle whose center is the origin, it's radius length is 1 and form vertices of an equilateral triangle.



4- $(\omega - \omega^2) = \pm\sqrt{3}\ i$, $(\omega^2 - \omega) = \pm\sqrt{3}\ i$

Example

1 If 1 , ω , ω^2 , are the cubic roots of one, find the value of :

a $5 + 5\omega + 5\omega^2$

b $(1 - \frac{2}{\omega} - \frac{2}{\omega^2})(3 + 5\omega + 5\omega^2)$

Solution

a The expression $= 5(1 + \omega + \omega^2)$ by taking 5 a common factor

$= 5 \times 0 = 0$

b The expression $= (1 - \frac{2}{\omega} - \frac{2}{\omega^2})(3 + 5\omega + 5\omega^2)$ by substituting $\frac{1}{\omega} = \omega^2, \frac{1}{\omega^2} = \omega$

$= (1 - 2\omega^2 - 2\omega)(3 + 5\omega + 5\omega^2) = (1 - 2(\omega^2 + \omega))(3 + 5(\omega + \omega^2))$

$= (1 - 2(-1))(3 + 5(-1)) = (1 + 2)(3 - 5) = -6$

Try to solve

1 If 1 , ω , ω^2 are the cubic roots of one, find the value of:

a $(2 + 5\omega + 2\omega^2)^5$

b $(\omega + \frac{1}{\omega})^2 (\omega^2 + \frac{1}{\omega^2})^3$

Example

2 Prove that $[\frac{5 - 3\omega^2}{5\omega - 3} - \frac{2 - 7\omega}{2\omega^2 - 7}]^4 = 9$

Solution

The expression $= [\frac{5 - 3\omega^2}{5\omega - 3} - \frac{2 - 7\omega}{2\omega^2 - 7}]^4 = [\frac{5\omega^3 - 3\omega^2}{5\omega - 3} - \frac{2\omega^3 - 7\omega}{2\omega^2 - 7}]^4$

$= [\frac{\omega^2(5\omega - 3)}{5\omega - 3} - \frac{\omega(2\omega^2 - 7)}{2\omega^2 - 7}]^4 = [\omega^2 - \omega]^4 = [\pm\sqrt{3}\, i]^4 = 9$

Try to solve

2 Prove that $= [\frac{a + \omega b + \omega^2 c}{\omega^2 a + b + c\omega} - \frac{c + a\omega^2 - b\omega}{\omega c + b\omega^2 + a}]^8 = 81$

Example

3 Prove that $x = \frac{-1 + \sqrt{-3}}{2}$ is one of the roots of the equation $x^{10} + x^5 + 1 = 0$

Solution

$\because x = \frac{-1 + \sqrt{-3}}{2} = \frac{-1 + \sqrt{3}\, i}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

i.e. x represents one of the complex roots of one

when $x = \omega$, then $\omega^{10} + \omega^5 + 1 = \omega^1 + \omega^2 + 1 = 0$

when $x = x^2$ then $\omega^{20} + \omega^{10} + 1 = \omega^2 + \omega + 1 = 0$

Try to solve

3) Form the quadratic equation whose roots are $(1 + \omega - \omega^2)^3$, $(1 - \omega + \omega^2)^3$

Exercises 2 - 3

If 1 ,ω ,ω^2 are the cubic roots of one :

Complete the following:

1) $(2 + 5\omega + 2\omega^2)^2 =$ ....

2) $(\omega - \omega^2)^4 =$ ....

3) $\left(\omega + \frac{1}{\omega}\right)^2 \left(\omega^2 + \frac{1}{\omega^2}\right)^2 =$ ....

4) If $x = \frac{-1 + \sqrt{3}\, i}{2}$, then $x^8 + x^4 =$ ....

5) $\left(\frac{1}{\omega + 1}\right)\left(1 + \omega - \frac{3}{\omega}\right) =$ ....

6) $1 + 3\omega + 3\omega^2 =$ ....

7) If $a = 2\omega - 3\omega^2$, $b = 3 + 5\omega^2$,then $a^2 + b^2 =$ ....

8) $\sum_{r=1}^{5} \omega^r =$ ....

Choose the correct Answer from the given:

9) The conjugate of ω is ....

a) ω b) ω^2 c) 1 d) $-\omega$

10) $\left(\omega^2 + \frac{1}{\omega}\right)\left(1 + \frac{1}{\omega^2}\right)^2 =$ ....

a) 2 b) 0 c) - 3 d) - 5

11) $(a + b\omega + a\omega^2)(a + b\omega^2 + a\omega^4) =$ ....

a) 1 b) $a - b$ c) $(a - b)^2$ d) $b^2 - a^2$

12) $\left(1 + 2\omega^5 + \frac{1}{\omega^2}\right)\left(1 + 2\omega + \frac{1}{\omega^4}\right) =$ ....

a) 0 b) 1 c) - 1 d) 2

13) $\frac{a - d\omega}{a\omega^2 - d} - \omega^2 =$ ....

a) 3i b) $\pm\sqrt{3}\, i$ c) - 3 d) 3

14) If $(1 + \omega)^7 = a + b\omega$ where a and b are real numbers, then (a , b) = ...

a) (0 , - 1) b) (1 , 1) c) (0 , 1) d) (1 , - 1)

15) If $(1 + \omega^2)^n = (1 + \omega)^n$, then the least value of the positive integer n is ....

a) 2 b) 3

c) 5 d) 6

16) $1 + \omega + \omega^2 + \omega^3 + ... + \omega^{100} =$ ....

a) 0 b) 1

c) ω d) $-\omega^2$

17) If $Z = \omega^x$ then $|Z| =$ .... where x is a positive integer

a) 1 b) ω

c) x d) ω^2

18) $\sum_{r=1}^{6} 1 + \omega^r =$ ....

a) 0 b) 6

c) 1 d) $1 + \omega$

19) Prove each of the following identities:

a) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)(1 - \omega^8 + \omega^{16}) = 2^4$

b) $\left(\frac{\omega}{1 + 2\omega}\right)^2 + \left(\frac{1 + 2\omega^2}{\omega^2}\right)^2 = \frac{-10}{3}\omega^2$

c) $\left[\frac{1}{1 + \omega i} - \frac{\omega + i}{1 + \omega^2 i}\right]^8 = 16$

d) $\left[\frac{5 - 3\omega^2}{5\omega - 3} - \frac{2 - 7\omega}{2\omega^2 - 7}\right]^2 = -3$

e) $(1 + \omega^2)^8 = \omega^2$

f) $(1 - \omega^2 + \omega^4)^2 + (1 + \omega^2 + \omega^4)^2 = 4\omega$

20) Find value of each of the following:

a) $5 + 3\omega + 3\omega^2$

b) $(1 + \omega + 2\omega^2)^2 + (1 + 2\omega + \omega^2)^2$

c) $\frac{\omega^2(\omega - 1)(\omega^2 - 1)}{(2\omega + 1)(\omega^2 + 2)}$

d) $\left[\frac{1}{1 + 3\omega^2} - \frac{1}{1 + 3\omega}\right]^2$

e) $(1 + \frac{1}{\omega} + i)(1 + \frac{1}{\omega^2} + i)$

(21) If $x = \frac{-1 + \sqrt{3}\,i}{2}$, prove that $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x = 0$.

(22) If $\frac{1}{1+\omega}$, $\frac{1}{1+\omega^2}$ are the two roots of a quadratic equation, find the equation .

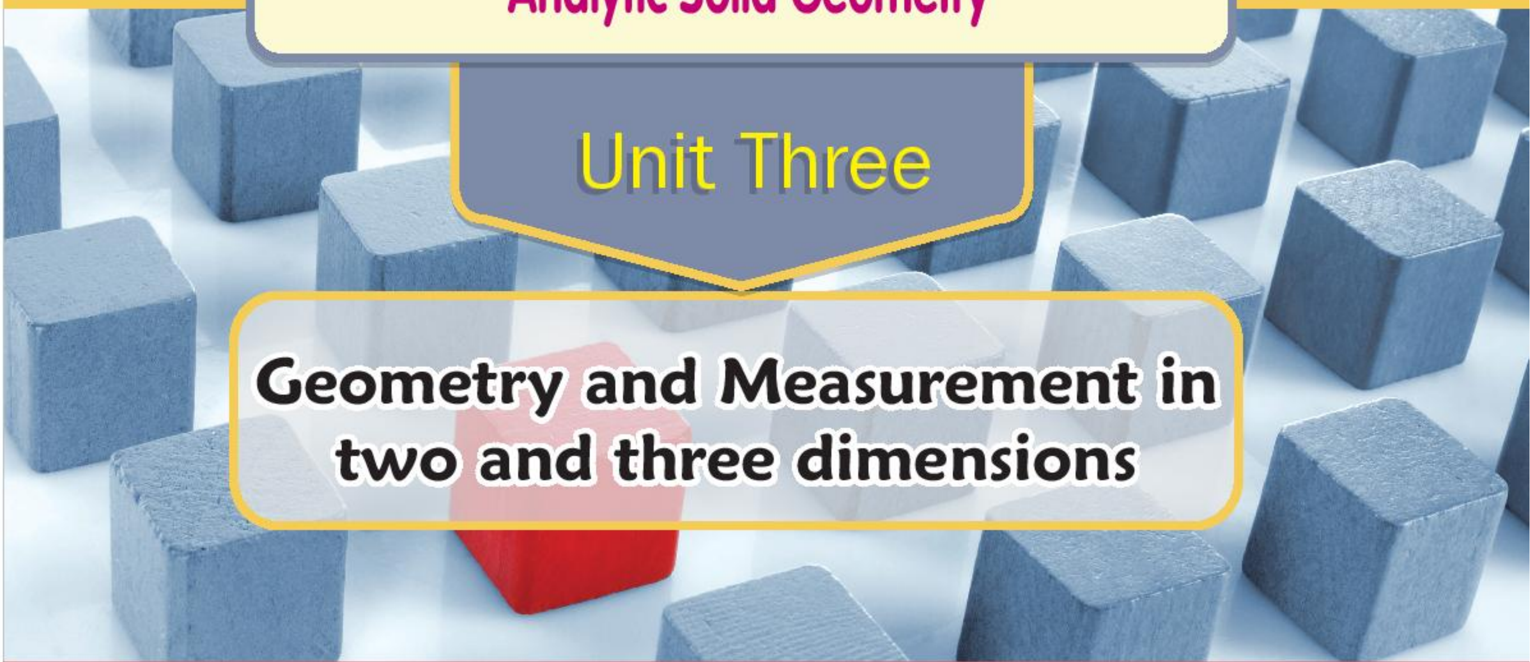
(23) If $Z = 2(\omega + i)(\omega^2 + i)$, find all different forms of the number Z. and the square roots of Z in the trigonometric form.

(24) **Creative thinking:** Find value of n which makes $(2 + 5\omega + 2\omega^2)^n = (2 + 2\omega + 5\omega^2)^n$

(25) Find : (a) $\sum_{r=zero}^{10} \omega^r$ (b) $\sum_{r=zero}^{10} (1 + \omega^r + \omega^{2r})$

Analytic Solid Geometry

Unit Three



Geometry and Measurement in two and three dimensions

Unit introduction

Geometry is the scince of studying the various types of shapes and describing them. It studies the relation of the shapes, angles and distances with each other. Besides, it is divided into two parts: The plane geometry which is specialized in studying the geometric shapes of 2-D only and the solid geometry which is specialized in studying the solids of 3-D (length - width- height). It deals with spaces such as cuboids, cylindric solids, spherical and conical bodies.

Greek had been the first to use geometry. Thales had discovered proofs for some theorems, then Euclid had assembled all the geometric corollaries and organized them in a book called "the principles". Later , this type of geometry have been developed into the analytic geometery, geometry of triangles, Minkowski geometry (4-D) and the non-Euclidean geometry and so on. In this unit, you learn the vectors to study the straight lines, planes and the relation among them in 3-D

Unit objectives

By the end of this unit and doing all activities included, the student should be able to:

- Recognize the coordinate system in three dimensions and resolve the vector in space.
- Find the distance between two points in space and the coordinate of the midpoint of a line segment in space .
- Recognize the vectors in space through:
  - Representing the vector by triple ordered.
  - The fundamental unit vectors in space $\hat{j} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$, $\hat{k} = (0, 0, 1)$.
  - Expressing any vector interims of $\hat{i}$, $\hat{j}$, $\hat{k}$
  - Expressing directing line segment in terms of its ends in space.
- Recognize the scalar product and vector product of two vectors in plane and space.
- Recognize the properties of the scalar product and vector product in plane and space.
- Recognize the angle between two vectors in space.
- Recognize the perpendicularity of two vectors in space
- Identify the direction angles and direction cosines of any vector in space.
- Use the scalar product to find the algebraic and vector companents of avector in direction of another vector
- Recognize the geometrical meaning of the norm of the vector product.
- Recognize the scalar triple product and its geometrical meaning.

Key terms

- space
- 3D
- projection
- right hand Rule
- 3D−vector
- plane
- scalar product
- vector product
- component
- work
- scalar triple product
- position vector
- unit vector
- the norm of vector

Materials

- Scientific calculator

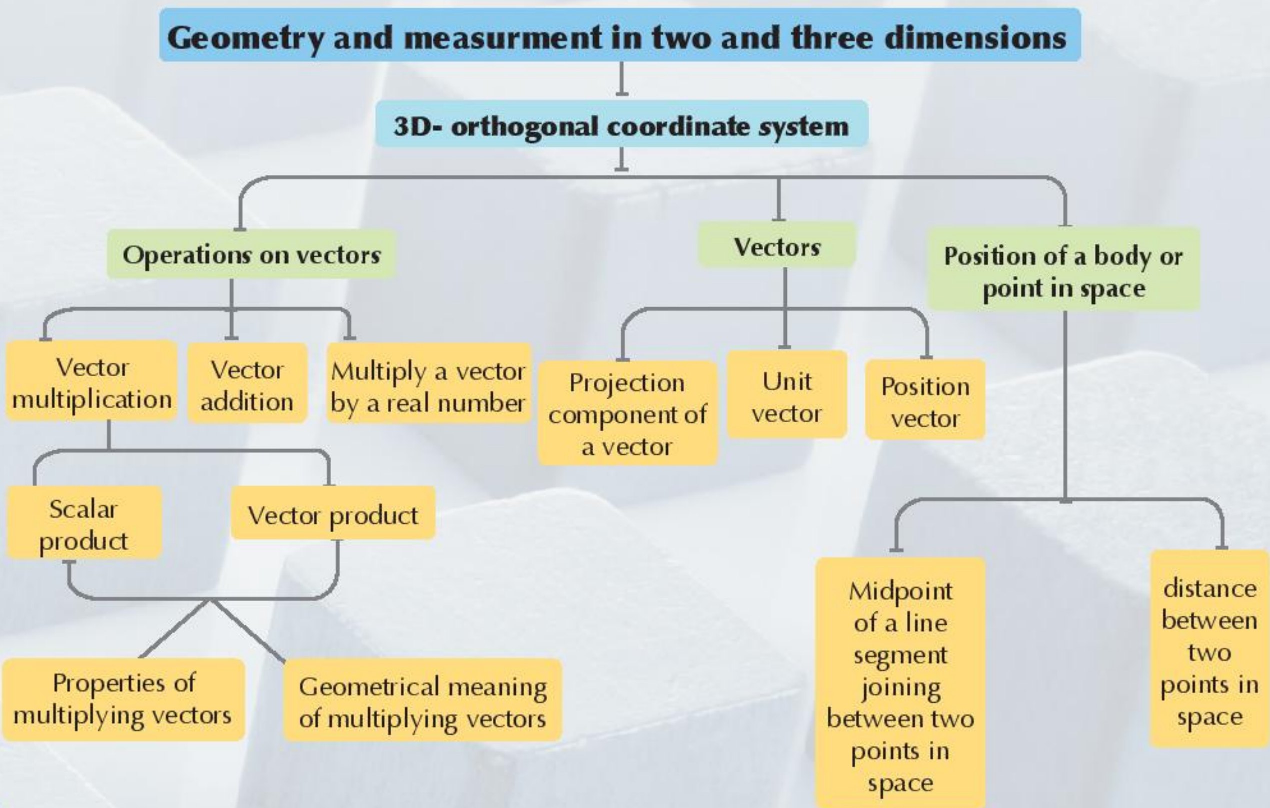
Unit Lessons

Lesson (3 - 1): The 3D orthogonal coordinate system.

Lesson (3 - 2): vectors in the space.

Lesson (3 - 3): Vector multiplication.

Unit chart



Unit Three

3 - 1 The three- dimensional orthogonal coordinate system

You will learn

- Determine the position of a point in 3D-coordinate system
- Identify the coordinates of a midpoint of a line segment joining between 2 points in space
- Find the distance between 2 points in the space

Key terms

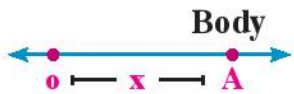
- Space
- 3D
- Projection
- Right hand rule
- Plane

Materials

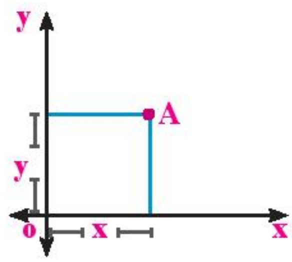
- Scientific calculator

Think and discuss

- To determine the position of a body on a straight line, we should know the distance between the body and a fixed (arbitrary) point on it called the origin (O)
  $OA = x \in R$



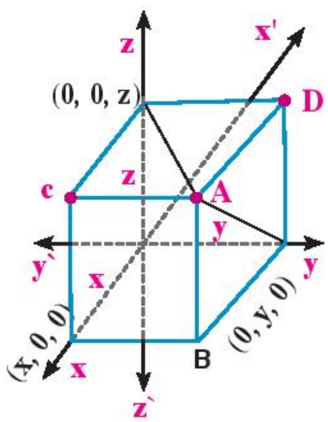
- To determine the position of a body in plane, you should know its projection on two dimensional orthognal axis.
  $A = (x, y) \in R^2$
- How can you determine the position of a body in space?



Learn

The three - dimensional orthogonal coordinate system R^3

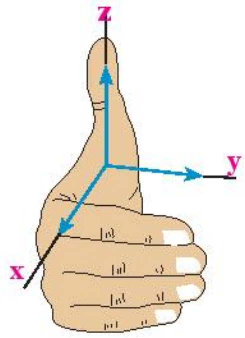
The coordinates of a point A in space are identified with respect to 3 mutually orthogonal axes intersect at a point by finding the projection of this point on each axis .



Think: In the previous 3 D coordinate system, find the coordinates of the points B, C, D,

basic definitions:

1- The right hand rule

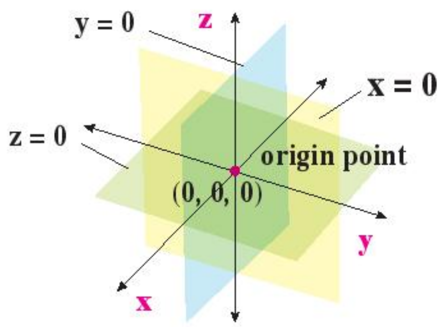


when the 3-dimensional orthogonal coordinate system is formed, we should follow the right hand rule.

where the curved fingers refer from the +ve direction of x-axis towards the positive direction of y-axis, your thumb points at the direction of positive z-axis.

2- Cartisian planes

- All points in space with coordinates (x, y, 0) are located in x y plane whose equation is $Z = 0$
- All points in space with coordinates (x, 0, z) are located in x z plane whose equation is $y = 0$
- All points in space with coordinates (0, y, z) are located in y z plane whose equation is $x = 0$



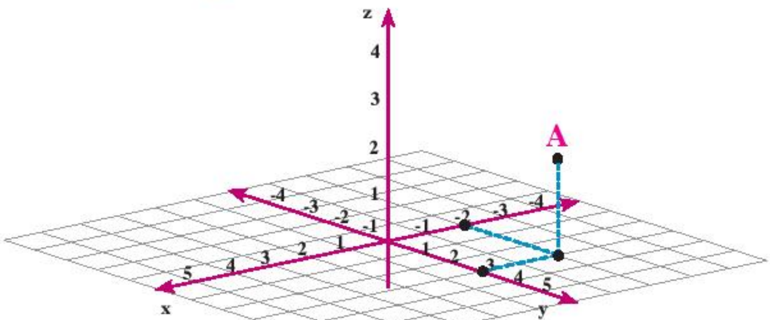
Example (Identifying the position of a point in space)

1) Identify the position of each of the following points using 3D-orthogonal coordinate system:

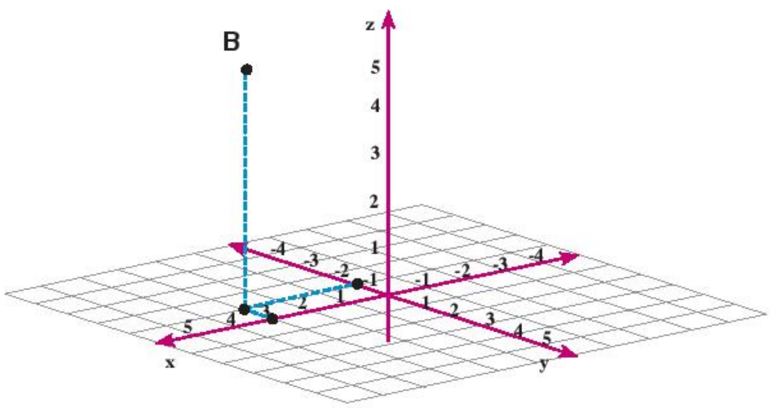
a) A (-2, 3, 2) b) B (3, -1, 5) c) C (4, 0, -1)

Solution

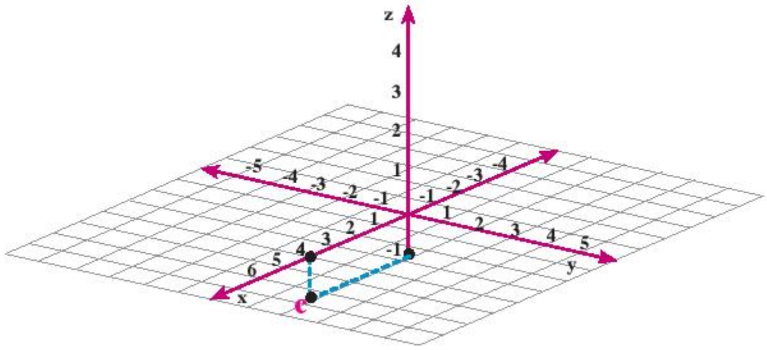
a) To identify the point A (-2, 3, 2), we identify the point (-2, 3) in the x y plane, then move 2 units in the +ve direction of z-axis to get point A (-2, 3 , 2)



b) To identify the point B (3, -1, 5), we identify the point (3, -1) in xy plane then move 5 units in the +ve direction of Z-axis to get point B.



c) To identify the point C (4, 0, -1), we identify point (4, 0) on x-axis, then move one unit in the -ve direction of z-axis.



Try to solve

1) a) Identify the position of each of the following points using 3-dimensional orthogonal coordinate system:

A (3, 2, 3) B (-1, 4, 3) C (0, 0, 4)

b) **Complete:**

1- The distance between point A (-1, 2, 3) and the Cartesian xy-plane = ... unit length.

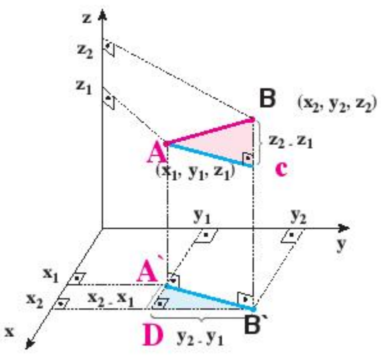
2- The distance between point B (4, -2, 1) and the cartesian yz-plane = ... unit length.

Learn

The distance between two points in space

If A (x_1, y_1, z_1), B (x_2, y_2, z_2) are two points in space, then the distance between A and B is given by the relation

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Example

2) Prove that the triangle ABC where A (2, -1, 3) , B(-4, 4, 2) and C (-2, 5, 1) is right angled at C.

Solution

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ **the distance rule**

$= \sqrt{(2 + 4)^2 + (-1 - 4)^2 + (3 - 2)^2} = \sqrt{62}$

$BC = \sqrt{(-4 + 2)^2 + (4 - 5)^2 + (2 - 1)^2} = \sqrt{6}$

$AC = \sqrt{(2 + 2)^2 + (-1 - 5)^2 + (3 - 1)^2} = \sqrt{56}$

$\because (AB)^2 = (\sqrt{62})^2 = 62,\ (BC)^2 + (AC)^2 = (\sqrt{6})^2 + (\sqrt{65})^2 = 6 + 56 = 62$

$\therefore (AB)^2 = (BC)^2 + (AC)^2$ $\qquad \therefore m(\hat{C}) = 90°$

Try to solve

2) Prove that the points A(4, 4, 0), B(4, 0, 4), C(0, 4, 4) are the vertices of an equilateral triangle and find its area.

Learn

The coordinates of a midpoint of a line segment

If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ are two points in space, then the coordinates of point C which lies at the midpoint of $\overline{AB}$ is :

$$C = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example

3) If A(1, -3, 2), B (4, -1, 4), find the coordinates of the midpoint of $\overline{AB}$

Solution

The coordinates of the midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

$= \left(\frac{1+4}{2}, \frac{-3-1}{2}, \frac{2+4}{2}\right)$

$= \left(\frac{5}{2}, -2, 3\right)$

Try to solve

3) Find the coordinates of the midpoint of $\overline{CD}$ where C (0, 4, -2), D(-6, 3, 4)

Critical thinking: If C (2, 2, 6) is the midpoint of $\overline{AB}$ where A (1, -4 , 0), find coordinates of the point B

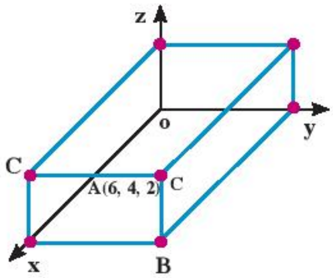
Exercises 3 - 1

Complete the following:

1. If point (x, y, z) lies in x y - plane, then z = ...

2. The two straight lines $\overleftrightarrow{xx`}$ and $\overleftrightarrow{zz`}$ form the coordinate plane .... whose equation is ....

3. The opposite figure represents a cuboid in an orthogonal coordinate system. One of it's vertices is the origion O(0, 0, 0) ,then



   the coordinates of the point B is ....

   Coordinates of the point C is ....

4. If A (1, -1, 4) , B (0, -3, 2), then the coordinates of the midpoint of $\overline{AB}$ is ....

Choose the correct answer from the following:

5. The distance between point (3, -1, 2) and the Cartesian xz plane is .... unit length
   a) 3 b) -1 c) 2 d) 1

6. The length of the perpendicular drawn from point (-2, 3, 4) to the x-axis is .... unit length.
   a) 2 b) 3 c) 5 d) 4

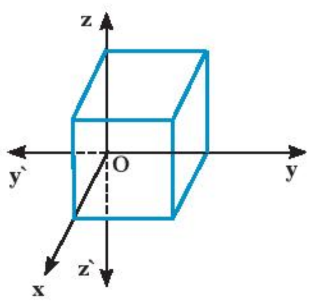
7. The coordinates of the midpoint of the line segment whose terminals (-3, 2, 4), (5, 1, 8) is
   a) $(1, \frac{3}{2}, 6)$ b) (2, -1, 4) c) (8, -1, 4) d) $(1, -\frac{3}{2}, 2)$

Answer the following questions :

8. Find the distance between the two points A and B in each of the following :
   a) A (7 , 0, 4) , B (1 , 0 , 0) b) A (4, 1, 9) , B (2 , 1 , 6)
   c) A (1 , 1 , -7) , B (-2 , -3 , -7)

9. Prove that the triangle whose vertices are the given points is a right angled triangle and find its Area:
   a) (-2 , 5 , 2), (0 , 0 , 2), (0 , 4 , 0) b) (-4 , 4 , 1), (2 , -1 , 2), (-2 , 5 , 0)

10. The opposite figure represents a cube whose volume is 27 cubic unit and one of its vertices is the origin, find coordinates of the other vertices .



11. If the points (7, 1, 3) , (5, 3, k), (3, 5, 3) are the vertices of a triangle , prove that this triangle is isosceles, then find the value (values) of k which make (s) the triangle equilateral

12. Find the coordinates of the midpoint of $\overline{AB}$ in each of the following:
    a. A(3, -1, 4) , B (2, 0, -1)
    b. A(-3, 5, 5) , B (-6, 4, 8)

13. If C (-1, 4, 0) is the midpoint of $\overline{AB}$ where B (4, -2, 1), find coordinates of point A.

14. **Creative thinking:**
    If A ∈ x -axis, B ∈ y -axis, C ∈ z-axis and if point(1, -1, 0) is the midpoint of $\overline{AB}$ and point (0, -1, 2) is the midpoint of $\overline{BC}$, find the coordinates of the midpoint of $\overline{AC}$

15. **Writing in math:** If all the points in the space in the form of (x, y, 0) lie in x y-plane whose equation z = 0, find the equation of the plane in which all of its points in the form (x, y, 2)

16. **Discover the error:** If point B (-1, 4, 2) is the midpoint of $\overline{AC}$ where A(1 , 0 , 2), find C

Ashraf's solution	Zeiad's solution
$C = \left(\left(\frac{x_1 + x_2}{2}\right), \left(\frac{y_1 + x_2}{2}\right), \left(\frac{z_1 + z_2}{2}\right)\right)$ $= \left(\left(\frac{-1 + 1}{2}\right), \left(\frac{4 + 0}{2}\right), \left(\frac{2 + 2}{2}\right)\right)$ $= (0, 2, 2)$	let C (x, y, z) $\therefore \frac{1 + x}{2} = -1 \longrightarrow x = -3$ $\frac{0 + y}{2} = 4 \longrightarrow y = 8$ $\frac{2 + z}{2} = 2 \longrightarrow z = 2$ $\therefore c(-3, 8, 2)$

Which answer is correct? Why?

Unit Three

3 - 2 Vectors in space

You will learn

- Representing the vector by three components
- Position vector in space
- Fundamental unit vectors in space
- Express the vector interms t of he unit vectors
- Expressing the directed line segment in space in terms of its terminals
- Equality of two vectors in space
- The norm of a vector in space
- Unit vector in the direction of a vector in space
- Adding vector in space
- Multiply a vector by a real number

Key terms

- Position vector in space
- The norm vector
- Unit vector
- Scalar product
- Vector product

Introduction:

You have previously studied the scalar quantities and vector quantities and have known that the vector is represented by a directed segment determined by a magnitude [norm of the vector] and a direction. We will learn in this lesson the vectors in space (a-three dimensional coordinate system).

Learn

Position vector in space

The position vector of point A (A_x, A_y, A_z) with respect to the origin O (0, 0, 0) is defined as the directed line segment whose starting point is O and end point is A.

- The position vector of point A is denoted by $\overrightarrow{A}$ i.e. $\overrightarrow{A} = (A_x, A_y, A_z)$
- A_x is called the component of $\overrightarrow{A}$ in direction of x axis.
- A_y is called the component of $\overrightarrow{A}$ in direction of y axis.
- A_z is called the component of $\overrightarrow{A}$ in direction of z axis .

The norm of a vector

It is the length of the directed line segment which represents the vector.

If $\overrightarrow{A} = (A_x, A_y, A_z)$, then from the distance between two points rule

$$\| \overrightarrow{A} \| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}$$

Example

1. If $\overrightarrow{A} = (2, -3,1)$, $\overrightarrow{B} = (0, 4, -3)$ then

- The component of the vector $\overrightarrow{A}$ in the direction of x-axis is 2
- The component of the vector $\overrightarrow{B}$ in the direction of Z-axis is -3

$$\| \overrightarrow{A} \| = \sqrt{2^2 + (-1)^2 + (3)^2} = \sqrt{14}$$

$$\| \overrightarrow{B} \| = \sqrt{(0)^2 + (4)^2 + (-3)^2} = 5$$

The vector $\overrightarrow{B}$ lies in the y z -plane (the component of the vector $\overrightarrow{A}$ vanishes in the direction of x-axis)

Try to solve

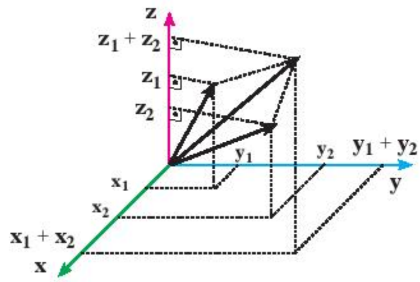
1. If $\overrightarrow{A} = (-1, 4, 2)$, $\overrightarrow{B} = (3, 1, 0)$, find

   a) $A_x + B_y$ b) $\|\overrightarrow{A}\| + \|\overrightarrow{B}\|$

Adding Vectors in space

If $\overrightarrow{A} = (A_x A_y, A_z)$, $\overrightarrow{B} = (B_x, B_y, B_z)$, then:

$\overrightarrow{C} = \overrightarrow{A} + \overrightarrow{B} = (A_x + B_x, A_y + B_y, A_z + B_z) = (C_x, C_y, C_z)$



Example

2. If $\overrightarrow{A} = (-2, 3, 1)$, $\overrightarrow{B} = (0, -2, 4)$, then:

   $\overrightarrow{A} + \overrightarrow{B} = (-2, 3, 1) + (0, -2, 4) = (-2+0, 3+(-2), 1+4) = (-2, 1, 5)$

Try to solve

2. If $\overrightarrow{A} = (4, -4, 0)$, $\overrightarrow{B} = (-1, 5, 2)$, find $\overrightarrow{A} + \overrightarrow{B}$

Properties of adding vectors in space

For any two vectors $\overrightarrow{A}$ and $\overrightarrow{B} \in R^3$, then:

1- **Closure property:** $\overrightarrow{A} + \overrightarrow{B} \in R^3$

2- **Commutative property:** $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{B} + \overrightarrow{A}$

3- **Associative property:** $(\overrightarrow{A} + \overrightarrow{B}) + \overrightarrow{C} = \overrightarrow{A} + (\overrightarrow{B} + \overrightarrow{C})$

4- **Identity element over addition (zero vector):** $\overrightarrow{O} = (0, 0, 0)$ is the neutral element of addition in R^3

i.e.: $\overrightarrow{A} + \overrightarrow{O} = \overrightarrow{O} + \overrightarrow{A} = \overrightarrow{A}$

5- **The additive inverse:** for every vector $\overrightarrow{A} = (A_x, A_y, A_z) \in R^3$ there is

$-\overrightarrow{A} = (-A_x, -A_y, -A_z) \in R^3$ such that: $\overrightarrow{A} + (-\overrightarrow{A}) = (-\overrightarrow{A}) + \overrightarrow{A} = \overrightarrow{O}$

Multiplying a vector by a scalar (a real number)

If $\overrightarrow{A} = (A_x, A_y, A_z) \in R^3$ and $K \in R$, then:

$$K\overrightarrow{A} = K(A_x, A_y, A_z) = (KA_x, KA_y, KA_Z) \in R^3$$

For example:

$3\,(2, -1, 4) = (6, -3, 12)$

$\frac{1}{2}\,(4, 9, 6) = (2, \frac{9}{2}, 3)$

$-2\,(1, -3, -4) = (-2, 6, 8)$

Properties of multiplying vectors by a real numbers

If $\overrightarrow{A}$, $\overrightarrow{B} \in R^3$ and K, L $\in$ R, then

1. Distributive property

- $K(\overrightarrow{A} + \overrightarrow{B}) = K\overrightarrow{A} + K\overrightarrow{A}$
- $(K + L)\overrightarrow{A} = K\overrightarrow{A} + L\overrightarrow{A}$

2. Associative property

- $K(L\overrightarrow{A}) = L(K\overrightarrow{A}) = (KL)\overrightarrow{A}$

Example

(3) If $\overrightarrow{A} = (-1, 5, 2)$, $\overrightarrow{B} = (4, -1, 3)$, then

1. $2\overrightarrow{A} - 3\overrightarrow{B} = 2(-1, 5, 2) -3\,(4, -1, 3)$

$= (-2, 10, 4) + (-12, 3, -9)$

$= (-14, 13, -5)$

2. Find the vector $\overrightarrow{C}$ where $2\overrightarrow{C} + 3\overrightarrow{A} = 2\overrightarrow{B}$

$\because 2\overrightarrow{C} + 3\overrightarrow{A} = 2\overrightarrow{B}$ — by adding $-3\overrightarrow{A}$ for both sides

$\therefore 2\overrightarrow{C} = 2\overrightarrow{B} - 3\overrightarrow{A}$

$\therefore 2\overrightarrow{C} = 2\,(4, -1, 3) -3\,(-1, 5, 2)$

$= (8, -2, 6) + (3, -15, -6)$

$= (11, -17, 0)$ — by multiplying by $\frac{1}{2}$

$\therefore \overrightarrow{C} = \frac{1}{2}\,(11, -17, 0) = (\frac{11}{2}, \frac{-17}{2}, 0)$

Try to solve

(3) If $\overrightarrow{C} = (2, -3, 1)$, $\overrightarrow{D} = (0, 2, -2)$

(a) find $5\overrightarrow{C} - 2\overrightarrow{D}$

(b) If $3\overrightarrow{A} - 4\overrightarrow{D} = \overrightarrow{C}$, find $\overleftarrow{A}$

Equality of vectors in the space

If $\overrightarrow{A} = (A_x, A_y, A_z)$, $\overrightarrow{B} = (B_x, B_y, B_z)$, then:

$$\overrightarrow{A} = \overrightarrow{B} \text{ if and only if: } A_x = B_x \ , \quad A_y = B_y \ , \quad A_z = B_z$$

Example

(4) Find the values of L, m , n which make the two vectors $\overrightarrow{A} = (L - 4, m^2 - 3 , 1)$, $\overrightarrow{B} = (5, 1, n^2)$ equal

Solution

$\because \quad \overrightarrow{A} = \overrightarrow{B}$

$\therefore \quad A_x = B_x \longrightarrow L - 4 = 5 \longrightarrow L = 9$

$A_y = B_y \longrightarrow m^2 - 3 = 1 \longrightarrow m^2 = 4 \longrightarrow m = \pm 2$

$A_z = B_z \longrightarrow n^2 = 1 \longrightarrow n = \pm 1$

Try to solve

(4) If $(2x + 1 \ , \ 5 \ , \ k + 4) = (-1 \ , \ y^2 - 4 \ , \ x + 1)$, find the value of x, y and k

The unit vector

The unit vector is a vector whose norm equals the unit length

For example :

$\overrightarrow{A} = (\frac{-3}{13}, \frac{4}{13}, \frac{12}{13})$ **is a unit vector because:** $\quad \|\overrightarrow{A}\| = \sqrt{(\frac{-3}{13})^2 + (\frac{4}{13})^2 + (\frac{12}{13})^2} = 1$

Try to solve

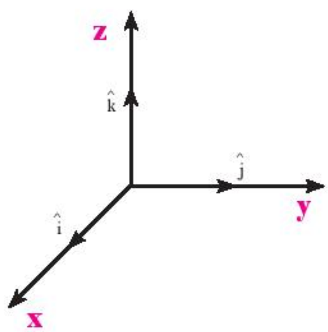
(5) Show which of the following vectors represents a unit vector

$\overrightarrow{A} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \qquad \overrightarrow{B} = (\frac{1}{5}, \frac{4}{5}, \frac{-\sqrt{5}}{5})$

Fundamental unit vectors ($\hat{i}$, $\hat{j}$, $\hat{k}$)

It is a directed segments whose starting point is the origin point and its norm is the unit length and its direction is the positive direction of x, y and z axes respectively:

$\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$, $\hat{k} = (0, 0, 1)$



Critical thinking

Express the vectors (-1, 0, 0) , (0, -1, 0), (0, 0, -1) interms of the fundamental unit vectors.

Expressing a vector in space in terms of the fundmental unit vectors

If $\overrightarrow{A} = (A_x, A_y, A_z) \in R^3$, then the vector $\overrightarrow{A}$ can be written in the form

$$\begin{aligned}\overrightarrow{A} &= (A_x, 0, 0) + (0, A_y, 0) + (0, 0, A_z)\\ &= A_x(1, 0, 0) + A_y(0, 1, 0) + A_z(0, 0, 1)\\ &= A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}\end{aligned}$$

Example

5 If $\overrightarrow{A} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\overrightarrow{B} = -\hat{i} - 2\hat{j}$, find

a $2\overrightarrow{A} - 3\overrightarrow{B}$ **b** $\|\overrightarrow{A} + \overrightarrow{B}\|$, $\|\overrightarrow{A}\| + \|\overrightarrow{B}\|$ What do you notice?

Solution

a $2\overrightarrow{A} - 3\overrightarrow{B} = 2(2\hat{i} - 3\hat{j} + \hat{k}) - 3(-\hat{i} - 2\hat{j})$

$= 4\hat{i} - 6\hat{j} + 2\hat{k} + 3\hat{i} + 6\hat{j}$

$\therefore 2\overrightarrow{A} - 3\overrightarrow{B} = 7\hat{i} + 2\hat{k}$

b $\overrightarrow{A} + \overrightarrow{B} = (2\hat{i} - 3\hat{j} + \hat{k}) + (-\hat{i} - 2\hat{j})$

$= \hat{i} - 5\hat{j} + \hat{k}$

$\therefore \|\overrightarrow{A} + \overrightarrow{B}\| = \sqrt{1^2 + (-5)^2 + 1^2} = \sqrt{27}$

$\|\overrightarrow{A}\| + \|\overrightarrow{B}\| = \sqrt{2^2 + (-3)^2 + 1^2} + \sqrt{(-1)^2 + (-2)^2}$

$= \sqrt{14} + \sqrt{5}$ **we notice that** $\|\overrightarrow{A} + \overrightarrow{B}\| \neq \|\overrightarrow{A}\| + \|\overrightarrow{B}\|$

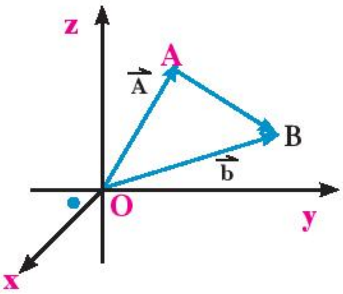
Try to solve

6 If $\overrightarrow{A} = -3\hat{j} - \hat{k} + 5\hat{i}$, $\overrightarrow{B} = -2\hat{k} + 3\hat{i}$, find

a $3\overrightarrow{A} - 5\overrightarrow{B}$ **b** $\|\overrightarrow{A} - \overrightarrow{B}\|$

Expressing a directed line segment in the space in terms of the coordinates of its terminals

let A and B be two points in space and their position vectors with respect to the origin be $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.



$\because \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

or $\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}$

Example

(6) If A (-2, 3, 1) , B (4, 0, 2) , then

$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}$

$= (4, 0, 2) - (-2, 3, 1) = (6, -3, 1)$

$\overrightarrow{BA} = \overrightarrow{A} - \overrightarrow{B}$

$= (-2, 3, 1) - (4, 0, 2) = (-6, 3, -1)$

Notice that: $\overrightarrow{AB} = - \overrightarrow{BA}$

Try to solve

(7) (a) If A (2, -3, 0), B (1, 4, -1) , find $\overrightarrow{AB}$

(b) If A (1, 1, -2), $\overrightarrow{AB}$ (4, -1, 2) , find coordinates of point B

The unit vector in a direction of a given vector

If $\overrightarrow{A} = (A_x, A_y, A_z) \in R^3$, then the unit vector in the direction of the vector $\overrightarrow{A}$ is denoted by $\overrightarrow{U_A}$ and given by the relation :

$$\overrightarrow{U_A} = \frac{\overrightarrow{A}}{\| \overrightarrow{A} \|}$$

Example

(7) If $\overrightarrow{A} = (-2, 2, 1)$, $\overrightarrow{B} = (3, 1, -2)$, find the unit vector in the direction of $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{AB}$

Solution

$$\overrightarrow{U_A} = \frac{\overrightarrow{A}}{\| \overrightarrow{A} \|} = \frac{(-2, 2, 1)}{\sqrt{4+4+1}} = \left(\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$\overrightarrow{U_B} = \frac{\overrightarrow{B}}{\| \overrightarrow{B} \|} = \frac{(3, 1, -2)}{\sqrt{9+1+4}} = \left(\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}\right)$$

$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}$

$= (3, 1, -2) - (-2, 2, 1) = (5, -1, -3)$

$$\therefore \overrightarrow{U_{AB}} = \frac{\overrightarrow{AB}}{\| \overrightarrow{AB} \|}$$

$$= \frac{(5, -1, -3)}{\sqrt{25+1+9}} = \left(\frac{5}{\sqrt{35}}, \frac{-1}{\sqrt{35}}, \frac{-3}{\sqrt{35}}\right)$$

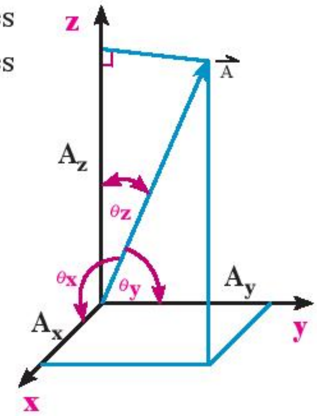
Try to solve

8) Find the unit vector in the direction of each of the following vectors:

a) $\overrightarrow{A} = (8, -4, -8)$ b) $\overrightarrow{B} = \hat{i} - 2\hat{j} - \hat{k}$ c) $\overrightarrow{C} = 3\hat{i} - 4\hat{k}$

Direction angles and direction cosines of a vector in space

If $\overrightarrow{A} = (A_x, A_y, A_z)$ is a vector in space and $(\theta_x, \theta_y, \theta_z)$ are the measures of the angles made by the vector with the +ve directions of x, y, z axes respectively, then:



$A_x = \| \overrightarrow{A} \| \cos\theta_x$, $A_y = \| \overrightarrow{A} \| \cos\theta_y$, $A_z = \| \overrightarrow{A} \| \cos\theta_z$

$\therefore \overrightarrow{A} = \| \overrightarrow{A} \| \cos\theta_x \hat{i} + \| \overrightarrow{A} \| \cos\theta_y \hat{j} + \| \overrightarrow{A} \| \cos\theta_z \hat{k}$

$= \| \overrightarrow{A} \| (\cos\theta_x \hat{i} + \cos\theta_y \hat{j} + \cos\theta_z \hat{k})$

$(\theta_x, \theta_y, \theta_z)$ **are called the direction angles of vector** $\overrightarrow{A}$

$\cos\theta_x, \cos\theta_y, \cos\theta_z$ **are called the direction cosines of vector** $\overrightarrow{A}$

Notice that: $\cos\theta_x \hat{i} + \cos\theta_y \hat{j} + \cos\theta_z \hat{k}$ represent the unit vector in the direction of the vector $\overrightarrow{A}$ i.e.

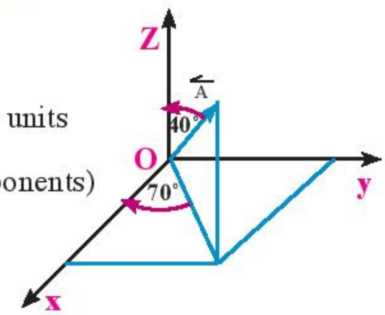
$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

Example

8) The opposite figure represents a vector $\overrightarrow{A}$ whose norm is 10 units

a) Express the vector $\overrightarrow{A}$ in algebraic form (Cartesian components)

b) Find the measure of the direction angles of $\overrightarrow{A}$



Solution

First resolve $\overrightarrow{A}$ into two components; the first in the direction of $\overrightarrow{OZ}$ with magnitude

$A_z = \| \overrightarrow{A} \| \cos\theta_z = 10 \cos 40 = 7.66$

The second in xy-plane

$A_{xy} = \| \overrightarrow{A} \| \sin\theta_z = 10 \sin 40 = 6.428$

Now, resolve the component A_{xy} into two components; the first is in the direction of $\overrightarrow{OX}$ with magnitude

$A_x = A_{xy} \cos 70 = 6.428 \cos 70 = 2.199$

the second is in the direction of $\overrightarrow{OY}$ with magnitude

$A_y = A_{xy} \sin 70 = 6.428 \sin 70 = 6.04$

So, the Cartesian form of the vector $\overrightarrow{A}$ is

$$\overrightarrow{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$
$$= 2.199 \hat{i} + 6.04 \hat{j} + 7.66 \hat{k}$$

Second : to find the measures of the direction angles, we get the unit vector in the direction of $\overrightarrow{A}$

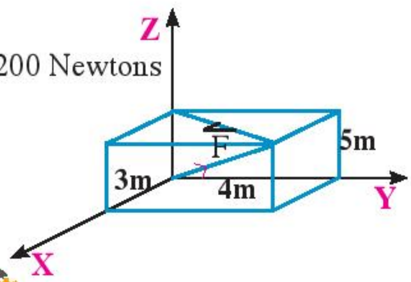
$$\overleftarrow{U_A} = \frac{\overrightarrow{A}}{\|\overrightarrow{A}\|} = \frac{1}{10}(2.199 \hat{i} + 6{,}04 \hat{j} + 7{,}66 \hat{k})$$
$$= 0.2199 \hat{i} + 0{,}604 \hat{j} + 0{,}766 \hat{k}$$

$\therefore \cos\theta_x = 0.2199$, **then** $\theta_x = \cos^{-1}(0.2199) = 77.3°$

$\cos\theta_y = 0.604$, **then** $\theta_y = \cos^{-1}(0.604) = 52.84°$

$\cos\theta_z = 0.766$, **then** $\theta_z = \cos^{-1}(0.766) = 40°$

Try to solve



9) The opposite figure represents the force $\overrightarrow{F}$ with a magnitude 200 Newtons
- a) Express the force $\overrightarrow{F}$ in an algebraic form.
- b) Find the measures of the direction angles of the force $\overrightarrow{F}$.

Exercises 3 - 1

Complete the following:

1) If $\overrightarrow{A} = (-3, 4, 2)$, then $\|\overrightarrow{A}\| =$ ....

2) If $\overrightarrow{A} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\overrightarrow{B} = 3\hat{i} - \hat{k}$, then $\overrightarrow{A} - \overrightarrow{B} =$ ....

3) The unit vector in the direction of $\overrightarrow{AB}$ where A (-1, 2, 0) , B (3, -1, 2) is ....

4) The vector $\overrightarrow{A} = 3\hat{i} + \hat{j} - 2\hat{k}$ makes an angle of measure .... with the +ve direction of x-axis.

5) The vector $\overrightarrow{B} = \hat{i} + 2\hat{j}$ makes an angle of measure .... with the +ve direction of z-axis.

Choose the correct answer from the following:

6) If $\overrightarrow{A} = (-2, k, 1)$ and $\|\overrightarrow{A}\| = 3$ unit, then k = ...
- a) 4
- b) -4
- c) ± 2
- d) $\sqrt{14}$

7) If 30°, 70°, θ are the direction angles of a vector, then one of the values of θ = ...

a) 100° b) 80° c) 260° d) 68,61°

8) If $\overrightarrow{A} = (-1, 5, -2)$, $\overrightarrow{B} = (3, 1, 1)$ and if $\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C} = \hat{i}$, then $\overrightarrow{C}$ = ...

a) $\hat{i} + 6\hat{j} - \hat{k}$ b) $-\hat{i} - 6\hat{j} + \hat{k}$

c) $\hat{i} + 4\hat{j} - 3\hat{k}$ d) $\hat{i} + 4\hat{j} - \hat{k}$

9) The direction cosines of the vector $\overrightarrow{A} = (-2, 1, 2)$ is ....

a) (-2, 1, 2) b) $(\frac{-2}{3}, \frac{1}{3}, \frac{2}{3})$ c) $(\frac{5}{-2}, 5, \frac{5}{2})$ d) (-1, 1, 1)

Answer the following :

10) If $\overrightarrow{A} = (2, -3, 1)$, $\overrightarrow{B} = (4, -2, 0)$, $\overrightarrow{C} = (-6, 0, 3)$, find each of the following vectors:

a) $\overrightarrow{A} + \overrightarrow{B}$ b) $3\overrightarrow{A} - \frac{1}{3}\overrightarrow{C}$ c) $\frac{3}{2}\overrightarrow{B} + \frac{2}{3}\overrightarrow{C}$

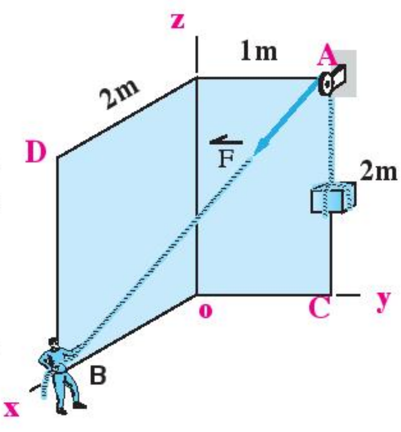
11) If $\overrightarrow{A} = 2\hat{i} - 3\hat{j} + 5\hat{k}$, $\overrightarrow{B} = 4\hat{j} - 2\hat{k}$, $\overrightarrow{C} = 4\hat{i} + 5\hat{j} - 6\hat{k}$, find each of the following vectors :

a) $2\overrightarrow{A} + \overrightarrow{B}$ b) $\frac{1}{2}\overrightarrow{B} - \overrightarrow{C}$ c) $3\overrightarrow{A} - 2\overrightarrow{C}$

12) Find the norm of each of the following vectors:

a) $\overrightarrow{A} = (2, -1, 0)$ b) $\overrightarrow{B} = (1, 2, -2)$ c) $\overrightarrow{C} = \hat{j}$ d) $\overrightarrow{D} = \hat{i} - 4\hat{j}$

13) If $\overrightarrow{A} = (k, 0, 0)$, $\hat{i} = (1, 0, 0)$, prove that $\| \overrightarrow{A} \| = |k| \, \| \hat{i} \|$



14) If the tension force in a stringe equals 21 newtons , fine the algebraic components of the force $\overrightarrow{F}$ in the directions of the Cartesian axes

15) **Open question:** What can you say about the coordinates of the vector $\overrightarrow{A}$ if the vector $\overrightarrow{A}$ is parallel to y z-plane ?

16) **Open question:** If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors in R^3. Is $\| \overrightarrow{A} + \overrightarrow{B} \| = \| \overrightarrow{A} \| + \| \overrightarrow{B} \|$? Which side is greater if both sides are unequal □

17) **Creative thinking:** Find the Algebric form of the vector $\overrightarrow{A}$ if its norm is 5 units and makes equal angles with the +ve directions of the Cartesian axes .

Vectors multiplication

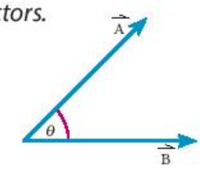
You have learned earlier how to make some operations on the vectors such as adding and multiplying a vector by scaler but you might ask yourself. Is it possible to multiply vectors? The answer is yes. There are two types of multiplying vectors; scalar product and vector product between two vectors. In this lesson, we will study the two types and their Algebraic and geometrical properties and their physical Applications to help you study mechanics.

You will learn

- Scalar product of two vectors in a plane and space
- Parallel and perpendicular vectors.
- The angle between two vectors
- The component of a vector in the direction of another vector
- The projection of a vector in the direction of another vector
- The cross product of two vectors in a plane and space
- The geometrical meaning of the cross product
- The right hand system of unit vectors
- The scaler triple product
- The geometrical meaning of scalar triple product

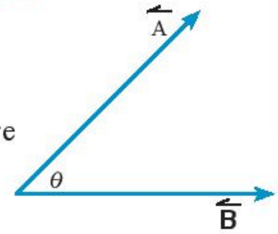
Key terms

- Scalar product
- vector product
- Component
- Unit vector
- Work
- Right hand rule
- Scalar triple product
- Scala product of two vectors.



Scalar product of two vectors (Dot product)

Think and discuss

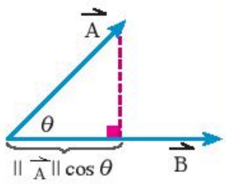


If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors and θ is the measure angle between them, find:

1- the component of $\overrightarrow{A}$ in the direction of $\overrightarrow{B}$.

2- the product of the norm of vector $\overrightarrow{B}$ and the component of $\overrightarrow{A}$ in the direction of vector $\overrightarrow{B}$.

From Think and discuss, we conclude that:



1- The component of the vector $\overrightarrow{A}$ in the direction of vector $\overrightarrow{B} = \| \overrightarrow{A} \| \cos\theta$

2- The product of the norm of vector $\overrightarrow{B}$ and the component of vector $\overrightarrow{A}$ in the direction of vector $\overrightarrow{B}$ equals $\| \overrightarrow{B} \| \, \| \overrightarrow{A} \| \cos\theta$

The absolute value of this quantity represents the area of a rectangle whose dimensions are the norm of the vector $\overrightarrow{B}$ and the component of vector $\overrightarrow{A}$ in the direction of vector $\overrightarrow{B}$.

Learn

The scalar product of two vectors (Dot product)

If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors and the measure of the angle between them is θ, then the area of the rectangle whose dimensions are the norm of one of them and the component of the other vector on it is known as the scaler product of the two vectors denoted by $\overrightarrow{A} \cdot \overrightarrow{B}$

i.e. $\overrightarrow{A} \cdot \overrightarrow{B} = \|\overrightarrow{A}\| \ \|\overrightarrow{B}\| \cos\theta$

Example

1. If $\overrightarrow{A}$ and $\overrightarrow{B}$ two vectors of the angle between them $60°$ and $\|\overrightarrow{A}\| = 2\|\overrightarrow{B}\| = 8$, find $\overrightarrow{A} \cdot \overrightarrow{B}$

Solution

$\|\overrightarrow{A}\| = 2\|\overrightarrow{B}\| = 8 \longrightarrow \|\overrightarrow{A}\| = 8 , \|\overrightarrow{B}\| = 4$

from the definition of the scaler product

$\overrightarrow{A} \cdot \overrightarrow{B} = \|\overrightarrow{A}\| \ \|\overrightarrow{B}\| \cos\theta$

$= 8 \times 4 \times \cos 60° = 16$

Try to solve

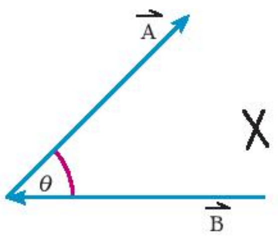
1. If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors and the measure of the angle between them is $135°$, $\|\overrightarrow{A}\| = 6$, $\|\overrightarrow{A}\| = 10$, find $\overrightarrow{A} \cdot \overrightarrow{B}$

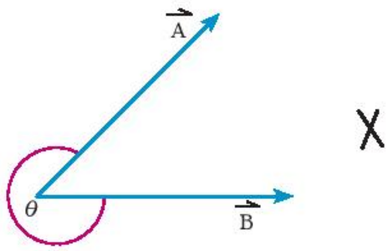
Critical thinking: What are the cases at which the value of the scaler product is equal to zero?

Important notes

1- To identify the angle between two vectors, the two vectors must be both getting in or out of the common point.

2- The measure of the angle between the two vectors $\in [0 , \pi]$





Example

2. If $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit vectors of a right hand system, find $\hat{i} \cdot \hat{i}$, $\hat{j} \cdot \hat{j}$, $\hat{k} \cdot \hat{k}$

Solution

$\hat{i} \cdot \hat{i} = \|\hat{i}\| \ \|\hat{i}\| \cos 0°$

$= 1 \times 1 \times 1 = 1$

similar $\hat{j} \cdot \hat{j} = 1$, $\hat{k} \cdot \hat{k} = 1$

Remember

The norm of the unit vector equals one.

Try to solve

2. If $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit vectors of a right hand system, find $\hat{i} \cdot \hat{j}$, $\hat{j} \cdot \hat{k}$, $\hat{k} \cdot \hat{i}$

Properties of the scaler product

From the previous example, we can conclude the properties of the scaler product:

1- $\overrightarrow{A} \cdot \overrightarrow{B} = \overrightarrow{B} \cdot \overrightarrow{A}$ — **Commutative property**

2- $\overrightarrow{A} \cdot \overrightarrow{A} = \| \overrightarrow{A} \|^2$

3- $\overrightarrow{A} \cdot \overrightarrow{B} = 0$ — **If and only if $\overrightarrow{A}$, $\overrightarrow{B}$ are perpendicular (condition of perpendicularity)**

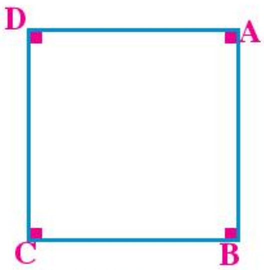
4- $\overrightarrow{A} \cdot (\overrightarrow{B} + \overrightarrow{C}) = \overrightarrow{A} \cdot \overrightarrow{B} + \overrightarrow{A} \cdot \overrightarrow{C}$ — **distributive property**

5- $(k\overrightarrow{A}) \cdot \overrightarrow{B} = \overrightarrow{A} \cdot (k\overrightarrow{B}) = k(\overrightarrow{A} \cdot \overrightarrow{B})$ — **K is a real number**

Example

(3) ABCD is a square with side length 10cm, find

(a) $\overrightarrow{AB} \cdot \overrightarrow{DC}$ (b) $\overrightarrow{AB} \cdot \overrightarrow{BC}$ (c) $\overrightarrow{AB} \cdot \overrightarrow{CA}$



Solution

(a) $\because$ $\overrightarrow{AB}$, $\overrightarrow{DC}$ are parallel and on the same direction

$\therefore$ the angle between them is = zero°

$\therefore \overrightarrow{AB} \cdot \overrightarrow{DC} = \|\overrightarrow{AB}\| \; \|\overrightarrow{DC}\| \cos 0°$

$= 10 \times 10 \times 1 = 100$

(b) $\overrightarrow{AB}$, $\overrightarrow{BC}$ are perpendicular **the measure of the angle between them is 90°**

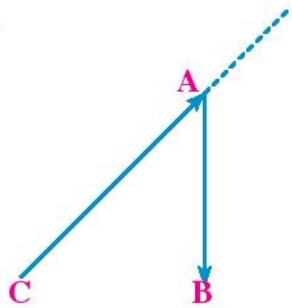
$\therefore \overrightarrow{AB} \cdot \overrightarrow{BC}$ = zero

(c) $\because$ $\overrightarrow{AB}$, $\overrightarrow{CA}$ **do not start at the same point**

$\therefore$ extend $\overrightarrow{CA}$ so the measure of the angle between them is 135°

$\overrightarrow{AB} \cdot \overrightarrow{CA} = \|\overrightarrow{AB}\| \, \|\overrightarrow{CA}\| \cos 135^o$

$= 10 \times 10\sqrt{2} \times \frac{-1}{\sqrt{2}} = -100$



Another solution (c)

$\overrightarrow{AB} \cdot \overrightarrow{CA} = \overrightarrow{AB} \cdot (-\overrightarrow{AC})$

$= -\overrightarrow{AB} \cdot \overrightarrow{AC}$

$= -\|\overrightarrow{AB}\| \, \|\overrightarrow{CA}\| \cos 45$

$= -10 \times 10\sqrt{2} \times \frac{1}{\sqrt{2}} = -100$

Try to solve

(3) ABC is an equilateral triangle with side length 8 cm, find each of:

(a) $\overrightarrow{AB} \cdot \overrightarrow{AC}$ (b) $\overrightarrow{AB} \cdot \overrightarrow{BC}$ (c) $(2\overrightarrow{AC}) \cdot (3\overrightarrow{CB})$

Learn

The scalar product of two vectors in an orthogonal coordindte system

If $\overrightarrow{A} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}), B = (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$ **then**

$\overrightarrow{A} \cdot \overrightarrow{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$ **using the distribution property**

$$= A_x B_x \hat{i} \cdot \hat{i} + A_x B_y \hat{i} \cdot \hat{j} + A_x B_z \hat{i} \cdot \hat{k}$$
$$+ A_y B_x \hat{j} \cdot \hat{i} + A_y B_y \hat{j} \cdot \hat{j} + A_y B_z \hat{j} \cdot \hat{k}$$
$$+ A_z B_x \hat{k} \cdot \hat{i} + A_z B_y \hat{k} \cdot \hat{j} + A_z B_z \hat{k} \cdot \hat{k}$$

$\because \quad \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

$\therefore \overrightarrow{A} \cdot \overrightarrow{B} = A_x B_x + A_y B_y + A_z B_z$

> **Note**
> **If** $\overrightarrow{A} = (A_x, A_y)$, $\overrightarrow{B} = (B_x, B_y)$ in the coordinates plane,
> **then** $\overrightarrow{A} \cdot \overrightarrow{B} = A_xB_x + A_yB_y$

Example

4 If $\overrightarrow{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\overrightarrow{B} = -\hat{i} - 2\hat{j} + \hat{k}$, find $\overrightarrow{A} \cdot \overrightarrow{B}$

Solution

$\overrightarrow{A} \cdot \overrightarrow{B} = (2, 3, 4) \cdot (-1, -2, 1)$
$= 2 \times -1 + 3 \times (-2) + 4 \times 1$
$= -2 - 6 + 4 = -4$

Try to solve

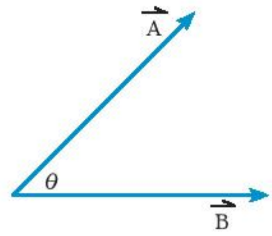
4 Find $\overrightarrow{A} \cdot \overrightarrow{B}$ in each of the following:

a $\overrightarrow{A} = (-1, 3, 2)$, $\overrightarrow{B} = 4\hat{i} - 2\hat{j} + 5\hat{k}$ What do you deduce?

b $\overrightarrow{A} = 2\hat{i} - \hat{j}$, $\overrightarrow{B} = \hat{j} - 3\hat{i}$

Learn

The angle between two vectors



You know that $\overrightarrow{A} \cdot \overrightarrow{B} = \|\overrightarrow{A}\| \, \|\overrightarrow{B}\| \cos\theta$

where θ is the measure of angle between two non zero vectors

$\overrightarrow{A}$, $\overrightarrow{B}$, $0^\circ \leqslant \theta \leqslant 180^\circ$

$$\therefore \cos\theta = \frac{\overrightarrow{A} \cdot \overrightarrow{B}}{\|\overrightarrow{A}\| \, \|\overrightarrow{B}\|}$$

Speical cases:

1- **If** $\cos\theta = 1$, **then** $\overrightarrow{A}$, $\overrightarrow{B}$ are parallel and in the same direction.

2- **If** $\cos\theta = -1$, **then** $\overrightarrow{A}$, $\overrightarrow{B}$ are parallel and in the opposite direction.

3- **If** $\cos\theta = 0$, **then** $\overrightarrow{A}$, $\overrightarrow{B}$ are perpendicular.

Example

(5) Find measure of the angle between the two vectors
$\overrightarrow{A} = 4\hat{i} + 3\hat{j} + 7\hat{k}$, $\overrightarrow{B} = 2\hat{i} + 5\hat{j} + 4\hat{k}$.

Solution

$\|\overrightarrow{A}\| = \sqrt{4^2 + 3^2 + 7^2} = \sqrt{74}$

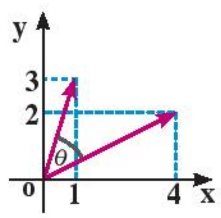
$\|\overrightarrow{B}\| = \sqrt{2^2 + 5^2 + 4^2} = \sqrt{45}$

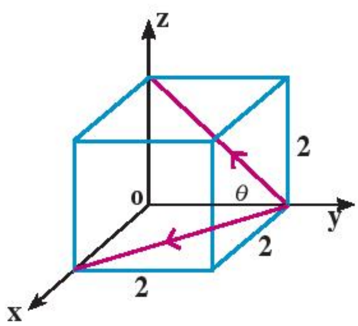
$$\cos\theta = \frac{\overrightarrow{A} \cdot \overrightarrow{B}}{\|\overrightarrow{A}\| \, \|\overrightarrow{B}\|} = \frac{(4, 3, 7) \cdot (2, 5, 4)}{\sqrt{74}\sqrt{45}} = \frac{8 + 15 + 28}{\sqrt{74}\sqrt{45}} = \frac{51}{\sqrt{74}\sqrt{45}}$$

$$\therefore \cos^{-1}\left(\frac{51}{\sqrt{74}\sqrt{45}}\right) = \cos^{-1}(0.8838) = 27.9^\circ$$

Try to solve

(5) Find θ in each of the following:



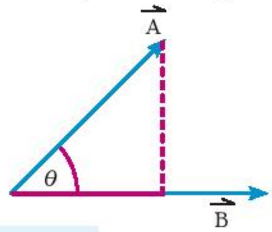


Learn

The componont (projection) of a vector in the direction of another vector.

If $\overrightarrow{A}$ and $\overrightarrow{B}$ are vectors, **then** the component of vector $\overrightarrow{A}$ is in the direction $\overrightarrow{B}$ (denoted A_B) is

$$A_B = \|\overrightarrow{A}\| \cos\theta = \frac{\overrightarrow{A} \cdot \overrightarrow{B}}{\|\overrightarrow{B}\|}$$



Note

the component of vector $\overrightarrow{A}$ is in the direction $\overrightarrow{A_B}$ (denoted $\overrightarrow{A_B}$) by

$$\overrightarrow{A_B} = \frac{\overrightarrow{B}}{\|\overrightarrow{B}\|}$$

Example

(6) Find the component of the force $\overrightarrow{F} = 2\hat{i} - 3\hat{j} + 5\hat{k}$ in the direction $\overrightarrow{AB}$ where A(1, 4, 0), B(-1, 2, 3)

Solution

$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}$

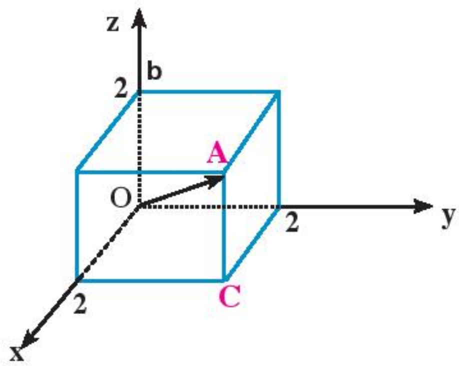
$= (-1, 2, 3) - (1, 4, 0) = (-2, -2, 3)$

The component of the force $\overrightarrow{F}$ in the direction of $\overrightarrow{AB} = \dfrac{\overrightarrow{F} \cdot \overrightarrow{AB}}{\|\overrightarrow{AB}\|}$

$$= \frac{(2, -3, 5) \cdot (-2, -2, 3)}{\sqrt{(-2)^2 + (-2)^2 + {}^2 3}} = \frac{17}{\sqrt{17}} = \sqrt{17}$$

Try to solve

(6) The opposite figure represents a cube with side length 2 length unit, find the projection of $\overrightarrow{OA}$ on the vector $\overrightarrow{CB}$



Critical thinking: when does the component of a vector in a direction of another vector vanish?

Learn

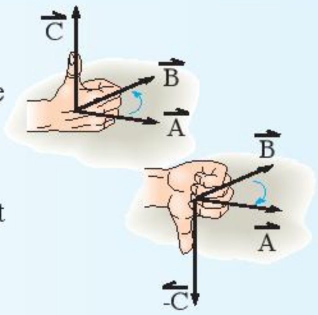
Vector product of two vectors (cross product)

If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors in a plane enclosing an angle of measure θ and $\overrightarrow{C}$ is a unit vector perpendicular to the plane which contains $\overrightarrow{A}$ and $\overrightarrow{B}$, **then** the cross product of $\overrightarrow{A}$ and $\overrightarrow{B}$ is given by the relation

$$\overrightarrow{A} \times \overrightarrow{B} = (\| \overrightarrow{A} \| \, \| \overrightarrow{B} \| \sin \theta) \, \overrightarrow{C}$$

The direction of the unit vector $\overrightarrow{C}$ is defined (up or down) According to the right hand rule where the curved fingers of the right hand show the direction of the relation from the vector $\overrightarrow{A}$ to the vector $\overrightarrow{B}$, then the thump shows the direction of the vector $\overrightarrow{C}$

Important notes



1- **If** $\overrightarrow{A} \times \overrightarrow{B}$ is in the direction of $\overrightarrow{C}$, then $\overrightarrow{B} \times \overrightarrow{A}$ is in the direction of $-\overrightarrow{C}$ **so** $\overrightarrow{A} \times \overrightarrow{B} = - \overrightarrow{B} \times \overrightarrow{A}$

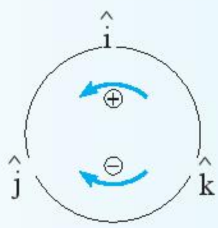
2- By applying the right hand rule on the right set of orthogonal unit vectors **then**

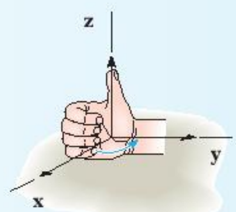
$\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$

$\hat{i} \times \hat{k} = -\hat{j}$, $\hat{k} \times \hat{j} = -\hat{i}$

$\hat{j} \times \hat{i} = -\hat{k}$





3- for any vector $\overrightarrow{A}$, $\overrightarrow{A} \times \overrightarrow{A} = \overrightarrow{O}$ Where $\overrightarrow{O}$ is the zero vector.

Example

(7) $\overrightarrow{A}$ and $\overrightarrow{B}$ Are two vectors in a plane and the measure of angle between them is 70°.

If $\| \overrightarrow{A} \| = 15$, $\| \overrightarrow{B} \| = 17.5$, find the norm of $\overrightarrow{A} \times \overrightarrow{B}$

Solution

$\because \overrightarrow{A} \times \overrightarrow{B} = (\| \overrightarrow{A} \| \, \| \overrightarrow{B} \| \sin \theta) \, \overrightarrow{C}$

$\therefore \| \overrightarrow{A} \times \overrightarrow{B} \| = \| \overrightarrow{A} \| \, \| \overrightarrow{B} \| \sin \theta = 15 \times 17.5 \times \sin 70 = 246.67$

Try to solve

7) If $\overrightarrow{A} \times \overrightarrow{B} = -65\ \overrightarrow{C}$ and $\|\overrightarrow{A}\| = 5$, $\|\overrightarrow{B}\| = 26$, find the measure of the angle between $\overrightarrow{A}$ and $\overrightarrow{B}$

The cross production in Cartisian coordinates

If $\overrightarrow{A} = (A_x, A_y, A_Z)$, $\overrightarrow{B} = (B_x, B_y, B_Z)$ two vectors, **then**

$$\overrightarrow{A} \times \overrightarrow{B} = (A_x \hat{i} + A_y \hat{j} + A_Z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_Z \hat{k})$$

$$= A_x B_x \hat{i} \times \hat{i} + A_x B_y \hat{i} \times \hat{j} + A_x B_Z \hat{i} \times \hat{k}$$

$$+ A_y B_x \hat{j} \times \hat{i} + A_y B_y \hat{j} \times \hat{j} + A_y B_Z \hat{j} \times \hat{k}$$

$$+ A_Z B_x \hat{k} \times \hat{i} + A_Z B_y \hat{k} \times \hat{j} + A_Z B_Z \hat{k} \times \hat{k}$$

where $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \overrightarrow{O}$

$\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$, then

$$\overrightarrow{A} \times \overrightarrow{B} = 0 + A_x B_y \hat{k} + A_x B_Z (-\hat{j})$$

$$+ A_y B_x (-\hat{k}) + 0 + A_y B_Z (\hat{i})$$

$$+ A_Z B_x (\hat{j}) + A_Z B_y (-\hat{i}) + 0$$

$$= (A_y B_Z - A_Z B_y) \hat{i} + (A_Z B_x - A_x B_Z) \hat{j} + (A_x B_y - A_y B_x) \hat{k}$$

The last form can be written in a form of determinant of order 3×3 as follows:

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Spiecial case

If $\overrightarrow{A} = (A_x , A_y)$, $\overleftarrow{B} = (B_x , B_y)$ in the xy-plane, **then**

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & 0 \\ B_x & B_y & 0 \end{vmatrix} = (A_x B_y - A_y B_x) \hat{k}$$

Example

8) If $\overrightarrow{A} = (-2, 3, 1)$, $\overrightarrow{B} = (1, 2, 4)$ find $\overrightarrow{A} \times \overrightarrow{B}$, then deduce the unit vector perpendicular to the plane which contains the two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$

Solution

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= (3 \times 4 - 2 \times 1)\ \hat{i}\ - (-2 \times 4 - 1 \times 1)\ \hat{j}\ + (-2 \times 2 - 3 \times 1)\ \hat{k}$$

$$= 10\ \hat{i}\ + 9\ \hat{j}\ - 7\ \hat{k}$$

The perpendicular unit vector on the plane of $\overrightarrow{A}$, $\overrightarrow{B}$ $= \dfrac{\overrightarrow{A} \times \overrightarrow{B}}{\| \overrightarrow{A} \times \overrightarrow{B} \|}$

$$\therefore\ \overrightarrow{C} = \frac{10\ \hat{i}\ + 9\ \hat{j}\ - 7\ \hat{k}}{\sqrt{10^2 + 9^2 + (-7)^2}} = \frac{10}{\sqrt{230}}\ \hat{i}\ + \frac{9}{\sqrt{230}}\ \hat{j}\ - \frac{7}{\sqrt{230}}\ \hat{k}$$

Try to solve

8) If $\| \overrightarrow{A} \| = 6$ and the direction cosines of the vector $\overrightarrow{A}$ are $\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}$ respectively and $\overrightarrow{B} = (-2, 3, 5)$, find $\overrightarrow{A} \times \overrightarrow{B}$

Properties of the vector product of two vectors

If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors, θ is the measure of the angle between them, then:

1- $\overrightarrow{A} \times \overrightarrow{B} = -\overrightarrow{B} \times \overrightarrow{A}$ **(the cross product is not commutative)**

2- $\overrightarrow{A} \times \overrightarrow{A} = \overrightarrow{B} \times \overrightarrow{B} = \overrightarrow{O}$

3- If $\overrightarrow{A} \times \overrightarrow{B} = \overrightarrow{O}$, then either $\overrightarrow{A} \mathbin{//} \overrightarrow{B}$ or one of them or both is equal to $\overrightarrow{O}$

4- $\overrightarrow{A} \times (\overrightarrow{B} + \overrightarrow{C}) = (\overrightarrow{A} \times \overrightarrow{B}) + (\overrightarrow{A} \times \overrightarrow{C})$ **distribution property**

5- $(k\overrightarrow{A}) \times \overrightarrow{B} = \overrightarrow{A} \times (k\overrightarrow{B}) = k(\overrightarrow{A} \times \overrightarrow{B})$ **where k is a real number**

Parallelism of two vectors

We saw in the properties of the cross multiplication that $\overrightarrow{A}$ and $\overrightarrow{B}$ are parallel if and only if: $\overrightarrow{A} \times \overrightarrow{B} = \overrightarrow{O}$

So $(A_y B_Z - A_Z B_y)\ \hat{i} + (A_Z B_x - A_x B_Z)\ \hat{j} + (A_x B_y - A_y B_x)\ \hat{k} = \overleftarrow{O}$

So $A_y B_Z = A_Z B_y$, $A_x B_Z = A_Z B_x$, $A_x B_y = A_y B_x$

So $\frac{A_y}{B_y} = \frac{A_z}{B_z}$, $\frac{A_x}{B_x} = \frac{A_z}{B_z}$, $\frac{A_x}{B_x} = \frac{A_y}{B_y}$

So $\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$

let any of the rations = K, then

$A_x = k B_x, A_y = k B_y , A_z = k B_z$

$\because \overrightarrow{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

$\therefore \overrightarrow{A} = k \overrightarrow{B}$

When $k > 0$, the two vectors are parallel and in the same direction and when $k < 0$, the two vectors are parallel and in opposite directions.

Example

9) If the vector $\overrightarrow{A} = 2\hat{i} - 3\hat{j} + m\hat{k}$ is parallel to vector $\overrightarrow{B} = \hat{i} + k\hat{j} + 8\hat{k}$, find value of m, k

Solution

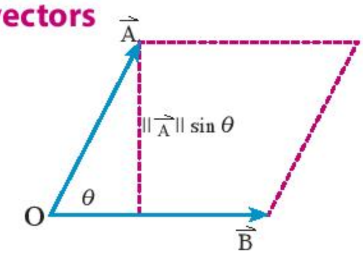
$\because \overrightarrow{A} \parallel \overrightarrow{B}$ $\therefore \frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$

$\therefore \frac{2}{1} = \frac{-3}{k} = \frac{m}{8}$ $\therefore k = \frac{1 \times -3}{2} = \frac{-3}{2}$, $m = \frac{2 \times 8}{1} = 16$

Try to solve

9) If $\overrightarrow{A} = (2, -3)$, $\overrightarrow{B} \parallel \overrightarrow{A}$ and if $\|\overrightarrow{B}\| = 3\sqrt{13}$, find $\overrightarrow{B}$.

The Geometrical meaning of the vector product of two vectors



We know that $\|\overrightarrow{A} \times \overrightarrow{B}\| = \|\overrightarrow{A}\|\,\|\overrightarrow{B}\| \sin\theta$

$= \|\overrightarrow{B}\| \times L$

where $L = \|\overrightarrow{A}\| \sin\theta$

= The area of the parallelogram in which $\overrightarrow{B}$ and $\overrightarrow{A}$ are two adjacent sides

= double the area of triangle in which $\overrightarrow{B}$ and $\overrightarrow{A}$ are two adjacent sides

Example

10 If $\overrightarrow{A} = (-3, 1, 2)$, $\overrightarrow{B} = (3, 4, -1)$, find the area of the parallelogram in which $\overrightarrow{A}$ and $\overrightarrow{B}$ are two adjacent sides.

Solution

$$\overrightarrow{A} \times \overrightarrow{B} = (-3, 1, 2) \times (3, 4, -1)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & 2 \\ 3 & 4 & -1 \end{vmatrix} = (-9)\,\hat{i} + 3\,\hat{j} - 15\,\hat{k}$$

$$\| \overrightarrow{A} \times \overrightarrow{B} \| = \sqrt{(-9)^2 + (3)^2 + (-15)^2} = 3\sqrt{35}$$

$\therefore$ the area of the parallelogram $= 3\sqrt{35}$ unit area.

Try to solve

10 If $\overrightarrow{A} = (1, 2, -4)$, $\overrightarrow{B} = (0, 5, -1)$, find the area of the triangle in which $\overrightarrow{A}$, $\overrightarrow{B}$ are two sides.

Learn

The scalar triple product

If $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$ are vectors, then the expression $\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C}$ is known as the scalar triple product which has a lot of applications in the statices filed)

(notice that the expression has no brackets where doing the scalar product first is meaningless)

let $\overrightarrow{A} = (A_x, A_y, A_z)$, $\overrightarrow{B} = (B_x, B_y, B_z)$, $\overrightarrow{C} = (C_x, C_y, C_z)$

then $\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C}$

$$= (A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= (A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}) \cdot [(B_y C_z - B_z C_y)\,\hat{i} - (B_x C_z - B_z C_x)\,\hat{j} + (B_x C_y - B_y C_x)\,\hat{k}\,]$$

$$A_x (B_y C_z - B_z C_y) - A_y (B_x C_z - B_z C_x) + A_z (B_x C_y - B_y C_x)$$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The properties of the scalat triple product

1- The value of scalar triple product does not change if the vectors are permuted in such a way that they are still read i the same cyclic order.

$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$

notice the cyclic order of $\vec{A}$, $\vec{B}$, $\vec{C}$

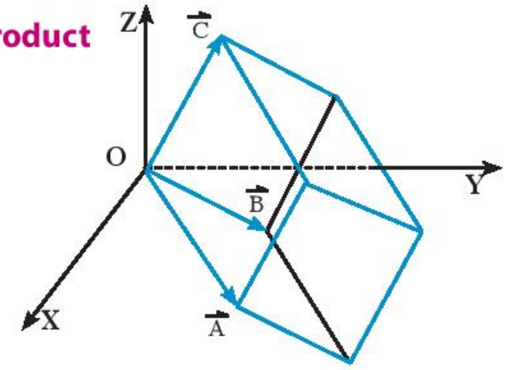
2- If the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ in the same plane, then the scalar triple product vanishes

So $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$

The geometrical meaning of the scalar triple product

If $\vec{A}$, $\vec{B}$, $\vec{C}$ are 3 vectors forming 3-non parallel sides of a parallelepiped, then the volume of the parallel piped = the absolute value of the scalar triple product.

Thus the volume of the parallelepiped = $|\vec{A} \cdot \vec{B} \times \vec{C}|$



Example

(11) Find the volume of the parallelepiped in which three adjacent sides are represented by the vectors $\vec{A}$ = (2, 1, 3), $\vec{B}$ = (-1 , 3 , 2), $\vec{C}$ = (1 , 1 , -2)

Solution

The volume of the parallelepiped = $|\vec{A} \cdot \vec{B} \times \vec{C}|$ **(1)**

$$\therefore \vec{A} \cdot \vec{B} \times \vec{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & 2 \\ 1 & 1 & -2 \end{vmatrix} = -28$$

then the volume of the parallelepiped= |-28| = 28 unit volume

Try to solve

(11) Find the volume of the parallelepiped in which three- non parallel edges are represented by the vectors $\vec{A}$ = (3 , -4 , 1), $\vec{B}$ = (0 , 2 , -3) , $\vec{C}$ = (3 , 2, 2)

Exercises 3 - 3

Complete the following: If $\hat{i}$, $\hat{j}$, $\hat{k}$ form a right hand system of unit vectors :

1. $\hat{i} \cdot \hat{j} =$ ....

2. $\hat{j} \times \hat{k} =$ ....

3. If $\overrightarrow{A} = (2, -1)$, $\overrightarrow{B} = (3, -4)$, then the component of $\overrightarrow{A}$ in the direction of $\overrightarrow{B}$ equals ....

4. If $\overrightarrow{A}$, $\overrightarrow{B}$ are non zero vectors and $\overrightarrow{A} \cdot \overrightarrow{B} = 0$, then $\overrightarrow{A}$, $\overrightarrow{B}$ are ....

5. If $\overrightarrow{A}$, $\overrightarrow{B}$ are non zero vectors and $\overrightarrow{A} \times \overrightarrow{B} = \overrightarrow{O}$, then $\overrightarrow{A}$, $\overrightarrow{B}$ are ....

6. The measure of the angle between the two vectors $3\hat{i} - \hat{j}$, $-4\hat{i} + 6\hat{j}$ equals ....

Choose the correct answer from the following:

7. $\hat{i} \times \hat{j} =$ ....
   - a) $\overrightarrow{O}$
   - b) 0
   - c) 1
   - d) $\hat{k}$

8. If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two perpendicular unit vectors, then $(\overrightarrow{A} - 2\overrightarrow{B}) \cdot (3\overrightarrow{A} + 5\overrightarrow{B}) =$
   - a) -8
   - b) -7
   - c) 24
   - d) 0

9. If $\overrightarrow{A}$ and $\overrightarrow{B}$ are unit vectors, then $\overrightarrow{A} \cdot \overrightarrow{B} \in$ ....
   - a) $]0, 1[$
   - b) $]-1, 1[$
   - c) $[-1, 1]$
   - d) R^+

10. The measure of the angle between the two vectors (2 , -2, 2) , (1, 1, 4) is ....
    - a) 57.02°
    - b) 35.26°
    - c) 134.37°
    - d) 0°

11. If the vectors (2 ,k, -3) , (4 , 6, -6) are parallel, then k = ...

a) 6
b) 3
c) -3
d) 1

Answer the following:

12. Find $\overrightarrow{A} \cdot \overrightarrow{B}$ in each of the following:

a) $\overrightarrow{A} = (5, 1, -2)$, $\overrightarrow{B} = (4, -4, 3)$
b) $\overrightarrow{A} = -3\hat{i} - 2\hat{j} - \hat{k}$, $\overrightarrow{B} = 6\hat{i} + 4\hat{j} + 2\hat{k}$
c) $\overrightarrow{A} = \hat{i}$, $\overrightarrow{B} = 2\hat{j} - \hat{k}$

13. Find the measure of the angle between the two vectors in each of the following:

a) (5 , 1, -2) , (1 , 1, -1)
b) (7 , 2 , -10) , (2 , 6 , 4)
c) (2 , 1 , 4) , (1, -2, 0)

14. Find $\overrightarrow{A} \times \overrightarrow{B}$ in each of the following:

a) $\overrightarrow{A} = (-2, 3, 1)$, $\overrightarrow{B} = (1, 3, -4)$
b) $\overrightarrow{A} = -\hat{i} - 2\hat{j}$, $\overrightarrow{B} = 3\hat{j} - 5\hat{k}$
c) $\|\overrightarrow{A}\| = 6$, $\|\overrightarrow{B}\| = 8$ the angle between them is 60°

15. ABCD is a square with side length 12cm. $\hat{e}$ is the unit vector perpendicular to its plane, find:

a) $\overrightarrow{AB} \cdot \overrightarrow{AC}$
b) $\overrightarrow{AB} \times \overrightarrow{CA}$
c) $\overrightarrow{BC} \cdot \overrightarrow{AD}$
d) $\overrightarrow{BD} \times \overrightarrow{AC}$
e) $\overrightarrow{AB} \cdot \overrightarrow{BC}$
f) $\overrightarrow{AB} \times \overrightarrow{BC}$

16. Find the unit vector perpendicular to the plane which contains the two vectors.
$\overrightarrow{A} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\overrightarrow{B} = -\hat{i} + 2\hat{j} - 3\hat{k}$

17. Calculate the area of the triangle DEF in each of the following:

a) D(5 , 1 , -2) , E (4 , -4 , 3) , F (2 , 4 , 0)
b) D (4 , 0 , 2) , E (2 , 1 , 5) , F (-1 , 0 , -1)

18. Calculate the area of the parallelogram LMNE in each of the following :

a) L (1 , 1) , M (2 , 3) , N(5 , 4)
b) L (2 , 1 , 3) , M (1 , 4 , 5) , N(2 , 5 , 3)

19 Find volume of the parallelepiped in which $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$ are three adjacent edges:

$\overrightarrow{A} = (1 , 1 , 3)$, $\overrightarrow{B} = (2 , 1 , 4)$, $\overrightarrow{C} = (5 , 1 , -2)$

20 In each of the following, show whether the two given vectors are parallel or perpendicular or otherwise:

a) $\overrightarrow{A} = (0 , 2, 2)$, $\overrightarrow{B} = (3 , 0 , -4)$

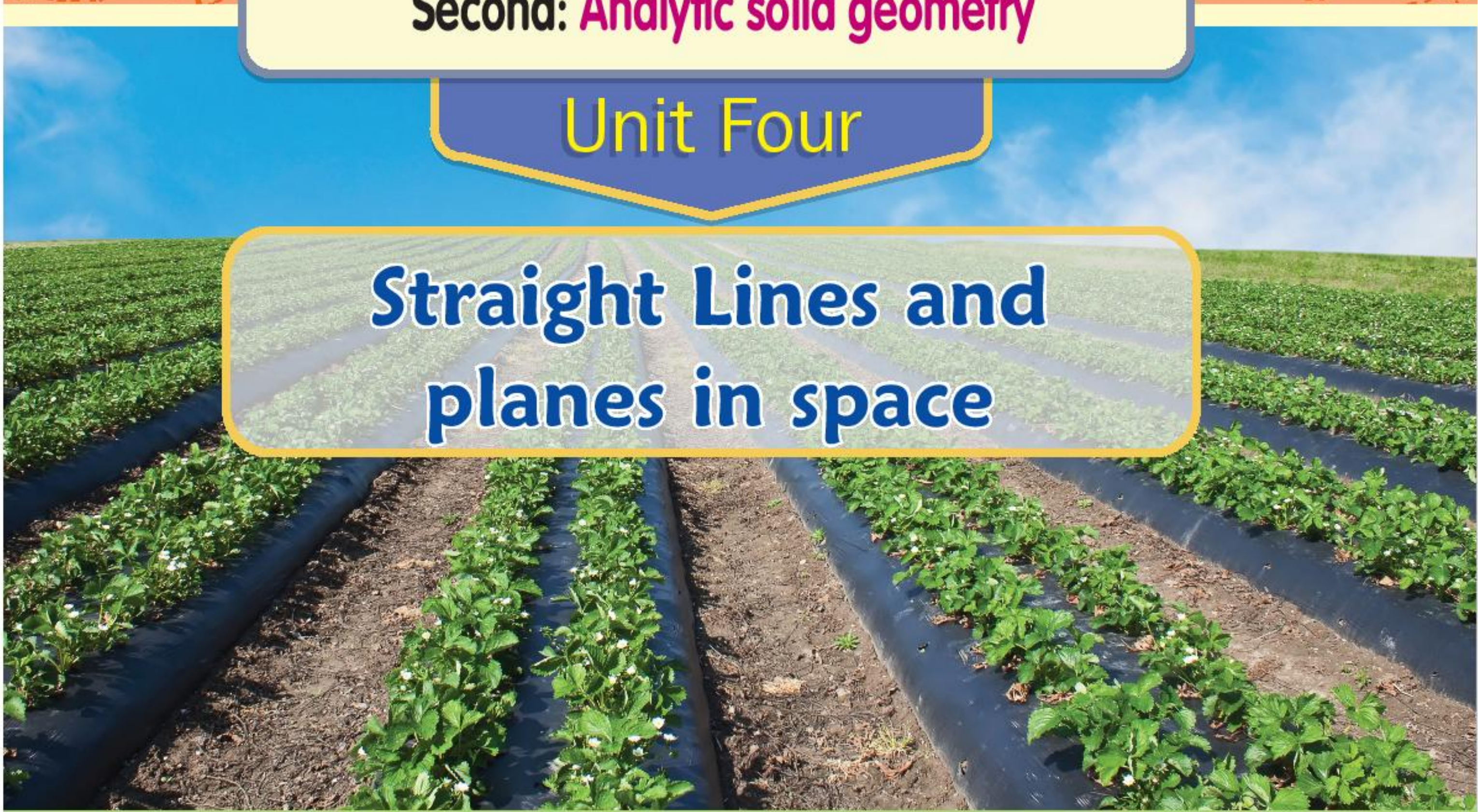
b) $\overrightarrow{E} = 10 \hat{i} + 40 \hat{j}$, $\overrightarrow{F} = -3 \hat{j} + 8 \hat{k}$

c) $\overrightarrow{A} = -2 \hat{i} + \hat{j} - 2 \hat{k}$, $\overrightarrow{B} = 8 \hat{i} - 4 \hat{j} + 8 \hat{k}$

Second: Analytic solid geometry

Unit Four

Straight Lines and planes in space



Unit introduction

In the previous unit , you learned how to identify a point in space and the position vectors and how to find their norms. These topics are fundamental in this unit since they are complementaries to what you learned in the previous unit and to what you learned last year. In this unit, you are going to learn the equation of the straight line in space and the equation of a plane in its different forms. The examples and how to solve them have been varied in order to fulfill the cognitive and skillful objectives which help the student learn the other concepts and knowledge related to the solid geometry in the next educational stages.

Unit obects

By the end of this unit and doing all activities included , the student should be able to:

- Find the direction vector of a straight line in space
- Find the parametric equations and the vector equation of a straight line in space
- Find the cartesian equation of a straight line in space
- Find the general equation of a plane in space
- Find the standard form of the equation of a plane in space
- Recognize the angle between two planes in space
- Conclude the condition of perpendicularity of two planes in space
- Conclude the condition of parallelism of two planes in space
- Find the equation of the intersection line of two planes in space
- Find the distance between a point and a straight line in space
- Find the distance between a point and a plane using the scalar product and the cartesian form
- Identify the distance between two parallel planes

Key terms

- Direction vector
- Direction angles
- Direction cosines
- Direction ratios
- Vector equation
- Parametric equations
- Cartesian equation
- General equation
- Proportional
- Parallel straight Lines
- Perpendicular straight Lines
- Intersecting straight Lines
- Skew straight Lines
- Perpendicular distance
- plane
- Standard form
- Parallel planes
- Perpendicular planes
- Intersecting planes
- Angle

Materials

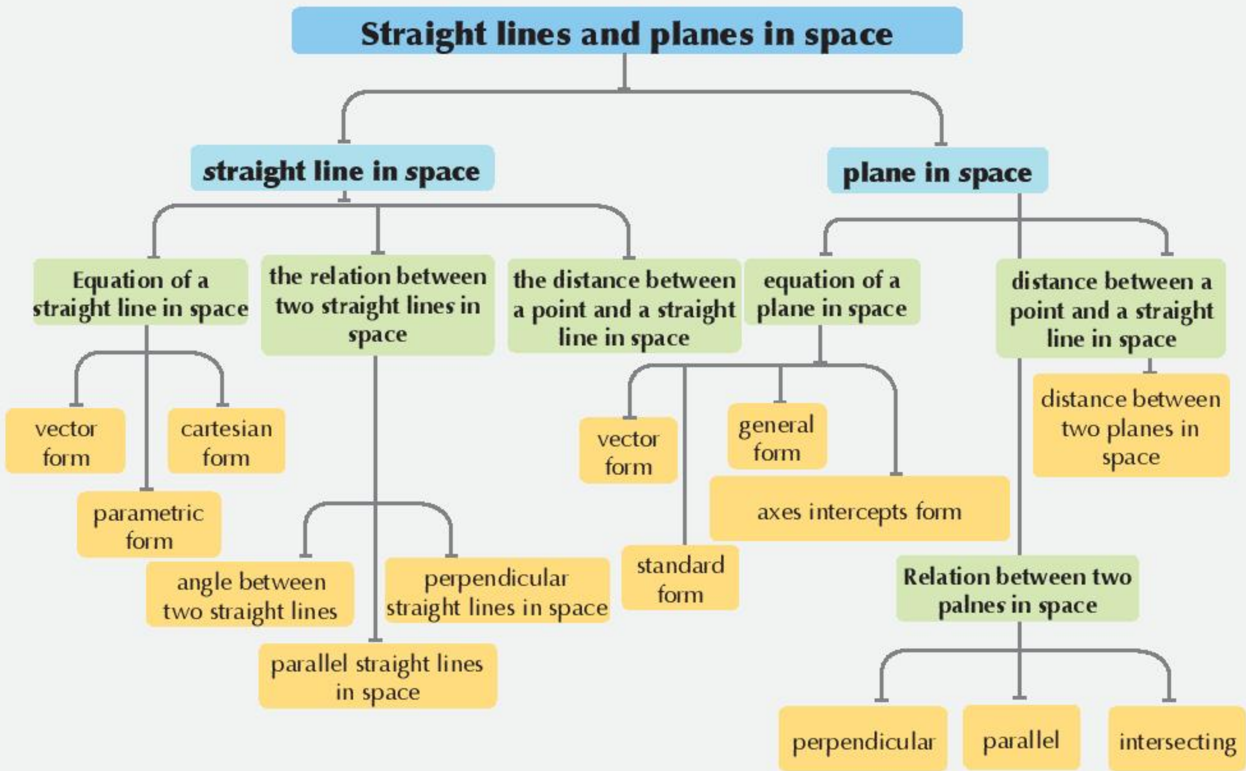
- Scientific calculator
- Computer 3D-program.

Unit Lessons

Lesson (4 - 1): The equation of a straight line in space.

Lesson (4 - 2): The equation of a plane in space.

Unit chart



Unit Four

4 - 1 Equation of a straight Line in space

You will learn

- Direction vector of a straight line
- Different forms of the equation of a straight line
- The angle between two straight lines
- The distance between a point and a straight line
- Parallel straight lines
- perpendicular straight lines

Key terms

- Direction vector
- Direction ratio
- Direction angles
- Cartesian equation
- Parametric equation

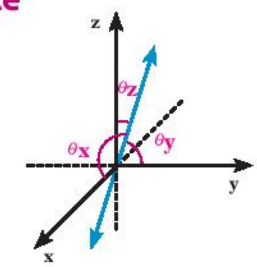
Materials

- Scientific calculator
- Computer 3-D program

You learned in the previous years the straight line in a plane and how to find the different forms of the equation of the straight line in a plane (vector -parametric-general) forms. In this lesson, we learn the straight line in space and how to get the equation of the straight line in space in different forms because it is much important in geometrical fields, architectural design and applications of space science.

Learn

Direction vector of a straight Line in space



If $\theta_x, \theta_y, \theta_z$ are the directed angles of a straight line in space, **then** $\cos\theta_x$, $\cos\theta_y$, $\cos\theta_z$ are directed cosines of these straight lines and they are usually denoted by ℓ , m , n.

$\ell = \cos\theta_x$, $m = \cos\theta_y$, $n = \cos\theta_z$

so $\ell^2 + m^2 + n^2 = 1$

so the vector $\overrightarrow{u} = \ell\,\hat{i} + m\,\hat{j} + n\,\hat{k}$ is the unit vector in the direction of the straight line.

And any vector parallel to the unit vector $\overrightarrow{u}$ is called the direction vector of the straight line and is denoted by $\overrightarrow{d}$

i.e. $\overrightarrow{d} = k(\ell\,\hat{i} + m\,\hat{j} + n\,\hat{k}) = (a, b, c)$

where a , b , c are proportional to ℓ , m , n , $k \in R^+$

a , b , c are called direction ratio.

for example: if $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ are the direction cosines of the straight line.

then the vector $\overrightarrow{d} = k(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ represents a direction vector of the straight line where $k \neq 0$

put $k = 3 \longrightarrow \overrightarrow{d} = (2, 1, 2)$

put $k = -6 \longrightarrow \overrightarrow{d} = (-4, -2, -4)$

i.e the straight line has infinite number of parallel direction vectors each is parallel to the straight line.

Example

(1) Find a direction vector of the straight line passing through A (-2 , 3, 1) , B (0 , 4 , -2)

Solution

A direction vector of the straight line $= \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (0 , 4 , -2) - (-2 , 3 , 1)$

$\therefore \overrightarrow{d} = (2 , 1 , -3)$

Try to solve

(1) Find a direction vector for each of the following straight lines:

(a) The straight line passing through the origin point and point (-1 , 2 , -2)

(b) The straight line passing through points C (0 , -2 , 3) and D (1 , 1 , -1)

Critical thinking:

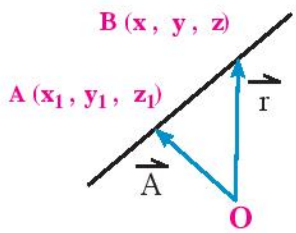
1- What can you say about the straight line with direction vector $\overrightarrow{d} = (a , b , 0)$?

2- Find a direction vector for each of the cartesian axes.

Learn

Vector form of the equation of a straight Line in space

If L is a straight line in space whose direction vector is $\overrightarrow{d} = (a , b , c)$ and passes through the point A whose position vector is $\overrightarrow{A} = (x_1, y_1, z_1)$ and if point B is any point on the straight line whose position vector is $\overrightarrow{r} = (x , y , z)$, **then**



From the figure: $\overrightarrow{r} = \overrightarrow{A} + \overrightarrow{AB}$

but $\overrightarrow{AB} \parallel \overrightarrow{d}$ $(\overrightarrow{AB} = t\,\overrightarrow{d})$

$\therefore$ $\overrightarrow{r} = \overrightarrow{A} + t\,\overrightarrow{d}$ ⟶ vector form of the equation of straight line.

Example

(2) Find the vector form of the equation of the straight line passing through point (3, -1, 0) and the vector (-2, 4 , 3) is a direction vector for it.

Solution

(3 , -1 , 0) **represents a point on the straight line** $\therefore \overrightarrow{A} = (3 , -1 , 0)$

(-2 , 4 , 3) **represents the direction vector of the straight line** $\therefore \overrightarrow{d} = (-2 , 4 , 3)$

the equation of the straight line is $\overrightarrow{r} = \overrightarrow{A} + t\,\overrightarrow{d}$

$\therefore \overrightarrow{r} = (3 , -1 , 0) + t(-2 , 4 , 3)$ ⟶ vector form of the equation of straight line.

Note: t is a real number does not express a unique constant number but it takes different real values and it is called a parameter and for each value of this parameter(t), we can find a point on the straight line.

for example: when t = 1 , **then** $\vec{r}$ = (1 , 3 , 3) represents the position vector of a point on the straight line .

and when t = 2 , **then** $\vec{r}$ = (-1 , 7 , 6) represents the position vector of another point on the straight line.

Try to solve

(2) Find the vector form of the equation of the straight line passing through point (4 , -2 , 5) and its direction vector is (1 , -2 , 2), then find another point on this straight line.

Learn

Parametric equations of a straight Line in space

From the vector equation of the straight line $\vec{r} = \vec{A} + t\,\vec{d}$

and by substituting for $\vec{r} = (x\ ,\ y\ ,\ z)$, $\vec{A} = (x_1\ ,\ y_1\ ,\ z_1)$, $\vec{d} = (a\ ,\ b\ ,\ c)$

then $(x\ ,\ y\ ,\ z) = (x_1\ ,\ y_1\ ,\ z_1) + t\,(a\ ,\ b,\ c)$

$\therefore$ $(x = x_1 + at\ ,\ y = y_1 + b\,t\ ,\ z = z_1 + c\,t)$ ⟶ the parametric equations of the straight line

Example

(3) Find the parametric equations for the straight line passing through point (2 , -1 , 3) and its direction vector is (4 , -2 , 5).

Solution

$\vec{r} = (2\ ,\ -1\ ,\ 3) + t\,(4\ ,\ -2\ ,\ 5)$ ⟶ **vector form of the equation of the straight line**

$\therefore\ (x\ ,\ y\ ,\ z) = (2\ ,\ -1\ ,\ 3) + t\,(4\ ,\ -2\ ,\ 5)$

$\therefore\ x = 2 + 4t$, $y = -1 - 2t$, $z = 3 + 5\,t$

Try to solve

(3) Find the parametric equations for the straight line passing through the origin point and its direction vector is (-2 , 3, 1).

Learn

The cartesian equation of a straight Line in space

From the parametric equations of the straight line

$x = x_1 + t\,a,\ y = y_1 + tb,\ z = z_1 + t\,c$ $\quad\therefore\ \frac{x - x_1}{a} = t\ ,\ \frac{y - y_1}{b} = t\ ,\ \frac{z - z_1}{c} = t$

$\therefore\ \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ ⟶ the cartesian form of the equation of the straight line.

Where each of a, b, c not equal to zero.

Note:

1- In the case a = 0 (say), then the cartesian form of the equation of the straight line takes the form of $x = x_1, \frac{y - y_1}{b} = \frac{z - z_1}{c}$

2- You learned in the previous years that the equation of the straight line in a plane is $a x + b y + c = 0$ and some people think that the equation of the straight line in space will be $a x + b y + c z + d = 0$ and that's a common mistake where the last equation represents the equation of a plane in space as we will see in the next lessons.

3- Since the direction ratios a , b , c are proportion to the direction cosine ℓ, m, n, then we can write the cartesian form of the equation of the straight line in the form

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Example

4 Find all the different forms of the equation of the straight line passing through the points (2 , -1 , 5) and (-3 , 1 , 4).

Solution

The direction vector of the straight line $\vec{d} = (-3 , 1 , 4) - (2 , -1 , 5) = (-5 , 2 , -1)$

∴ **The vector form of the equation of the straight line** $\vec{r} = (2 , -1 , 5) + t(-5 , 2 , -1)$

The parametric equations $x = 2 - 5t ,\ y = -1 + 2t ,\ z = 5 - t$

The Cartesian form $\frac{x - 2}{-5} = \frac{y + 1}{2} = \frac{z - 5}{-1}$

Try to solve

4 Find all the different forms of the equation of the straight line passing through point (3 , 2 , 0) and (-1 , 3 , 4)

Example

5 Find all the different forms of the equation of the straight line $\frac{3x + 1}{2} = \frac{y - 1}{2} = \frac{5 - z}{3}$

Solution

let $\frac{3x + 1}{2} = \frac{y - 1}{2} = \frac{5 - z}{3} = t$

∴ $\frac{3x + 1}{2} = t$, then $x = \frac{-1}{3} + \frac{2}{3}t$

$\frac{y - 1}{2} = t$, then $y = 1 + 2t$

$\frac{5 - z}{3} = t$, then $z = 5 - 3t$

} **The parametric form of the equation of the straight line**

from the parametric equation, we can write the equation

$$(x , y , z) = (-\frac{1}{3} , 1 , 5) + t(\frac{2}{3} , 2 , -3)$$

i.e $\vec{r} = (-\frac{1}{3} , 1 , 5) + t(\frac{2}{3} , 2 , -3)$ vector form

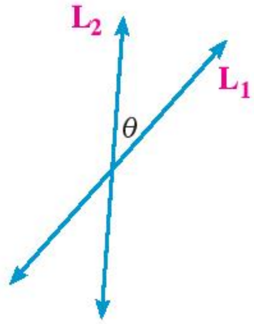
Notice that: the direction ratios of the straight line are $(\frac{2}{3} , 2 , -3)$ or (2 , 6 , -9)

Try to solve

5. Find all the different forms of the equation of the straight line $\frac{x+4}{3} = \frac{2y+5}{2} = \frac{4-z}{4}$, then find a point belongs to the straight line.

Learn

The angle between two straight lines in space



If L_1 , L_2 are two straight lines in space whose direction vectors are $\overrightarrow{d_1} = (a_1, b_1, c_1)$ and $\overrightarrow{d_2} = (a_2, b_2, c_2)$, then the smallest angle between the two straight lines L_1, L_2 is given by the relation:

$$\cos\theta = \frac{|\overrightarrow{d_1} \cdot \overrightarrow{d_2}|}{\|\overrightarrow{d_1}\| \, \|\overrightarrow{d_2}\|}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

and if (ℓ_1, m_1, n_1), (ℓ_2, m_2, n_2) are the direction cosines for the two straight lines, then:

$$\cos\theta = |\ell_1 \ell_2 + m_1 m_2 + n_1 n_2|$$

Example

6. Find the measure of the angle between the two straight lines $\overrightarrow{r_1} = (2, -1, 3) + t_1(-2, 0, 2)$ and $x = 1$, $\frac{y-4}{3} = \frac{z+5}{-3}$

Solution

From the equation of the first straight line $\overrightarrow{d_1} = (-2, 0, 2)$

From the equations of the second straight line $\overrightarrow{d_2} = (0, 3, -3)$

$$\therefore \cos\theta = \frac{|\overrightarrow{d_1} \cdot \overrightarrow{d_2}|}{\|\overrightarrow{d_1}\| \, \|\overrightarrow{d_2}\|} = \frac{|(-2, 0, 2) \cdot (0, 3, -3)|}{\sqrt{(-2)^2 + 0^2 + 2^2}\sqrt{0^2 + 3^2 + (-3)^2}}$$

$$= \frac{6}{\sqrt{8}\,\sqrt{18}} = \frac{1}{2} \qquad \therefore \theta = 60^\circ$$

Try to solve

6. Find the measure of the angle between the two straight lines
$L_1 : x = 2 - 5t,\ y = 1 - t,\ z = 3 + 4t$, $L_2 : \frac{x+1}{3} = \frac{2-y}{4} = \frac{z}{2}$

Example

7. Find the measure of the angle between the two straight lines whose direction cosines are

$$\left(\frac{5}{13\sqrt{2}}, \frac{-12}{13\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Solution

$(\ell_1, m_1, n_1) = (\frac{5}{13\sqrt{2}}, \frac{-12}{13\sqrt{2}}, \frac{1}{\sqrt{2}})$ $(\ell_2, m_2, m_2) = (\frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}})$

$\therefore \cos\theta = |\ell_1\ell_2 + m_1m_2 + n_1n_2|$

$= |\frac{5}{13\sqrt{2}} \times \frac{-3}{5\sqrt{2}} + \frac{-12}{13\sqrt{2}} \times \frac{4}{5\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}|$

$= |\frac{-15}{130} + \frac{-48}{130} + \frac{1}{2}| = \frac{1}{65}$

$\therefore \theta = \cos^{-1}(\frac{1}{65}) = 89° 7' 6''$

Try to solve

7) Find the measure of the angle between the two straight lines with direction cosines $(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$

Learn

Parallel lines in space

If $\overrightarrow{d_1} = (a_1, b_1, c_1)$, $\overrightarrow{d_2} = (a_2, b_2, c_2)$ are the direction vectors of the two straight lines L_1 and L_2, then $L_1 // L_2$ if and only if $\overrightarrow{d_1} // \overrightarrow{d_2}$. This condition can be satisfied by several forms

1- $\overrightarrow{d_1} = k\,\overrightarrow{d_2}$ **2-** $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_{-2}}$ **3-** $\overrightarrow{d_1} \times \overrightarrow{d_2} = \overrightarrow{o}$

Note

1- If the two straight lines are parallel and there is a point on one of them satisfying the equation of the other, then the two straight lines are coincident .

2- If $\overrightarrow{d_1}$ is not parallel to $\overrightarrow{d_2}$, then L_1 and L_2 are either intersect or skew.

Example

8) Prove that the two straight lines $\overrightarrow{r_1} = \hat{j} + t_1(\hat{i} + 2\hat{j} - \overrightarrow{k})$

$\overrightarrow{r_2} = (\hat{i} + \hat{j} + \hat{k}) + t_2(-2\hat{i} - 2\hat{j})$

intersect at a point, then find their intersection point.

Solution

$\overrightarrow{d_1} = (1, 2, -1)$, $\overrightarrow{d_2} = (-2, -2, 0)$

$\because \frac{a_1}{a_2} = \frac{1}{-2} = -\frac{1}{2}$, $\frac{b_1}{b_2} = \frac{2}{-2} = -1$ $\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$\therefore$ The two straight lines are not parallel to prove that the two straight lines intersect at a point, look for a value for t_1, and t_2 which make $\overrightarrow{r_1} = \overrightarrow{r_2}$

$\therefore \hat{j} + t_1(\hat{i} + 2\hat{j} - \hat{k}) = \hat{i} + \hat{j} + \hat{k} + t_2(-2\hat{i} - 2\hat{j})$ by equating the cofficients

$\therefore t_1 = 1 - 2t_2$, then $t_1 + 2t_2 = 1$ **(1)**

$2t_1 = -2t_2$, then $t_1 + t_2 = 0$ **(2)**

$-t_1 = 1$, then $t_1 = -1$ **(3)**

by substitution from (3) in (1) $t_2 = 1$

This values satisfy equation (2)

$\therefore$ Thses two straight lines intersect at a point and the position vector of the point of the intersection is

$\overrightarrow{r} = \hat{j} - 1(\hat{i} + 2\hat{j} - \hat{k}) = -\hat{i} - \hat{j} + \hat{k}$ **i.e** (-1 , -1 , 1)

Try to solve

8 Prove that the two straight lines $\overrightarrow{r_1} = (3, -3, 5) + t_1(0, -5, 5)$

$\overrightarrow{r_2} = (-2, 3, 1) + t_2(5, -1, -1)$

are perpendicular and intersect at a point, find the coordinates of their intersection point.

Learn

Perpendicular lines in space

If $\overrightarrow{d_1} = (a_1, b_1, c_1)$, $\overrightarrow{d_2} = (a_2, b_2, c_2)$ are the direction vectors of the two straight lines L_1 and L_2 , then

$L_1 \perp L_2$ if and only if $\overrightarrow{d_1} \cdot \overrightarrow{d_2} = 0$

Example

9 Prove that the two lines $\overrightarrow{r_1} = (1,2,4) + t_1(2, -1, 1)$ & $\overrightarrow{r_2} = (1, 1, 1) + t_2(-2, 7, 11)$ are orthogonal, then show that they are skew.

Solution

$\overrightarrow{d_1} = (2, -1, 1)$ $\longrightarrow$ direction vector of the first straight line

$\overrightarrow{d_2} = (-2, 7, 11)$ $\longrightarrow$ direction vector of the second straight line

$\therefore$ $\overrightarrow{d_1} \cdot \overrightarrow{d_2} = (2, -1, 1) \cdot (-2, 7, 11)$

$= 2 \times (-2) + (-1) \times 7 + 1 \times 11$

$= -4 - 7 + 11$

$= 0$ $\therefore$ the two straight lines are perpendicular

To prove that the two straight lines are skew, we prove that there are not any values for t_1, t_2 make $\overrightarrow{r_1} = \overrightarrow{r_2}$

i.e. $(1, 2, 4) + t_1(2, -1, 1) = (1, 1, 1) + t_2(-2, 7, 11)$ by equating the coefficients

$\therefore$ $1 + 2t_1 = 1 - 2t_2$, then $t_1 + t_2 = 0$ **(1)**

$2 - t_1 = 1 + 7t_2$, then $-t_1 - 7t_2 = -1$ **(2)**

$4 + t_1 = 1 + 11t_2$, then $t_1 - 11t_2 = -3$ **(3)**

By solving the two equations 1 and 2, we get $t_1 = \frac{-1}{6}$, $t_2 = \frac{1}{6}$ and these values do not satisfy the third equation

$\therefore$ the two straight lines are skew

Try to solve

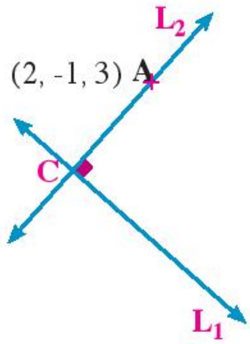
(9) Prove that the two straight lines

$\overrightarrow{r_1} = (3, -1, 2) + t_1 (4, 1, 3)$, $\overrightarrow{r_2} = (0, 4, -1) + t_2 (1, -1, 2)$ are skew.

Example

(10) Find the equation of the straight line passing through the point (2, -1, 3) and intersects the straight line $\overrightarrow{r_1} = (1, -1, 2) + t (2, 2, -1)$ orthogonally.

Solution



Let C be the point of intersection of the two straight lines

$\because$ $C \in$ straight line L_1 (given straight line)

$\therefore$ C can be written in the form

C $(1 + 2t, -1 + 2t, 2 - t)$

the direction vector of L_2 (required straight line) is

$\overrightarrow{d_2} = \overrightarrow{AC} = \overrightarrow{C} - \overrightarrow{A}$

$\therefore$ $\overrightarrow{d_2} = (2t - 1, 2t, -t - 1)$

$\because$ $\overrightarrow{d_1} = (2, 2, -1)$

$\because$ The two straight lines are perpendicular $\therefore \overrightarrow{d_1} \cdot \overrightarrow{d_2} = 0$

$\therefore$ $(2, 2, -1) \cdot (2t - 1, 2t, -t - 1) = 0$

$\therefore$ $4t - 2 + 4t + t + 1 = 0$

$\therefore$ $9t = 1$ then $t = \frac{1}{9}$

$\therefore$ $\overrightarrow{d_2} = (\frac{-7}{9}, \frac{2}{9}, \frac{-10}{9}) = (-7, 2, -10)$

$\therefore$ the equation of L_2 is $\overrightarrow{r} = (2, -1, 3) + t_2 (-7, 2, -10)$

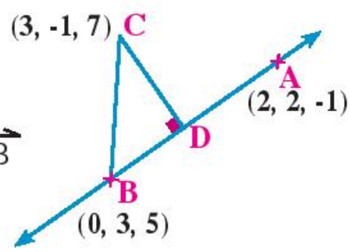
Try to solve

(10) Find the equation of the straight line passing through the origion point and intersects the straight line $\overrightarrow{r} = (3, 1, 4) + t (2, 1, 3)$ orthogonally.

Example **(distance between a point and a straight line in space)**

(11) Find the perpendicular distance from point (3, -1, 7) to the straight line passing through the two points (2, 2, -1) and (0, 3, 5)

Solution



Let A (2, 2, -1), B (0, 3, 5), C (3, -1, 7)

$\overrightarrow{BC} = \overrightarrow{C} - \overrightarrow{B} = (3, -1, 7) - (0, 3, 5) = (3, -4, 2)$

The direction vector of the straight line $\overrightarrow{d} = \overrightarrow{BA} = \overrightarrow{A} - \overrightarrow{B}$

$\therefore$ $\overrightarrow{d} = (2, -1, -6)$

B D is the absolute value of the projection $\overrightarrow{BC}$ on the straight line $\overleftrightarrow{AB} = \frac{|\overrightarrow{BC} \cdot \overrightarrow{BA}|}{\|\overrightarrow{BA}\|}$

$$\therefore \quad BD = \frac{|(3, -4, 2) \cdot (2, -1, -6)|}{\sqrt{2^2+(-1)^2+(-6)^2}} = \frac{2}{\sqrt{41}}$$

$$\text{but} \quad \|\overrightarrow{BC}\| = \sqrt{3^2+(-4)^2+2^2} = \sqrt{29}$$

$$\therefore \text{ The perpendicular distance } CD = \sqrt{(BC)^2 - (DB)^2}$$

$$= \sqrt{29 - \frac{4}{41}} = \sqrt{\frac{1185}{41}} \simeq 5.3 \text{ unit length}$$

Try to solve

(11) Find the length of the perpendicular drawn from point (2 , 1 , -4) to the straight line $\vec{r} = (1, -1, 2) + t\,(2, 3, -2)$

Critical thinking: Can you prove the following relation which identifies the distance from point B to the straight line $\vec{r} = \vec{A} + t\,\vec{d}$ the perpendicular distance $= \frac{\|\overrightarrow{AB} \times \vec{d}\|}{\|\vec{d}\|}$

Exercises 4 - 1

Complete:

(1) The vector equation of the straight line passing through point (2 , -1 , 3) and the vector (-1 , 4 , 2) is a direction vector of it is ....

(2) The measure of the angle between the two straight lines 2x = 3y = -z and 6x = -y = -4z equals

(3) The measure of the angle between the two straight lines whose direction ratios are (1, 1, 2) and($\sqrt{3}$ -1, -$\sqrt{3}$ -1, 4) equals ....

(4) If θ_z is the angle made by the straight line passing through point (3, -1 , 1) and the origin point with the +ve direction of z axis, then cos θ_z = ...

(5) A direction vector of the straight line passing through the two points (7 , -5 , 4)and (5 , -3 , 3) is..........

Answer the following:

(6) Find the direction cosines of the straight line with its direction ratios

(a) -1 , 2 , 3 (b) 1 , 1 , 1

(7) Find all the different forms of the equation of the straight line.

(a) Passes through point (4 , -2 , 5) and the vector $\vec{d}$ = (2 , 1 , -1) is a direction vector of it.

(b) Passes through point (3 , -1 , 5) and parallel to the vector $\overrightarrow{AB}$ where $\overrightarrow{AB}$ = (4 , -2 , 2)

c) Passes through the two points (3 , -2 , 0) and (0 , 4 , 1)

d) Passes through point (3 , 2 , 5) and makes equal angles with the +ve directions of the coordinated axes.

8) Find the vector form of the equation of the straight line $x - 3 = \frac{y+2}{4} = \frac{2-z}{3}$

9) If $\overrightarrow{OA} = \hat{i} - 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = -\hat{i} - 3\hat{j}$,
$\overrightarrow{OC} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\overrightarrow{OD} = 8\hat{i} + \hat{j} + 4\hat{k}$.

Find the vector equation for each of the following straight lines

a) Passes through the two points A, B

b) Passes thorough point D and parallel to $\overleftrightarrow{BC}$

c) Passes through point C and intersects $\overleftrightarrow{AB}$ orthogonally

10) Find the measure of the angles between the two straight lines

a) L_1 : passing through the two points (-3 , 2 , 4) and (2 , 5 , -2)
L_2 : passing through the two points (1 , -2 , 2) and (4 , 2 , 3)

b) L_1 : $\vec{r} = (2, -1, 3) + t_1(-1, 4, 2)$
L_2 : $\vec{r} = (0, 2, -1) + t_2(1, 1, 3)$

c) L_1 : $2x = 3y = 4z$
L_2 : $\frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{-5}$

11) State the necessary condition(s) to make the two straight lines
L_1: $x = x_1 + a_1t_1$, $y = y_1 + b_1t_1$, $z = z_1 + c_1t_1$
L_2 : $x = x_2 + a_2t_2$, $y = y_2 + b_2t_2$, $z = z_2 + c_2t_2$

a) parallel b) perpendicular c) intersect at a point

12) Find the vector equation of the straight line passing through point A (1 , -1 , 0) and parallel to the straight line passing through the two points B (-3 , 2, 1), C (2 , 1 , 0), then show that point D (-14 , 2 , 3) belongs to the straight line.

13) Find the value of n which makes the two straight lines L_1 : $\vec{r_1} = (3, -1, n) + t_1(4, 1, 3)$
L_2 : $x = \frac{y-4}{-1} = \frac{z+1}{2}$ intersect at a point, then find the point of their intersection

14) **Discover the error:**

a) The sum of the squares of direction ratios for any straight line equals 1

b) The direction cosines of the straight line passing through the two points (x_1, y_1, z_1) , (x_2, y_2, z_2) is $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

c) If (a_1, b_1, c_1) and (a_2, b_2, c_2) are the direction ratios of the two straight lines L_1 and L_2 , then the measure of the angle between them is given by the relation $\cos\theta = |a_1a_2 + b_1b_2 + c_1c_2|$

Unit Two

4 - 2

The equation of a plane in space

You will learn

- Vector equation of a plane in space
- Standard equation of a plane in space
- General equation of a plane in space
- Angle between two planes
- Condition of parallel planes
- Condition of orthogonal planes
- Equation of intersection line of two planes in space
- The distance between a point and a plane
- The distance between two parallel planes

Key terms

- plane
- Standard form
- Parallel planes
- Perpendicular planes
- Interecting planes
- Angle

Materials

- Scientific calculator
- 3-D computer programmes

Think and discuss

1- If $\overrightarrow{A}$ and $\overrightarrow{B}$ two orthogonal vectors, then $\overrightarrow{A} \cdot \overrightarrow{B} =$ ....

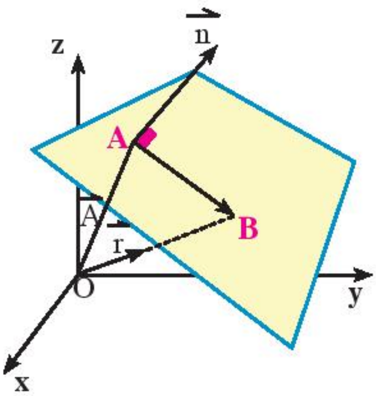
2- The direction vector of the straight line passing through the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is ....

3- The z -coordinate for all the points lying on x y-plane equals ....

Learn

The vector form of the equation of a plane in space

If point $A(x_1, y_1, z_1)$ belongs to the plane and its position vector is $\overrightarrow{A}$ and the normal direction vector to the plane is $\overrightarrow{n} = (a, b, c)$ and $B(x, y, z)$ any point on the plane its position vector is $\overrightarrow{r}$ then :



$$\overrightarrow{n} \cdot \overrightarrow{AB} = 0$$

$\therefore \quad \overrightarrow{n} \cdot (\overrightarrow{B} - \overrightarrow{A}) = 0 \qquad (\overrightarrow{B} = \overrightarrow{r})$

$\therefore \quad \overrightarrow{n} \cdot \overrightarrow{r} = \overrightarrow{n} \cdot \overrightarrow{A}$ $\longrightarrow$ **the vector form of the equation of the plane.**

i.e: to find the vector equation of the plane, we must know a point on the plane and the perpendicular direction vector to the plane.

Example

1 Find the vector form of the equation of the plane that has normal vector $\overrightarrow{n} = \hat{i} + \hat{j} + \hat{k}$ and contains point (0 , 1, 1) .

Solution

The vector equation $\overrightarrow{n} \cdot \overrightarrow{r} = \overrightarrow{n} \cdot \overrightarrow{A}$ where $\overrightarrow{A} = (0, 1, 1)$

$\therefore \quad (1, 1, 1) \cdot \overrightarrow{r} = (1, 1, 1) \cdot (0, 1, 1)$

$(1, 1, 1) \cdot \overrightarrow{r} = 2$

Try to solve

1. Find the vector form of the equation of the plane passing through point (2 , -3 , 1) and the vector $\overrightarrow{n} = (1 , -2 , 3)$ is normal to the plane.

Learn

The standard form and general form of the equation of a plane in space

From the vector form of the equation of the plane

$$\overrightarrow{n} \cdot (\overrightarrow{r} - \overrightarrow{A}) = 0$$

where $\overrightarrow{n} = (a , b , c)$, $\overrightarrow{r} = (x , y , z)$, $\overrightarrow{A} = (x_1 , y_1 , z_1)$

$\therefore (a , b , c) \cdot (x - x_1 , y - y_1 , z - z_1) = 0$

$\therefore a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \longrightarrow$ **the standard form for the equation of the plane**

by expanding

$\therefore ax + by + cz + (-ax_1 - by_1 - cz_1) = 0$

consuming $-ax_1 - by_1 - cz_1 = d$, **then**

$ax + by + cz + d = 0 \longrightarrow$ **the general form of the equation of the plane**

Example

2. Find the standard form and the general form of the equation of the plane passing through point (3 , -5 , 2) and the vector $\overrightarrow{n} = (2 , 1, 1)$ is normal to the plane.

Solution

The standard form $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\therefore 2(x - 3) + (y + 5) + (z - 2) = 0 \longrightarrow$ **the standard form**

Expand and collect the like terms

$\therefore 2x + y + z - 3 = 0 \longrightarrow$ **general form**

Try to solve

2. Find all the different forms of the equation of the plane passing through point (-3 , 4 , 2) and the vector $\overrightarrow{n} = (1 , -1, 3)$ is normal to the plane.

Example **(Equation of a plane passing through three non-colinear points)**

3. Find the different forms of the equation of the plane passing through points (3 , -1 , 0), (2 , 1 , 4) and (0 , 3 , 3).

Solution

First : we must make sure that the points are non-colinear

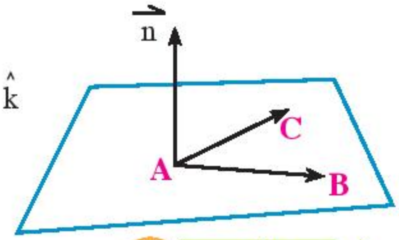
Assuming A (3 , -1 , 0), B (2 , 1 , 4), C (0 , 3 , 3)

$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (-1, 2, 4)$, $\overrightarrow{AC} = \overrightarrow{C} - \overrightarrow{A} = (-3 , 4 , 3)$

$\because \frac{-1}{-3} \neq \frac{1}{2}$ $\therefore \overrightarrow{AB} \nparallel \overrightarrow{AC}$ $\therefore$ the points are non-colinear

To find the equation of the plane, we need the normal vector to the plane by getting the vector product of the two vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$.

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 4 \\ -3 & 4 & 3 \end{vmatrix} = -10\,\hat{i} - 9\,\hat{j} + 2\,\hat{k}$$



$\therefore$ **The vector form of the equation of the plane**

$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{A}$

$\therefore$ $(-10 , -9 , 2) \cdot \vec{r} = (-10 , -9 , 2) \cdot (3 , -1, 0)$

$\therefore$ $(-10 , -9 , 2) \cdot \vec{r} = -21$

The standard form of the equation of the plane

$a (x - x_1) + b (y - y_1) + c (z - z_1) = 0$

$\therefore$ $-10 (x - 3) - 9 (y + 1) + 2z = 0$

The general form of the equation of the plane

$(-10 , -9 , 2) \cdot (x , y, z) = -21$

$\therefore$ $-10x - 9y + 2 z + 21 = 0$

Tip

The equation of the plane passing through the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Try to solve

3) Find all the different forms of the equation of the plane passing through points (1 , 0, 0), (0 , 2, 0) and (0 , 0 , 3).

Example (a plane contains two straight lines)

4) Prove that the two straight lines $\vec{r_1} = (3\,\hat{i} + \hat{j} - \hat{k}) + t_1(\hat{i} + 2\,\hat{j} + 3\,\hat{k})$

$\vec{r_2} = (2\,\hat{i} + 5\,\hat{j}) + t_2(\hat{i} - \hat{j} + \hat{k})$

are intersecting , then find the equation of the plane containing them.

Solution

If the two straight lines intersect, then $r_1 = r_2$

$\therefore$ $(3\,\hat{i} + \hat{j} - \hat{k}) + t_1(\hat{i} + 2\,\hat{j} + 3\,\hat{k}) = (2\,\hat{i} + 5\,\hat{j}) + t_2(\hat{i} - \hat{j} + \hat{k})$

by equating the coefficients ,we find

$3 + t_1 = 2 + t_2$, then $t_1 - t_2 = -1$ **(1)**

$1 + 2t1 = 5 - t_2$, then $2t_1 + t_2 = 4$ **(2)**

$-1 + 3t_1 = t_2$, then $3t_1 - t_2 = 1$ **(3)**

by solving 1 , 2 $t_1 = 1$, $t_2 = 2$

by substituting with these values in the equation (3), we find it satisfies the equation

$\therefore$ The two straight lines intersect.

the normal vector to the plane is $\overrightarrow{n}$ where

$$\overrightarrow{n} = \overrightarrow{d_1} \times \overrightarrow{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = 5\ \hat{i} + 2\ \hat{j} - 3\ \hat{k}$$

The vector equation of the plane $\overrightarrow{n} \cdot \overrightarrow{r} = \overrightarrow{n} \cdot \overrightarrow{A}$

$\therefore (5\ ,\ 2\ ,\ -3) \cdot \overrightarrow{r} = (5\ ,\ 2\ ,\ -3) \cdot (3\ ,\ 1\ ,\ -1)$

$\therefore (5\ ,\ 2\ ,\ -3) \cdot \overrightarrow{r} = 20$

The general form

$(5\ ,\ 2\ ,\ -3) \cdot (x\ ,\ y\ ,\ z) = 20$

$\therefore\ 5x + 2y - 3z = 20$

Try to solve

4) Prove that the two straight lines $L_1: 2x = 3y = 4z$, $L_2: 3x = 2y = 5z$ are intersecting, then find the equation of the plane containing the two straight lines .

Example

5) Find the point of intersection of the straight line $2x = 3y - 1 = z - 4$ with the plane $3x + y - 2z = 5$

Solution

From the equation of the plane

$$y = 5 + 2z - 3x$$

by substituting in the straight line equation

$$2x = 14 + 6z - 9x$$
$$= z - 4$$

$$11x - 6z = 14 \quad (1)$$

$$-5z + 9x = 18 \quad (2)$$

by solving the equations (1) , (2), we get

$$x = -38\ ,\ z = -72$$

by substituting in the plane equation

$$\therefore\ y = -25$$

$\therefore$ the point of intersection is $(-38\ ,\ -25\ ,\ -72)$

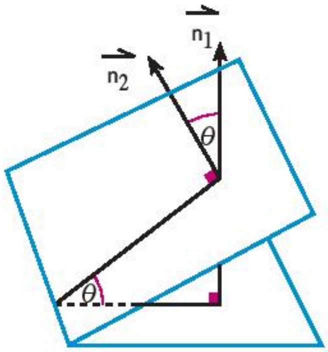
Try to solve

5) Find the intersection point of the straight line $\overrightarrow{r} = (1\ ,\ 4\ ,\ 2) + t\ (3\ ,\ 2,\ 2)$ with the plane $(3\ ,\ 2,\ 2)\ .\ \overrightarrow{r} = -2$

Learn

the angle between two planes

The measure of the angle between two planes is the measure of the angle between their two normal vectors i.e. $\vec{n_1}$ and $\vec{n_2}$ are the two normal vectors on the two planes, then the measure of the angle between the two planes is given by the relation



$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{\|\vec{n_1}\| \, \|\vec{n_2}\|} \quad \text{where } 0 \leqslant \theta \leqslant 90^\circ$$

Example

6 Find the measure of the angle between the two planes $P_1: (2, -1, 4) \cdot \vec{r} = 5$ and $P_2: 3x - y + 2z = 4$

Solution

The normal vector to the first plane $\vec{n_1} = (2, -1, 4)$

the normal vector to the second plane $\vec{n_2} = (3, -1, 2)$

$\therefore$ The measure angle between the two planes is θ where

$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{\|\vec{n_1}\| \, \|\vec{n_2}\|} = \frac{|(2, -1, 4) \cdot (3, -1, 2)|}{\sqrt{2^2 + (-1)^2 + 4^2}\sqrt{3^2 + (-1)^2 + 2^2}} = \frac{15}{7\sqrt{6}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{15}{7\sqrt{6}}\right) = 28^\circ 58'$$

Try to solve

6 Find the measure of the angle between the two planes $x - 3y + 2z = 0$ and $2x + y - z = 3$

Parallel planes and perpendicular planes

If $\vec{n_1}$ and $\vec{n_2}$ are the normal vectors to the two planes, then

1- The two planes are parallel if $\vec{n_1} \,//\, \vec{n_2}$ i.e $\left(\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}\right)$

2- the planes are perpendicular if $\vec{n_1} \cdot \vec{n_2} = 0$ i.e. $(a_1 a_2 + b_1 b_2 + c_1 c_2) = 0$

Example

7 If the plane $2x - y + kz = 5$ is parallel to the plane $x + Ly + 4z = 1$, find the value of k, L.

Solution

$\because$ The two plane are parallel $\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$\therefore \frac{2}{1} = \frac{-1}{L} = \frac{k}{4}$ $\therefore L = \frac{-1}{2}$, $k = 8$

Try to solve

7 If the plane $x - 3y + z = 4$ is perpendicular to the plane $ax + 2y + 3z = 2$, find the value of a

Example **(The equation of the line of intersection of two planes)**

8 Find the equation of the line of intersection of the two planes $x + 2y - 2z = 1$, $2x + y - 3z = 5$

Solution

by eliminating x from the two equations by multiplying the first equation by -2 and adding to the second

$\therefore\ -3y + z = 3$, then $z = 3y + 3$ **(1)**

by eliminating y from the two equations by multiplying the second equation by -2 and adding to the first

$\therefore\ -3x + 4z = -9$, then $z = \frac{3x - 9}{4}$

$\therefore\ \boxed{\frac{3x - 9}{4} = \frac{3y + 3}{1} = \frac{z}{1}}$ **the equation of the line of intersection**

Another solution:

$x + 2y - 2z = 1$ **(1)**

$2x + y - 3z = 5$ **(2)**

by eleminating x

$-3y + z = 3$ **(3)**

let z = k

(3) $y = \frac{k - 3}{3}$, **(2)** $x = \frac{9 + 4k}{3}$

$\therefore$ **the parametric equations of the line of intersection are**

$x = 3 + \frac{4}{3}k$, $y = -1 + \frac{1}{3}k$, $z = k$

Third solution:

The line of intersection is perpendicular to the two normal vectors $\overrightarrow{n_1}$, $\overrightarrow{n_2}$.

$\therefore$ The direction vector of the intersection line $\overrightarrow{d}$ can be calculated using the vector product of the two vectors $\overrightarrow{n_1}$, $\overrightarrow{n_2}$

$$\overrightarrow{d} = \overrightarrow{n_1} \cdot \overrightarrow{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 2 & 1 & -3 \end{vmatrix} = -4\ \hat{i} - \hat{j} - 3\ \hat{k}$$

To find a point on the intersection line, put $x = 1$

By substituting in the equation of the first plane $2y - 2z = 0$ **(1)**

By substituting in the equation of the second plane $y - 3z = 3$ **(2)**

By solving equations (1), (2), we get $z = -\frac{3}{2}$, $y = -\frac{3}{2}$

$\therefore$ The point $(1 , -\frac{3}{2} , -\frac{3}{2})$ lies on the line of intersection .

The equation of the intersection line $\overrightarrow{r} = (1 , -\frac{3}{2} , -\frac{3}{2}) + t(-4 , -1 , -3)$

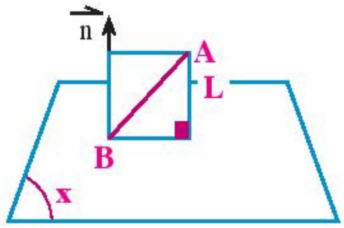
Try to solve

8 Find the equation of the intersection line of the two planes $3x - y + 2z = 3$ and $x - 2y + 5z = 2$

Learn

the length of the perpendicular drawn from a point to a plane

If A (x_1 , y_1 , z_1) is a point outside the plane x and B is a point on the plane , $\vec{n}$ is the normal vector to the plane, then the distance from the point A to the plane equals the length of the projection of $\overrightarrow{BA}$ to $\vec{n}$



$$L = \frac{|\overrightarrow{BA} \cdot \vec{n}|}{\|\vec{n}\|}$$

Example

9 Find the length of the perpendicular drawn from the point (1 , -1 , 3) to the plane whose equation is $\vec{r} \cdot (2 , 2 , -1) = 5$

Solution

We must get a point on the plane and the normal vector to the plane $\vec{r} \cdot (2 , 2 , -1) = 5$ then $\vec{n} = (2 , 2 , -1)$

To find a point on the plane, we assume that the plane cuts z-axis at the point z (0 , 0 , z)

$\therefore$ $(0 , 0 , z) \cdot (2 , 2 , -1) = 5$, **then** $z = -5$

$\therefore$ The point B (0 , 0 , -5) lies on the plane

$\overrightarrow{BA} = \vec{A} - \vec{B} = (1 , -1 , 8)$ where A (1 , -1 , 3)

the length of the perpendicular $(L) = \frac{|\overrightarrow{BA} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|(1 , -1 , 8) \cdot (2 , 2 , -1)|}{\sqrt{2^2 + 2^2 + (-1)^2}} = \frac{8}{3}$ units

Try to solve

9 Find the distance between point (-2 , 1 , 4) and the plane whose equation $\vec{r} \cdot (1 , -3 , 2) = 4$

Cartisian form of the perpendicular length drawn from a point and a plane

You notice that the perpendicular length from point A (x_1 , y_1 , z_1) to the plane passing through point b (x_2 , y_2 , z_2) and $\vec{n} = (a , b , c)$ is the normal vector to the plane is given by the relation

$$L = \frac{|\overrightarrow{BA} \cdot \vec{n}|}{\|\vec{n}\|}$$

$$\therefore L = \frac{|(x_1 - x_2 , y_1 - y_2 , z_1 - z_2) . (a , b , c)|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax_1 + b\,y_1 + c\,z_1 + (-a\,x_2 - b\,y_2 - cz_2)|}{\sqrt{a^2+b^2+c^2}}$$

$\because$ point B (x_2 , y_2 , z_2) lies on the plane $a\,x + b\,y + c\,z + d = 0$

$\therefore\ - a\,x_2 - b\,y_2 - c\,z_2 = d$

$\therefore\ L = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+ b^2 + c^2}}$ **the Cartesian form of the length of the perpendicular**

Example

(10) Find the length of the perpendicular drawn from point (1 , 5, -4) to the plane whose equation $3x - y + 2z = 6$

Solution

$$L = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|3\,(1) - (5) + 2\,(-4) - 6|}{\sqrt{3^2 + (-1)^2 + 2^2}} = \frac{16}{\sqrt{14}} \text{ unit length}$$

Try to solve

(10) Find the length of the perpendicular drawn from point (-1 , 4 , 0) to the plane with equation $x - 2y - z = 4$

Example **(The distance between two parallel planes)**

(11) Prove that the two planes $x + 3y - 4z = 3$, $2x + 6y - 8z = 4$ are parallel, then find the distance between them.

Solution

To prove that the palanes are parallel, we prove that their normal vectors are parallel.

$\overrightarrow{n_1} = (1 , 3 , -4)$, $\overrightarrow{n_2} = (2 , 6 , -8)$

$\because\ \dfrac{a_1}{a_2} = \dfrac{1}{2}$ $\qquad \dfrac{b_1}{b_2} = \dfrac{3}{6} = \dfrac{1}{2}$, $\qquad \dfrac{c_1}{c_2} = \dfrac{-4}{-8} = \dfrac{1}{2}$

$\therefore\ \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ $\qquad \therefore$ The two planes are parallel

To find the distance between them, we find a point on one of them, then we find the distance between this point and the other plane.

To get a point on the first plane, we assume $x = 0$, $y = 0$

by substituting in the first plane equation

$\therefore z = \dfrac{-3}{4}$

$\therefore$ point $(0 , 0 , \dfrac{-3}{4})$ lies on the first plane

Then the perpendicular length from this point to the second plane is

$$L = \frac{|2\,(0) + 6\,(0) - 8\,(\frac{-3}{4}) - 4|}{\sqrt{2^2 + 6^2 + (-8)^2}} = \frac{\sqrt{26}}{14} \text{ unit length}$$

Try to solve

(11) Prove that the two planes $3x + 6y + 6z = 4$, $x + 2y + 2z = 1$ are parallel and find the distance between them.

Learn

Equation of a plane using the intercepted parts from the coordinate axes

If a plane cuts the coordinates axes at points $(x_1 , 0 , 0)$, $(0 , y_1 , 0)$ $(0 , 0 , z_1)$, then the equation of the plane is in the form

$$\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$$ ⟶ **the equation of the plane interms of the intercepted parts from the coordinate axes**

Ask your teacher to prove the previous form of the plane equation.

Example

(12) Find the equation of the plane which intercepts the coordinate axes x, y, z the parts 2, -3, 0 respectively.

Solution

The equation of the plane is $$\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$$

i.e. $$\frac{x}{2} + \frac{y}{-3} + \frac{z}{5} = 1$$

Try to solve

(12) Find the intercepted parts made by the plane $2x + 3y - z = 6$ with the coordinate axes.

Critical thinking:

If the plane $3x + 2y + 4z = 12$ intersects the coordinate axes x , y , z at the points a , b , c respectively, find the area of the triangle A B C

Exercises 4 - 2

Choose the correct answer :

(1) Which of the following points belongs to the plane $2x + 3y - z = 5$

(a) (1 , 1 , 1) (b) (1 , 2 , 0) (c) (0 , 2 , 1) (d) (3 , 2 , -1)

(2) The plane $3x - 2y + 4z = 12$ intercepts from x -axis a part of length

(a) 3 (b) -4 (c) 4 (d) 6

(3) If the intercepted parts from the coordinate axes by the plane $x + 5y - 6z = 30$ are a , b , c, then a + b + c =

(a) 0 (b) 30 (c) 31 (d) 41

4) The equation of the plane which passes through the point (1 , 2 , 3) and parallel to the coordinate axes x , y is

a) $x + y = 3$ b) $z = 3$ c) $x = 1$ d) $y = 2$

5) Then equation of the plane passing through points (2 , 3 , 5) , (-1 , 3 , 1) , (4 , 3 , -2) is

a) $x + y - z = 0$ b) $x = -1$ c) $y = 3$ d) $z = -2$

6) The equation of the plane passing through point (1 , -2 , 5) and the vector (2 , 1 , 3) is perpendicular to it is

a) $2x + y + 3z = 1$ b) $2x + y + 3z = 15$
c) $x - 2y + 5z = 15$ d) $x + y + z = 4$

Answer the following:

7) Find all the different forms of the equation of the plane passing through point (1 , -1 , 4) and its normal vector is $\vec{n} = (2, -3, 4)$, then show the following:

a) Is point (2 , 2 , 1) lying on the plane?
b) Is the vector $\vec{u} = (3, -5, -2)$ parallel to the plane?

8) Find three points in space belonging to each of the following planes:

a) $x = 3$ b) $y = -2$ c) $x + 3y = 5$ d) $2x - y + 3z = 4$

9) Find the general equation of the plane passing through the origin point and the vector $\vec{n} = \hat{i} + 2\hat{j} - 3\hat{k}$ is normal to it.

10) Find all the different forms of the equation of the plane passing through point (2 , -1 , 0) and the vector $\vec{n} = 4\hat{i} + 10\hat{j} - 7\hat{k}$ is normal to it.

11) Find all the different forms of the equation of the plane passing through the three points a (2 , -1 , 0) b (-1 , 3 , 4) , c (3 , 0 , 2)

12) Prove that the straight line $\vec{r} = \hat{k} + t(2\hat{i} + 3\hat{j} + 4\hat{k})$ is perpendicular to the plane $x + \frac{3}{2}y + 2z = 5$

13) Prove that point A (2 , 3 , 1) & the straight line L: $\vec{r} = (3\hat{i} + \hat{j} + 3\hat{k}) + t(\hat{i} - 2\hat{j} + 2\hat{k})$ lying on the plane whose equation is $\vec{r} \cdot (2\hat{i} - \hat{k}) = 3$

14) Find the equation of the plane passing through point (2 , 1 , 4) and satisfies the following condition:

a) Parallel to the plane $2x + 3y + 5z = 1$
b) Perpendicular to the straight line passing through the two points (3 , 2 , 5) , (1 , 6 , 4)
c) Perpendicular to each of the planes $7x + y + 2z = 6$, $3x + 5y - 6z = 8$

15) Find the coordinates of the point of intersection of the straight line $\vec{r} = \hat{k} + t(2\hat{i} + \hat{j} + \hat{k})$ with the plane $\vec{r} \cdot \hat{i} = 4$

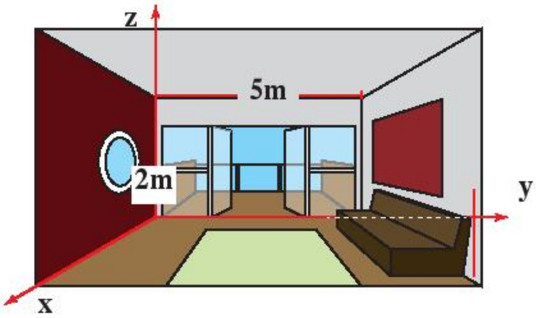
16 Find all the different forms of the equation of the plane that intercepts 2 , 4 , 5 from the coordinate axes x , y , z respectively.

17 **Environment:** In the opposite figure, find the equation for each of:

a The floor plane.

b The cieling plane.

c The lateral walls planes.



18 Find the equation of the plane which contains the straight line
L_1: $\vec{r} = (0 , 3 , -5) + t_1 (6 , -2 , -1)$ and parallel to the straight line
L_2 : $\vec{r} = (1 , 7 , -4) + t_2 (1 , -3 , 3)$

19 Find the measure of the angle between the following pairs of planes:

a $P_1 : 2x - y + z = 5$, $P_2: 3x + 2y - 2z = 1$

b $P_1: \vec{r} \cdot (2 , 1 , -1) = 4$, $P_2: \vec{r} \cdot (3 , -2 , 0) = 7$

c $P_1 : y = 4$, $P_2: x - 3y + 5z = 1$

Questions of multi-requirements

20 If the points A , B , C , D are in space whose position vectors are $-\hat{j} + \hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$, $-\vec{x} - 2\vec{y} + 2\vec{z}$, $7\vec{x} - 4\vec{y} + 2\vec{z}$ respectively

a Find the normal vector to the plane ABC

b Show that the length of the perpendicular from D to the plane A B C equals $2\sqrt{6}$

c Show that the two planes A B C , D B C are orthogonal.

d Find the equation of the line of the intersection of the two planes A B C , O D B

21 If the plane X contains points A (1 , 4 , 2) B (1 , 0 , 5), C (0 , 8 , -1) and the plane Y contains point D (2 , 2 , 3) and the vector $\vec{n} = \hat{j} + 2\hat{j} + 2\hat{k}$ is perpendicular to it, find:

a The cartisian equation of X

b The cartisian equation of the plane Y

c What is the values of t , f if point (t , 0 , f) belongs to each of the two planes X , Y?

d Find the vector equation of the line of intersection of the two planes X , Y

e If point (1 , 1, p) equidistant from the two planes X , Y, find all possible values of p.

Second:

Differential & integral Calculus

Pre- requisites for differential and integral calculus

Learn

1 - Derivative of cotangent function

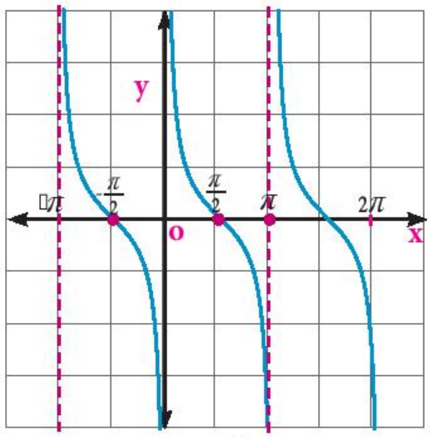
If $y = \cot x$ where $x \in R$, $x \neq n\pi$, $n \in z$

then: $\frac{d}{dx}(\cot x) = -\csc^2 x$

Notice that :

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right]$$

$$= \frac{\sin x \times (-\sin x) - \cos x \times \cos x}{(\sin x)^2} = -\frac{1}{(\sin x)^2} = -\csc^2 x$$



y = cot x

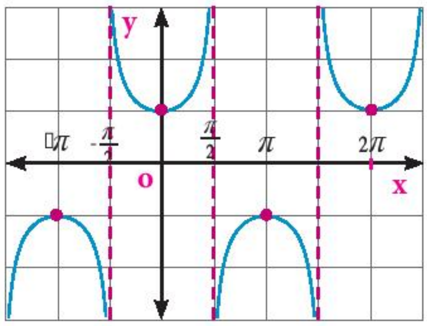
2- Derivative of the secant function

If $y = \sec x$ where :

$x \in R$, $x \neq \frac{(2n+1)\pi}{2}$, $n \in Z$

then:

$\frac{d}{dx}(\sec x) = \sec x \tan x$ **(check)**



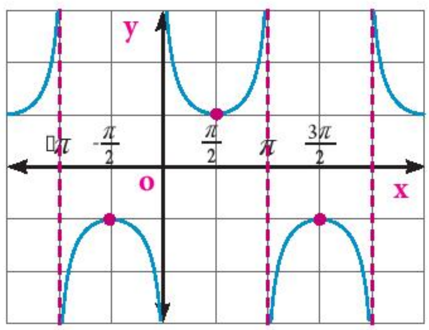
y = sec x

3- Derivative of cosecant function:

If $y = \csc x$ where

$x \in R$, $x \neq n\pi$, $n \in Z$

Then : $\frac{d}{dx}(\csc x) = -\csc x \cot x$ **check**



y = csc x

Example

1. Find $\frac{dy}{dx}$ for each of the following:

a) $y = 3x^5 + 4\cot x$

b) $y = 3\sec x - 5\tan x$

c) $y = x^3 \csc x$

Solution

a) $\frac{dy}{dx} = 3 \times 5x^4 + 4(-\csc^2 x) = 15x^4 - 4\csc^2 x$

b) $\frac{dy}{dx} = 3(\sec x \tan x) - 5\sec^2 x = \sec x[3\tan x - 5\sec x]$

c) $\frac{dy}{dx} = 3x^2 \csc x + x^3(-\csc x \cot x) = x^2 \csc x[3 - x\cot x]$

Integrd of Tigonometric Functions:

$\int \sin x \, dx = -\cos x + c$	
$\int \cos x \, dx = \sin x + c$	
$\int \sec^2 x \, dx = \tan x + c$	$x \neq \frac{2n+1}{2}\pi$, $n \in Z$
$\int \csc^2 x \, dx = -\cot x + c$	$x \neq n\pi$, $n \in Z$
$\int \sec x \tan x \, dx = \sec x + c$	$x \neq \frac{2n+1}{2}\pi$, $n \in Z$
$\int \csc x \cot x \, dx = -\csc x + c$	$x \neq n\pi$, $n \in Z$

Unit One: Differentiation and its Applications

Unit introduction

In your previous study to the functions, you have identified explicit functions in one variable in the form y = f (x) and the operations on these functions and their formations. You have also investigated the differentiability of the continuous function on a domain and you could find the first derivative of the algebraic functions and some trigonometric functions.

In this unit, you are going to complete the study of differentiating the trigonometric functions and identify the other functions which their variables cannot be separated where the variables relate with an implicit relation or by its definition throughout the parametric variable. This requires to learn different patterns of differentiation such as implicit differentiation and parametric differentiation which depend on the derivative of the composite function (chain rule) to differentiate the function. You will also investigate the existence of the second derivative of the function in regard to learning the higher derivatives of the function in regard to learning the higher derivatives of the function which give a hand to learn several life applications.

This unit interests in some necessary applications of the differentiation in various domains of mathematics, physics, economy and the biological sciences throughout learning the two equations of the tangent and normal on a tangent of a curve and the related time rates to help you model and solve some daily life problems.

Learning outcomes

By the end of the unit and carrying out the related activities, the student should be able to:

- Find the differentiation of implicit functions (explicit, implicit and parametric)
- Find the higher derivatives (second and third) of different functions and identify how to express them.
- Find the related time rates including the physical application
- Model and solve life and economic problems.
- Find the derivatives of the exponential functions $y = e^x$, $y = a^x$, and the derivative of the logarithmic function $y = \ln x$, $y = \log_a x$

Key terms

- Differentiation
- First Derivative
- Explicit Function
- Implicit function
- Parameter
- Implicit Defferentiation
- Parametric Defferentiation
- Higher Derivatives
- Rate
- Related Rates

Materials

- Graphical calculator
- Geogebra and graph computer

Unit lessons

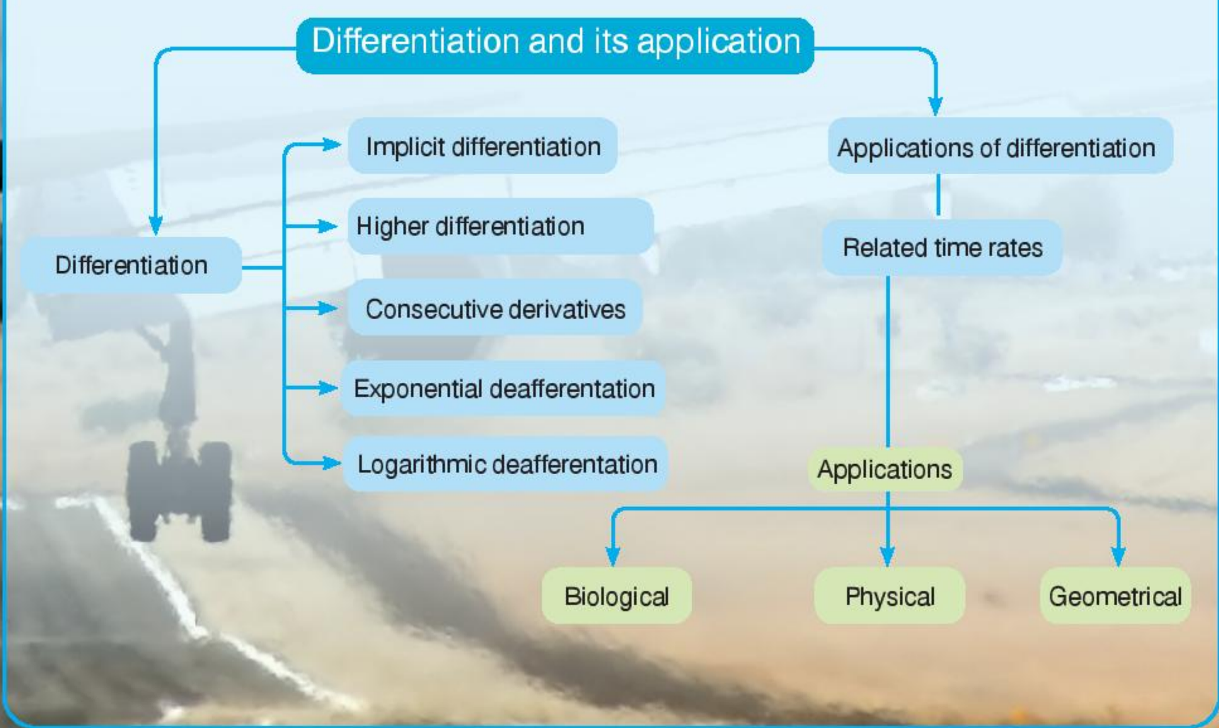
Lesson (1 - 1): Implicit and parametric differentiation

Lesson (1 - 2): Heigher deivatives of the function

Lesson (1 - 3): Derivative of the exponential and logarithmic function.

Lesson (1 - 4): Related time rates

Unit planning guide



1 - 1 Implicit and Parametric Differentiation

You will learn

- implicit differentiation
- Parametric differentiation

Key terms

- Relation
- Explicit function
- Implicit function
- Parameter

Materials

- Scientific calculator

Implicit Defferentiation

You have previously found a defined function in the form $y = f(x)$; it is an explicit function of the independent variable x where it determines the value of y directly whenever the value of x is known such as :

$y = 4x^3 - 5x + 2$, $y = \sin(2x + 3)$, $y = \frac{x+1}{x-1}$, ...

and $y' = 12x^2 - 5$, $y' = 2\cos(2x + 3)$, $y' =$ ....

But if y is related to the variable x with an equation containing x and y together such as:

$xy + y - 4 = 0$ (1) , $x^2 + y^2 - 9 = 0$ (2)

then each equation defining an implicit relation between x , y expresses the relation between the two coordinates of point (x, y) lying on its graphical curve .

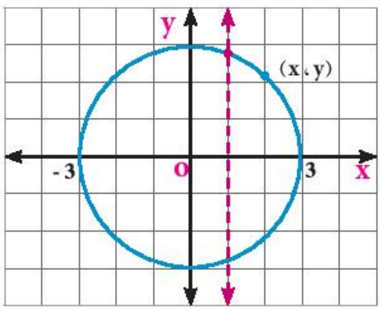
Notice that:

1- the equation $xy + y - 4 = 0$ can be written in the form:

$y(x + 1) = 4$ $\therefore y = \frac{4}{x+1}$ where $x \neq -1$

in this case, the implicit relation is defined as an explicit function.

2- The set of points (x , y) which satisfies the equation $x^2 + y^2 = 9$ draws a circle whose center is the origin point and its radius length is 3 units. From testing the vertical line, we notice the relation $x^2 + y^2 = 9$ doesnot represent a function but $y^2 = 9 - x^2$

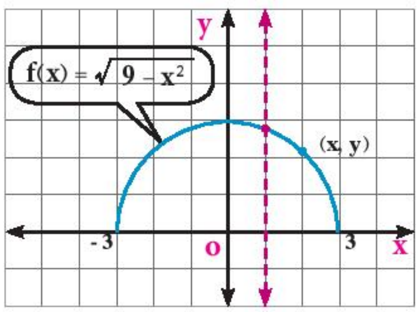


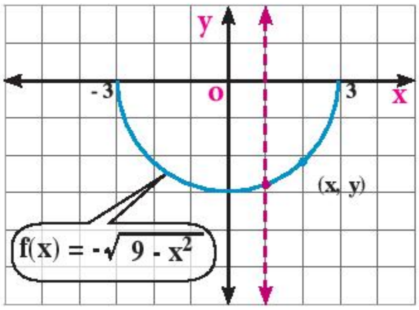
$\therefore y = \pm\sqrt{9 - x^2}$

The implicit relation $x^2 + y^2 = 9$ can be defined as two explicit functions

the first $y = \sqrt{9 - x^2}$

its domain is $[-3 , 3]$ range is $[0 , 3]$ and it is differentiable for each $x \in]-3 , 3[$





The second: $y = -\sqrt{9 - x^2}$
its domain [-3, 3] range [-3, 0]
and it is differentiable for each $x \in]-3 , 3[$
in many equations in the form f(x , y) = 0 it is difficult to express y in terms of x directly since the variable y doesnot represent an explicit function with respect to x. This function is called the implicit function.
The process of differentiating the implicit function (implicit differentiation) requires to differentiate both sides of the equation with respect to one of the two variables x or y according to the chain rule to get $\frac{dy}{dx}$ or $\frac{dx}{dy}$ respectively.

Example

1) Find $\frac{dy}{dx}$ if:

a) $x^3 + y^2 - 7x + 5y = 8$

b) $3xy + y^2 = x^2 - 7$

Solution

Remember

If y is a function in x and is differentiable ,then:

$$\frac{d}{dx}(y)^n = n\, y^{n-1}\frac{dy}{dx}$$

a) Notice that the equation doesnot give y explicity in terms of x. To find $\frac{dy}{dx}$, we differentiate the two sides of the equation with respect to x taking into consideration that y is a function to the variable x and it is differentiable then :

$$3x^2 + 2y\frac{dy}{dx} - 7 + 5\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y + 5) = 7 - 3x^2 \qquad \therefore \frac{dy}{dx} = \frac{7 - 3x^2}{2y + 5}$$

b) $\because 3xy + y^2 = x^2 - 7$ by differentiating the two sides of the equation with respect to x.

$$\therefore \frac{d}{dx}(3xy) + 2y\frac{dy}{dx} = 2x$$

$$3x\frac{dy}{dx} + y \times 3 + 2y\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}[3x + 2y] = 2x - 3y \qquad \therefore \frac{dy}{dx} = \frac{2x - 3y}{3x + 2y}$$

Try to solve

1) Find $\frac{dy}{dx}$ if:

a) $x^3 - 5xy + y^3 = 4x$

b) $x^2y + y^2x = 25$

Example

2) Find $\frac{dy}{dx}$ if:

a) $\sin 2y = y\cos 3x$

b) $\tan 2x + \cot y = xy$

Solution

a By differentiating both sides of the equation with respect to x

$\therefore \frac{d}{dx}(\sin 2y) = \frac{d}{dx}(y \cos 3x)$

$\cos 2y \times 2 \frac{dy}{dx} = y[-\sin 3x \times 3] + \cos 3x [\frac{dy}{dx}]$

$\frac{dy}{dx}[2\cos 2y - \cos 3x] = -3y \sin 3x \qquad \therefore \frac{dy}{dx} = \frac{3y \sin 3x}{\cos 3x - 2\cos 2y}$

b By differentiating both sides of the equation with respect to x

$\frac{d}{dx}(\tan 2x) + \frac{d}{dx}(\cot y) = \frac{d}{dx}(xy)$

$2\sec^2 2x - \csc^2 y \frac{dy}{dx} = x\frac{dy}{dx} + y$

$\frac{dy}{dx}[x + \csc^2 y] = 2\sec^2 2x - y \qquad \therefore \frac{dy}{dx} = \frac{2\sec^2 2x - y}{x + \csc^2 y}$

Try to solve

2 Find $\frac{dy}{dx}$ if:

a $x \cos y + y \cos x = 1$

b $3y = \sin x \cos 2y$

Notice: The final formula of the derivative $\frac{dy}{dx}$ in the implicit differentiation contains both x and y. This makes the process of calculating it over difficult at one of the values of x because we need first to know the corresponding value of y which is so difficult to be determined from the implicit relation.

Parametric Defferentiation

If the x-coordinate and y- coordinate of point (x , y) can be expressed as a function in a third variable t (it is called the parameter) by the two equations:

$x = f(t)$ and $y = g(t)$ where f and g have the same domain.

the two equation together represent an equation to one curve expressed in the parametric form

Learn

For the curve given in the parametric form $x = f(t)$, $y = g(t)$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ where f and g are two differentiable functions with respect to t

Example

3 Find $\frac{dy}{dx}$ for the following curves at the given values:

a $x = 5t + 3$, $y = 16t^2 + 9$, $t = 5$

b $x = 3\cos 2\theta$, $y = 4\sin 3\theta$, $\theta = \frac{\pi}{4}$

Solution

a) $x = 5t + 3 \qquad \frac{dx}{dt} = 5, \qquad y = 16t^2 + 9 \qquad \frac{dy}{dt} = 32t$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{32t}{5} \qquad$ and $\left[\frac{dy}{dx}\right]_{t=5} = 32$

b) $x = 3\cos 2\theta \qquad \frac{dx}{d\theta} = 3 \times -\sin 2\theta \times 2 = -6\sin 2\theta$

$y = 4\sin 3\theta \qquad \frac{dy}{d\theta} = 4 \times \cos 3\theta \times 3 = 12\cos 3\theta$

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{12\cos 3\theta}{-6\sin 2\theta} = \frac{-2\cos 3\theta}{\sin 2\theta}$

at $\theta = \frac{\pi}{4}$ then $\frac{dy}{dx} = \frac{-2\cos\frac{3\pi}{4}}{\sin\frac{\pi}{2}} = -2 \times \frac{-1}{\sqrt{2}} = \sqrt{2}$

Try to solve

3) Find $\frac{dy}{dx}$ for the following curves at the given values

a) $x = (t+7)(t-2) \quad , \quad y = (t^2+1)(t-2) \quad , \quad t = 1$.

b) $x = \sec^2\theta - 1, \quad y = \tan\theta, \quad \theta = \frac{-3\pi}{4}$

c) $x = \sqrt{3t-2} \quad , \quad y = \sqrt{4t+1}, \quad t = 2$

Critical thinking: find the value of the parameter z at which the curve $x = 2z^3 - 5z^2 - 4z + 12 \quad , \quad y = 2z^2 + z - 4$ has a horizontal tangent and another vertical tangent.

Example

4) Find the derivative of $(4x^3 - 9x^2 + 5)$ with respect to $(3x^2 + 7)$

Solution

Putting $y = 4x^3 - 9x^2 + 5, \quad z = 3x^2 + 7$ then $y = f(x), \qquad z = g(x)$

the two function f, g are differentiable with respect to x considering x is parameter for each of the two variables y and z

$\therefore$ from the parametric differentiation , we find that:

$\frac{dy}{dz} = \frac{y'}{z'} = \frac{12x^2 - 18x}{6x} = 2x - 3 \qquad$ **i.e.** $\frac{d}{d(3x^2+7)}[4x^3 - 9x^2 + 5] = 2x - 3$

Try to solve

4) Use the parametric differentiation to find:

a) derivative $x^2 + 1$ with respect to $\sqrt{x^2 - 1}$

b) derivative $\sqrt{8 + x^2}$ with respect to $\frac{x}{x+1}$ at $x = 1$

c) derivative $x - \sin x$ with respect to $1 - \cos x$ at $x = \frac{\pi}{3}$

Exercises 1 - 1

First : choose the correct answer:

1. If $x^2 + y^2 = 1$, then $\frac{dy}{dx}$ equals:

   a) x b) $\frac{1}{y}$

   c) $-\frac{x}{y}$ d) $-\frac{y}{x}$

2. If $x^2 + y^2 = 2xy$, then $\frac{dy}{dx}$ equals:

   a) -1 b) 0

   c) 1 d) 2

3. If $y^2 - 2\sqrt{x} = 0$, then $\frac{dy}{dx}$ equals:

   a) $\frac{2y}{\sqrt{x}}$ b) $\sqrt{x}$

   c) $\frac{x}{y^2}$ d) $\frac{1}{y^3}$

4. If $x = 2t^2 + 3$, $y = \sqrt{t^3}$, $n = 1$, then $\frac{dy}{dx}$ equals:

   a) $\frac{3}{8}$ b) $\frac{3}{4}$

   c) 2 d) 6

5. the slope of the tangent to the curve $xy^2 = 3$ at point $(3, 1)$ equals:

   a) -3 b) $-\frac{1}{6}$

   c) $\frac{1}{3}$ d) $\frac{3}{2}$

Second: find $\frac{dy}{dx}$ for each of the following:

6. $x^2 - 4y^2 + 7 = 0$
7. $x^4 + 3y^4 - 2 = 0$
8. $x^2 - 2xy = 5 - y^2$
9. $x^3 + 6xy = 4y + 3$
10. $\frac{x}{y} + \frac{y}{x} = 1$
11. $xy + \sin y = 5$

(12) $x \sin y + y \cos x = 0$

(13) $x \csc y = y \cot x$

(14) $x^2 \sin y - y^2 \sin x = 9$

(15) $\sin 2x \cos 2y = \frac{3}{4}$

Third: find $\frac{dy}{dx}$ for the following curves at the given values:

(16) $x = 13 - 2t$, $y = 4t^2 - \sqrt{t}$, $t = 4$

(17) $x = \sin 2\pi\theta$, $y = \cos 2\pi\theta$, $\theta = \frac{1}{6}$

(18) $x = 5 + \sec^2 3\theta$, $y = 1 - \tan 3\theta$, $\theta = \frac{\pi}{4}$

(19) Find the slope of the tangent to the curve $\cos\sqrt{\pi y} = 3x + 1$ at point $(-\frac{1}{3}, \frac{\pi}{4})$

(20) Find the derivative of $\frac{x+1}{x-1}$ with respect to $\sqrt{2x+1}$ at $x = 4$

(21) Find the value of parameter t at which the curve $x = 2t^3 - 5t^2 + 4t - 9$, $y = 2t^2 + n - 5$ has:

(a) a vertical tangent.

(b) a horizontal tangent.

1 - 2 Higher Derivatives of a Function

You will learn

- Finding the higher order derivatives of a function

Key terms

- Order
- Derivative

Materials

- Scientific calculator

Think and discuss

if $y = f(x)$ where $y = x^4 + 5x^3 - 2x + 3$ find the derivative of function f, Can you repeat the process of differentiation with respect to x? why? Does the process of differentiation end up? Explain.

Learn

Higher - Order Derivative

- **If** $y = f(x)$ where f is a differentiable function with respect to x, then its first derivative is $y' = \frac{dy}{dx} = f'(x)$ and it represents a new function.
- **If** the first derivative is differentiable with respect to x, then its derivative $\frac{d}{dx}(\frac{dy}{dx})$ is called the second derivative of the function f and it represents another function. It is denoted by the symbol $y'' = \frac{d^2y}{dx^2} = f''(x)$
- By repeating the process of differentiation, we get the third derivative of the function f and it is denoted by the symbol $\frac{d^3y}{dx^3}$, and so on.

The derivatives of a function starting from the second derivative are called the higher derivatives and the derivative is written from the order n as follows:

$y^{(n)} = \frac{d^ny}{dx^n} = f^{(n)}(x)$ where n is a positive integer number

Notice that :

1- $\frac{d^2y}{dx^2}$ is read d two y by dx squared

2- There is a difference between $\frac{d^2y}{dx^2}$ and $\left(\frac{dy}{dx}\right)^2$, the first refers to the second derivative of the function whereas the second refers to the square of the first derivative.

Example

1 Find the second derivative for each of the following:

a $y = 2x^4 + 3x - 5$ **b** $y = \frac{x+1}{x-1}$

c) $y = \sin(3x - 2)$ d) $y = \sqrt{3x - 2}$

Solution

a) $\because y = 2x^4 + 3x - 5 ,\ x \in R$ $\therefore \frac{dy}{dx} = 8x^3 + 3$, $\frac{d^2y}{dx^2} = 24x^2$

b) $\because y = \frac{x+1}{x-1} ,\ x \neq 1$ $\therefore \frac{dy}{dx} = \frac{x - 1 - (x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2} ,\ x \neq 1$

$\frac{d^2y}{dx^2} = \frac{d}{dx}[-2(x-1)^{-2}] = \frac{4}{(x-1)^3} ,\ x \neq 1$

c) $\because y = \sin(3x - 2) ,\ x \in R$ $\therefore \frac{dy}{dx} = 3\cos(3x - 2) ,\ \frac{d^2y}{dx^2} = -9\sin(3x - 2)$

d) $\because y = \sqrt{3x - 2} ,\ x > \frac{2}{3}$ $\therefore \frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}} ,\ x > \frac{2}{3}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}[\frac{3}{2}(3x-2)^{-\frac{1}{2}}] = -\frac{9}{4\sqrt{(3x-2)^3}} ,\ x > \frac{2}{3}$

Try to solve

1) Find the third derivative for each of the following:

a) $y = x^4 - 2x^2 + 5$ b) $z = (2n - 1)^4$

c) $f(x) = \cos(2x + \pi)$ d) $f(x) = \frac{x}{x-1}$

Critical thinking: if $y = \sin ax$, investigate the consecutive differentiation pattern and find $y^{(25)}$

Example

2) If $y^2 + 2xy = 8$, prove that : $(x + y)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 = \text{zero}$

Solution

a) $\because y^2 + 2xy = 8$, **by differentiating both sides with respect to x**

$\therefore 2y\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$ **by dividing by 2**

$(x + y)\frac{dy}{dx} + y = 0$ **by differentiating both sides with respect to x**

$\therefore (x + y)\frac{d^2y}{dx^2} + \frac{dy}{dx}(1 + \frac{dy}{dx}) + \frac{dy}{dx} = 0$

and $(x + y)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + (\frac{dy}{dx})^2 = \text{zero}$

Try to solve

2) a) If $x^2 + y^2 = 9$, **prove that:** $y\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 + 1 = 0$

b) If $y = \tan x$, **prove that:** $\frac{d^2y}{dx^2} = 2y(1 + y^2)$

Parametric equations:

Example

(3) If $x = 2n^3 - 5$, $y = 6n^2 + 1$ find $\frac{d^2y}{dx^2}$ at $n = 1$

Solution

By differentiating each of x and y with respect to parameter n

$$\therefore \frac{dx}{dn} = 6n^2 \quad , \quad \frac{dy}{dn} = 12n$$

$$\because \frac{dy}{dx} = \frac{dy}{dn} \times \frac{dn}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{12n}{6n^2} = 2n^{-1} \quad , \quad n \neq 0$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}[2n^{-1}] = -2n^{-2} \times \frac{dn}{dx}$$

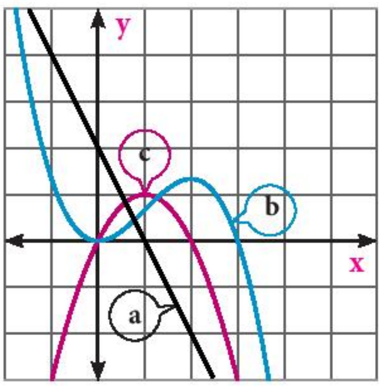
$$= \frac{-2}{n^2} \times \frac{1}{6n^2} = -\frac{1}{3n^4} \quad , \quad n \neq 0$$

at $n = 1$ $\qquad \therefore \frac{d^2y}{dx^2} = -\frac{1}{3}$

Try to solve

(3) If $x = z^2 - 2z$, $y = z^2$

Find $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ at $z = 2$



Critical thinking: the opposite figure shows a graphical representation to the curves of the functions f(x), f' (x) , f'' (x) where f(x) is polynomial . Determine the curve of each function.

Activity

Use the geogebra program or any other program to graph the following functions and their first and second derivatives, then record your observations.

(a) $f(x) = x^3 - 4x^2 + 12$ $\qquad$ (b) $g(x) = \frac{1}{4}x^2 + 4$

Do your observations match with your decision in critical thinking?

Exercises 1 - 2

Find the third derivative for each of the following:

1. $y = x^5 - 4x^3 + 3$
2. $y = \frac{2x}{x+1}$
3. $y = \sin(2x - 7)$
4. $y = \cos(\pi - 3x)$
5. $y = \sin x \cos x$
6. $y = \sqrt{2x - 5}$

Answer the following:

7. If $3x^2 + 5 = 2xy$, prove that: $x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 3$
8. If $x^2 + y^2 = 4$, prove that: $y^3 \frac{d^2y}{dx^2} - 4 = 0$
9. If $y = 3 \cos(2x + 1)$, prove that: $\frac{d^2y}{dx^2} + 4y = 0$
10. If $xy = \sin x \cos x$, prove that: $x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 4xy = 0$
11. If $y = x \sin x$, prove that: $x \frac{d^3y}{dx^3} + x \frac{dy}{dx} + 2y = 0$
12. If $y = \sec x$, prove that: $y \frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 = y^2 (3y^2 - 2)$
13. If $\frac{dy}{dx} = 2x - 3$, $\frac{dz}{dx} = x^2 - 1$, find: $\frac{d^2y}{dz^2}$ at $x = 2$
14. If $x = 3n^2 - 1$, $y = n^3 + 2$, find: $\frac{d^2y}{dx^2}$ at $n = 4$
15. If $x = \frac{z+1}{z-1}$, $y = \frac{z-1}{z+1}$, find: $\frac{dy}{dx}$ at $z = 2$
16. If $x = \sec z$, $\sqrt{y} = \tan z$, prove that: $\frac{d^2y}{dx^2} = 2$

1 - 3 Derivatives of Exponential and Logarithmic functions

You will learn

- Derivatives of exponential functions.
- Derivatives of logarithmic functions.
- Logarithmic differentiation.
- Higher derivatives of exponential and logarithmic functions.
- Modeling the problems.

Key terms

- Derivative
- Chain Rule
- First Derivative
- Logarithmic Differentiation

Materials

- Scientific calculator.

Definition: The number e is defined by the relation:

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots\ldots \infty = \sum_{n=0}^{\infty} \frac{1}{\lfloor n}$$

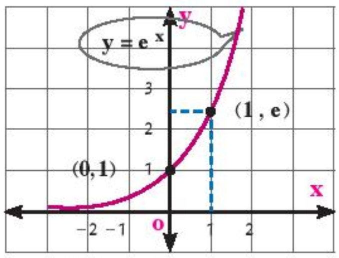
Learn

The natural exponential function

It is an exponential function whose base is e
$f(x) = e^x \ , \ x \in \mathbb{R}$

Notice that

1) The domain of the function f where $f(x) = e^x$ is $\mathbb{R}$ and its range is $]0, \infty[$
2) The curve of the function passes through points (0 , 1) , (1 , e)
3) $f(x) = e^x$ is *One - to - One* function that accepts the existance of an inverse function called the natural logarithmic function.
4) The symbol *exp (x)* is used in any graphic program to graph the function
5) $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots \infty = \sum_{n=0}^{\infty} \frac{x^n}{\lfloor n}$



$\lim_{x \to \infty} e^x = \infty$,

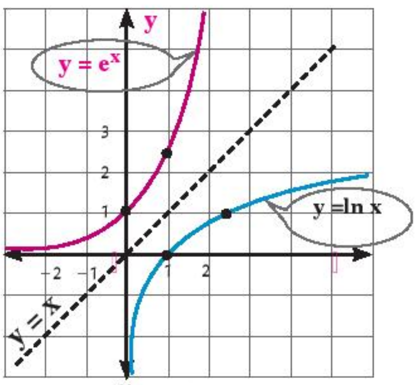
$\lim_{x \to -\infty} e^x = 0$

The natural logarithmic function

It is a logarithmic function whose base is e , $f(x) = \ln x$, $x \in \mathbb{R}^+$

Notice that:

1) The domain of the function f where $f(x) = \ln x$ is $\mathbb{R}^+$ and its range is $\mathbb{R}$
2) The curve of the function passes through points (1 , 0) and (e , 1)
3) it is an inverse function for the function $y = e^x$
4) The symbol *ln(x)* is used in any graphic program to graph the function.



$\lim_{x \to \infty} \ln x = \infty$,

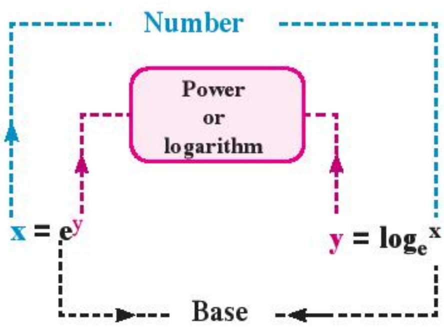
$\lim_{x \to 0} \ln x = -\infty$

5) To find the value of ln 10 for example, press the following buttons:

Start → ln 1 0 =

We find that ln 10 = 2.302585093 to the nearest 9 decimals.

Some properties of the natural logarithm



The natural logarithm has the same logarithms properties which you have previously learned.

If $x \in \mathbb{R}^+$, $y \in \mathbb{R}$, $a \in \mathbb{R}^+ - \{1\}$ than:

1) The form $\ln x = y$ is equivalent to the form $e^y = x$

2) $e^{\ln x} = x$

3) $\ln e = 1$

4) $\ln 1 = 0$

5) $\log_a x = \frac{\ln x}{\ln a}$ **(the property of changing the base)**

For each x and $y \in \mathbb{R}^+$ and $n \in \mathbb{R}$

6) $\ln xy = \ln x + \ln y$

7) $\ln \frac{x}{y} = \ln x - \ln y$

8) $\ln x^n = n \ln x$

9) $\log_e x \times \log_x e = 1$

Derivative of the natural exponential function

if $f(x) = e^x$ then $f'(x) = e^x$

$$\because e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots\ldots \infty$$

Differentiating both sides with respect to x

$$\frac{d}{dx}(e^x) = \frac{1}{\lfloor 1} + \frac{2x}{\lfloor 2} + \frac{3x^2}{\lfloor 3} + \ldots\ldots \infty$$

$$= 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots\ldots \infty$$

$$\frac{d}{dx}(e^x) = e^x$$

Example *Derivative of the natural exponential function*

(1) Find the first derivative for each of:

(a) $y = x^2 + 3e^x$ (b) $y = x^3 e^x$ (c) $y = \frac{2e^x}{x+1}$

Solution

a) $\because y = x^2 + 3e^x \quad \therefore \frac{dy}{dx} = 2x + 3\frac{d}{dx}(e^x) = 2x + 3e^x$

b) $\because y = x^3 e^x \qquad \therefore \frac{dy}{dx} = x^3 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^3)$

$= x^3 e^x + 3x^2 e^x = x^2 e^x (x + 3)$

c) $\because y = \frac{2e^x}{x+1} \qquad \therefore \frac{dy}{dx} = \frac{(x+1)\frac{d}{dx}(2e^x) - 2e^x \frac{d}{dx}(x+1)}{(x+1)^2} = \frac{2xe^x}{(x+1)^2}$

Try to solve

1) Find $\frac{dy}{dx}$ for each of the following:

a) $y = 2e^x + \cos 2x$ b) $y = e^x \sin x$ c) $y = \frac{e^x}{\tan x}$

Critical thinking: What is the relation between the slope of the tangent to the curve $y = e^x$ at any point on it and the y - coordinate of this point? Explain

Chain rule

If Z is a differentiable function of x , $f(z) = e^z$

then: $\frac{d}{dx}(e^z) = e^z \cdot \frac{dz}{dx}$

Example

2) Find the first derivative for each of:

a) $y = e^{3x^2+5}$ b) $y = 3e^{\sec x}$ c) $y = (e^{3x} - e^{-2x})^5$

Solution

a) $\because y = e^{3x^2+5} \qquad \therefore \frac{dy}{dx} = e^{3x^2+5} \times \frac{d}{dx}(3x^2 + 5) = 6x\, e^{3x^2+5}$

b) $\because y = 3e^{\sec x} \qquad \therefore \frac{dy}{dx} = 3e^{\sec x} \times \frac{d}{dx}(\sec x) = 3e^{\sec x} \cdot \sec x \tan x$

c) $\because y = (e^{3x} - e^{-2x})^5 \qquad \therefore \frac{dy}{dx} = 5(e^{3x} - e^{-2x})^4 \, [3e^{3x} + 2e^{-2x}]$

Try to solve

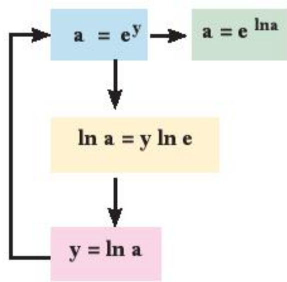
2) Find $\frac{dy}{dx}$ for each of the following:

a) $y = 2x + e^{6x}$ b) $y = \frac{1}{2} e^{7-x^2}$ c) $y = (e^{2x} + e^{-2x})^3$

Learn

Derivative of exponential function to the base a

If $f(x) = a^x$ **than** $f'(x) = a^x \ln a$



Notice that $a = e^{\ln a}$ **(from the logarithm properties)**

$\therefore a^x = \left[e^{\ln a}\right]^x = e^{x \ln a}$

and $\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln a}) = e^{x\ln a} \times \ln a = a^x \times \ln a$

in general: $\frac{d}{dx}(a^z) = a^z \ln a \bullet \frac{dz}{dx}$

Example *The derivative of the exponential function*

(3) Find $\frac{dy}{dx}$ for each of the following:

(a) $y = 5 \times 6^x$ (b) $y = 3^{(3x^2 - 5x + 2)}$ (c) $y = e^{\sin x} \times 2^{-5x}$

Solution

(a) $\because y = 5 \times 6^x$ $\quad \therefore \frac{dy}{dx} = 5 \frac{d}{dx}(6^x) = 5 \times 6^x \ln 6$

(b) $\because y = 3^{3x^2 - 5x + 2}$ $\quad \therefore \frac{dy}{dx} = (6x - 5) \times 3^{(3x^2 - 5x + 2)} \times \ln 3$

(c) $\because y = e^{\sin x} \times 2^{-5x}$ $\quad \therefore \frac{dy}{dx} = e^{\sin x} \frac{d}{dx}(2^{-5x}) + 2^{(-5x)} \frac{d}{dx} e^{\sin x}$

$$= e^{\sin x}[-5 \times 2^{-5x} \ln 2] + 2^{-5x}[e^{\sin x} \cos x]$$

$$= 2^{-5x} e^{\sin x}[\ln 2^{-5} + \cos x]$$

Try to solve

(3) Find $\frac{dy}{dx}$ for each of the following:

(a) $y = 5^{x^2 + 2x}$ (b) $y = 2^{\sec^2 x}$ (c) $y = e^{2x} a^{x^2 - 5}$

Learn

Derivative of natural logarithm function

if $f(x) = \ln x$, $x > 0$ **then** $f'(x) = \frac{1}{x}$

Notice that the logarithmic function is an inverse function of the exponential function

If $y = \ln x$ **then** $x = e^y$ **(1)**

by differentiating both sides of relation **(1)** with respect x $\therefore 1 = e^y \frac{dy}{dx}$ **(2)**

From **(1)** and **(2)**: we de deduce that $\frac{dy}{dx} = \frac{1}{x}$ **I.e:** $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Example

Derivative of natural logarithm function

(4) Find the first derivative for each of the following:

(a) $y = 3x + \ln x$ (b) $y = (2x^5 - 3)\ln x$ (c) $y = \frac{\ln x - 1}{\ln x + 1}$

Solution

(a) $\because y = 3x + \ln x$ $\therefore \frac{dy}{dx} = 3 + \frac{d}{dx}(\ln x) = 3 + \frac{1}{x}$

(b) $\because y = (2x^5 - 3)\ln x$ $\therefore \frac{dy}{dx} = (2x^5 - 3)\frac{d}{dx}(\ln x) + (\ln x)\frac{d}{dx}(2x^5 - 3)$

$= (2x^5 - 3) \times \frac{1}{x} + 10x^4 \ln x$

$= \frac{1}{x}[(2x^5 - 3) + 10x^5 \ln x]$

(c) $\because y = \frac{\ln x - 1}{\ln x + 1}$ $\therefore \frac{dy}{dx} = \frac{(\ln x + 1)\left(\frac{1}{x}\right) - (\ln x - 1) \times \frac{1}{x}}{(\ln x + 1)^2} = \frac{2}{x(\ln x + 1)^2}$

Try to solve

(4) Find $\frac{dy}{dx}$ for each of the following:

(a) $y = 5 - 3\ln x$ (b) $y = x^2 \ln x$ (c) $y = \frac{1 - 2\ln x}{\ln x}$

Critical thinking: What is the relation between the slope of tangent to the curve $y = \ln x$ at any point on it and the x-coordinate to the tangent point? Explain.

Chain rule

- **If** z is a differentiable function with respect to x and $f(z) = \ln z$

than: $\frac{d}{dx}[\ln z] = \frac{1}{z} \bullet \frac{dz}{dx}$

- **If** $x < 0$, **then:** $\frac{d}{dx}[\ln(-x)] = \frac{1}{-x} \times -1 = \frac{1}{x}$

➲ **In general**

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{for each } x \neq 0$$

, $\frac{d}{dx}[\ln|z|] = \frac{z'}{z}$ where z is a differentiable function at x.

Example

5 Find $\frac{d}{dx}$ for each of the following:

a $y = \ln(2x^3 + 9)$ b $y = x^4 \ln x^3$ c $y = \ln \frac{x^2}{x+7}$

Solution

a $\because y = \ln(2x^3 + 9)$ $\therefore \frac{dy}{dx} = \frac{1}{2x^3+9} \times \frac{d}{dx}(2x^3 + 9) = \frac{6x}{2x^3+9}$

b $\because y = x^4 \ln x^3$ $\therefore \frac{dy}{dx} = x^4 \frac{d}{dx}(\ln x^3) + \ln x^3 \frac{d}{dx}(x^4)$

$= x^4 \times \frac{1}{x^3} \times 3x^2 + \ln x^3 \times 4x^3$

$= 3x^3 + 4x^3 \ln x^3 = x^3[3 + 4\ln x^3]$

c $\frac{d}{dx}\left(\ln \frac{x^2}{x+7}\right) = \frac{x+7}{x^2} \times \frac{(x+7)(2x) - x^2 \times 1}{(x+7)^2} = \frac{x+14}{x(x+7)}$

Try to solve

5 Find $\frac{dy}{dx}$ for each of the following:

a $y = \ln(7x - 3)^2$ b $y = 2x^2 \ln x^3$ c $y = \frac{x}{\ln x}$

Learn

Derivative of logarithmic function to the base a

$$\text{If } f(x) = \log_a x, \text{ then } f'(x) = \frac{1}{x \ln a}$$

Notice $\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{1}{\ln a}\frac{d}{dx}[\ln x] = \frac{1}{x} \cdot \frac{1}{\ln a}$

and $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$

in general $\boxed{\frac{d}{dx}(\log_a z) = \frac{1}{z \ln a} \cdot \frac{dz}{dx}}$

Remember

From the properties of logarithms

$\log_a x = \frac{\ln x}{\ln a}$

$\log_e a \times \log_a e = 1$

Derivative of the logarithmic function

Example

6 Find $\frac{dy}{dx}$ for each of the following:

a $y = \log_3 x$ b $y = \log_5 (3x - 2)$ c $y = \log (2x - 3)^2$

Solution

a $\because y = \log_3 x$ $\therefore \frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\ln 3} = \frac{1}{x \ln 3}$

b $\because y = \log_5 (3x - 2)$ $\therefore \frac{dy}{dx} = \frac{3}{(3x - 2)\ln 5}$

c $\because y = 2\log |2x - 3|$ $\therefore \frac{dy}{dx} = \frac{2 \times 2}{(2x - 3)\ln 10} = \frac{4}{(2x - 3)\ln 10}$

Try to solve

6 Find the slope of the tangent for each of the following curves at the given values of x:

a $y = \log_2 5x$, $x = 2$ b $y = 4\log (3x + 1)$, $x = 1$

c $y = \log_2 (2x^2 - 3)^4$, $x = 1$ d $y = 3(\log x)^2$, $x = 3$

7 **Geometric applications:** If $\overleftrightarrow{AB}$ is a tangent to the curve $y = \ln \frac{x}{2}$ at point C (1 , y) and intersects x - axis at point A and y-axis at point B, find the length of $\overline{AB}$

Solution

length of $\overline{AB}$ ← intersection points A, B ← equation of tangent $\overleftrightarrow{AB}$

To find the length of $\overline{AB}$, follow the opposite graph:

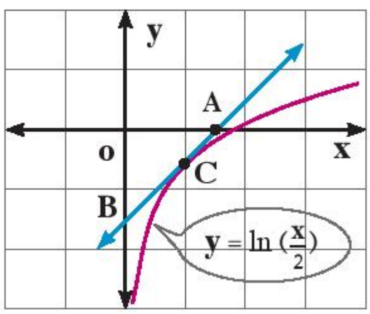
The slope of the tangent at any point: $\frac{dy}{dx} = \frac{1 \times 2}{x} \times \frac{1}{2} = \frac{1}{x}$

$\because \overleftrightarrow{AB}$ touches the curve at any point C (1 , y)

then $y = \ln \frac{1}{2} = -\ln 2$ i.e C (1 , - ln 2), at which

$\frac{dy}{dx} = 1$, and the equation of the tangent $\overleftrightarrow{AB}$ at C is:

$y + \ln 2 = x - 1$



$\because \overleftrightarrow{AB}$ intersects X -axis at point A $\therefore A(1 + \ln 2 ,\ 0)$

and y - axis at point B $\therefore B(0 ,\ -1 - \ln 2)$

Then $(AB)^2 = (1 + \ln 2)^2 + (1 + \ln 2)^2$ $\therefore AB = \sqrt{2}\,(1 + \ln 2)$

Try to solve

7) If the normal to the curve $y = \ln 2x$ at point A $(1 , \ln 2)$ intersects x-axis at point B, find the length of $\overline{AB}$ to the nearest three decimals.

Mathematical Application

Logarithmic differentiation

The relation among the variables can be expressed in a logarithmic form by taking off the natural logarithm of its both sides and using the properties of logarithms to simplify the relation before doing the differentiation operations.

Example

8) Find $\frac{dy}{dx}$ for each of the following:

a) $y = (x^3 + 5)^x$ b) $y = [\sin x]^{\tan x}$

Solution

a) $\because y = (x^3 + 5)^x$ **By taking the natural logarithm of both sides of the relation**

$\therefore \ln y = x \ln (x^3 + 5)$ **By differentiating both sides of the relation with respect to x**

$$\frac{1}{y}\frac{dy}{dx} = \ln (x^3 + 5) + \frac{x}{x^3 + 5} \times 3x^2$$ **by multiplying both sides $\times\, y = (x^3 + 5)^x$**

$$\therefore \frac{dy}{dx} = (x^3 + 5)^x \left[\frac{3x^3}{x^3 + 5} + \ln (x^3 + 5)\right]$$

b) $\because y = [\sin x]^{\tan x}$ **By taking the natural logarithm of both sides of the relation**

$\ln y = \tan x \ln \sin x$ **By differentiating both sides of the relation with respect to x**

$$\frac{1}{y}\frac{dy}{dx} = \tan x \times \frac{d}{dx}(\ln \sin x) + \ln \sin x \times \frac{d}{dx}(\tan x)$$

$$= \frac{\sin x}{\cos x} \times \frac{1}{\sin x} \times \cos x + \ln \sin x \times \sec^2 x$$

$$= 1 + \sec^2 x \log_e \sin x$$ **by multiplying both sides $\times\, y = [\sin x]^{\tan x}$**

$$\therefore \frac{dy}{dx} = [\sin x]^{\tan x}(1 + \sec^2 x \ln \sin x)$$

Try to solve

8) Find $\frac{dy}{dx}$ for each of the following

a) $y = x^{2x}$ b) $y = (\sin x)^x$ c) $y^2 = 3^x \times 2^y$

9) **Satisfying a relation:** If $y = e^{-x}\sqrt{\frac{1+x}{1-x}}$ where $-1 < x < 1$, prove that: $(1 - x^2)\, y' = x^2 y$

Solution

$\because y = e^{-x}\sqrt{\frac{1+x}{1-x}}$ **By taking the logarithm of both sides to base e**

$\therefore \ln y = \ln e^{-x} + \frac{1}{2}\ln\frac{1+x}{1-x}$

$\ln y = -x + \frac{1}{2}[\ln(1+x) - \ln(1-x)]$

By differentiating both sides of the relation with respect to x

$\frac{1}{y}\times y' = -1 + \frac{1}{2}\left[\frac{1}{1+x} - \frac{-1}{1-x}\right]$

$\frac{y'}{y} = -1 + \frac{1}{2}\left[\frac{1-x+1+x}{1-x^2}\right]$

$\frac{y'}{y} = -1 + \frac{1}{1-x^2} = \frac{-1+x^2+1}{1-x^2}$

$\frac{y'}{y} = \frac{x^2}{1-x^2}$ $\therefore (1-x^2)\, y' = x^2 y$

Try to solve

9) If $y = ae^{\frac{b}{x}}$, prove that: $x\,y\,y'' + 2\,y\,y' - x\,y'^2 = 0$

Exercises 1 - 3

Choose the correct answer

1) If $f(x) = e^{3x}$, then $f'(x)$ equals:

a) e^{3x} b) $3e^{3x}$ c) $9e^{3x}$ d) $3e^{2x}$

2) If $f(x) = ae^x$, then $f'(-2)$ equals:

a) $-f(2)$ b) $-f'(2)$ c) $-f(-2)$ d) $f(-2)$

3) The curve of the function f: $f(x) = 1 + \ln(x - 2)$ is the same curve of the function g : $g(x) = \ln x$ by translation:

a) (1 , 2) b) (1 , -2) c) (-2 , 1) d) (2 , 1)

4) The ratio between the slope of the tangent to the curve $y = \ln 3\sqrt{x+1}$ and the slope of the tangent to the cure $y = \ln 5\sqrt{x+1}$ when $x = a$ is

a) 3 : 5 b) 5 : 3 c) 1 : 1 d) ln 3 :ln 5

Find the first derivative for each of the following:

5) $y = e^{3x^5}$

6) $y = e^{x^2 - x}$

7) $y = (3^{x-1})^{-2}$

8) $y^2 = e^{5x^2 - 3}$

9) $y = \ln(2x - 7)$

10) $y = \ln(\frac{1}{2}x^2 + x)$

11) $y = \ln \frac{x^2}{x + 7}$

12) $y = x^2 \ln x$

13) $y = \log_3 (4x + 9)^2$

14) $y = \frac{e^{3x}}{\log x}$

15) $y = \sec e^x$

16) $y = 2e^{3x} - 5 \log_2 \frac{x}{5}$

Find the slope of the tangent for each of the following curves at the given values:

17) $y = \sqrt{x} - 2e^x \quad , \quad x = \frac{1}{4}$

18) $y = x^2 - 3 \ln x \quad , \quad x = 2$

19) $y = \frac{1}{4} e^{2x} - 2 \ln x \quad , \quad x = \frac{1}{2}$

Find $\frac{dy}{dx}$ for each of the following:

20) $y e^x = e^3$

21) $x \ln y = 58$

22) $y = x^{\sin x}$

23) $y = e^{e^x}$

24) $y = e^{x^e}$

25) $y = x^{\frac{1}{x}}$

Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ for each of the following:

26) $x = e^{2n} \quad , \quad y = n^3$

27) $x = 6 \ln n \quad , \quad y = n^2$

Answer the following:

28) If $y = x^2 \ln \frac{x}{a}$, find $\frac{d^3y}{dx^3}$ when $x = 4$

29) If $y = \sqrt{\frac{x^2 + 1}{x^2 - 1}}$, prove that $(x^4 - 1) y' + 2x y = 0$

30) Find the values of x at which the tangent of the curve $y = 9x^3 - 8\ln x$ is parallel to x-axis.

31) Find the equation of the normal to the curve $y = 3e^x$ at a point lying on it and its x-coordinate equals -1

1 - 4 Related time rates

You will learn

- The concept of the related time rates
- Methods of solvingthe equations of the related time rates
- Modeling and solving mathematical , physical and life problems

Key terms

- Rate
- Related Rates

Materials

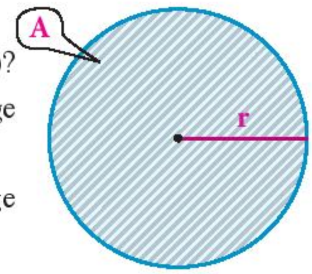
- Scientific calculator
- Computer graphics

Identifying and naming the variables
↓
Graphing the incomes
↓
Find the relation
↓
Differentating the relation with respect to time
↓
Substituting the values to find the required rate

Think and discuss

When a circular lamina is exposed to a heat source for a period of time (t) second

- Does its radius length (r) change over time (t)?
- Does the surface area (A) of the lamina change over time (t)?
- Does the surface area (A) of the lamina change as its radius length(r) changes? Explain.



Note that :

1 - The two variables a and r change overtime (a function of time) and they are related by the relation
$A = \pi r^2$ i.e : $A = f(r)$

2 - The differentiation of both sides of the relation above with respect to the time leads to a new equation relating between the related time rate of each variable and it is known as the equation of the related rates

Where: $\frac{dA}{dt} = f'(r) \times \frac{dr}{dt}$

3 - The time rate is positive if the variable increases by the time increasing and it is negative if the variable decreases by the time increasing.

Oral expression : Which of the following rates is positive?
(expansion - shrinking - approaching - diverge- molding- leaking - melting - accumulation - decreasing- increasing)

Example Inflating the balloon

1 When inflating a spherical balloon with gas, the rate of increase in its volume was 8π cm^3/sec when its radius length was 4cm. Find at this moment :

a The rate of increasing the radius length.

b The rate of increasing the surface area.

Solution

Let the volume of the balloon v, the radius length (r) and the surface area of the ballon (A) be differentiable functions at t.

a $V = \frac{4}{3}\pi r^3$ by differentiating both sides of the equation with respect to the time

$$\frac{dv}{dt} = \frac{4}{3}\pi \times 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad (1)$$

$\because \frac{dv}{dt} = 8\pi$ cm^3/sec , r = 4cm by substituting in the equation

$\therefore 8\pi = 4\pi(4)^2 \frac{dr}{dt}$ **i.e** $\frac{dr}{dt} = \frac{1}{8}$ cm/sec

b $A = 4\pi r^2$ by differentiating both sides of the equation with respect to the time

$$\frac{dA}{dt} = 4\pi \times 2r \frac{dr}{dt} = 8\pi r \frac{dr}{dt} \qquad (2)$$

$\because \frac{dr}{dt} = \frac{1}{8}$ cm/sec , r = 4cm by substituting in the equation

$\therefore \frac{dA}{dt} = 8\pi \times 4 \times \frac{1}{8} = 4\pi$ cm^2/sec

Try to solve

1) **Volume:** A cube extends by heat so its edge length increases at a rate of 0,02 cm/min, and its surface area increases at a rate of 0,72 cm^2/min, at a moment . Find the cube edge length at this moment and the rate of the increase in its volume at this time .

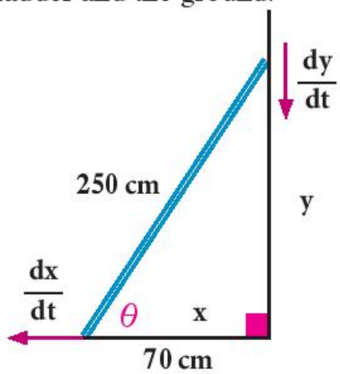
Example **Ladder motion**

2) A 250cm ladder is leaning against a vertical wall . If the top of the ladder slid down at a rate of 10cm/sec when the base of the ladder is 70cm from the wall . Find:

a The rate of sliding the base of the ladder.

b The rate of change of the measure of the angle between the ladder and the ground.

Solution



a Let y be the distance between the top of the ladder and the ground and x the distance between the base of the ladder and the vertical wall,

From pythagorean theorem $x^2 + y^2 = (250)^2$ (1)

By differentiating both sides of the equation with respect to time

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ $\therefore \frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$ (2)

$\because$ The top of the ladder slides down , then y decreases

$\therefore \frac{dy}{dt} = -10$ cm /sec

when x = 70 cm from equation (1) we find that: y = 240 cm

By substituting in equation (2) we deduce that: $\frac{dx}{dt} = -\frac{240}{7} \times -10 = \frac{240}{7}$ cm/sec

i.e the base of the ladder slides away from the wall at a rate of $\frac{240}{7}$ cm /sec

b Let θ be the measure of the angle of inclination of the ladder on the ground

$\sin\theta = \frac{y}{250}$ by differentiating both sides with respect to t

$\therefore \cos\theta \frac{d\theta}{dt} = \frac{1}{250} \frac{dy}{dt}$ but $\frac{dy}{dt} = -10$ cm/sec when x = 70 cm

$\frac{70}{250} \times \frac{d\theta}{dt} = \frac{1}{250} \times -10$ $\therefore \frac{d\theta}{dt} = -\frac{1}{7}$ rad/sec

i.e. The measure of the angle decreases at a rate of $\frac{1}{7}$ radian /sec

Try to solve

(2) **ladder motion:** The base of a ladder is placed on horizontal ground and its top is leaning against a vertical wall. If the base of the ladder slid away from the wall at a rate of 30cm/sec , find the rate of sliding the top of the ladder when the measure of the angle between the ladder and the ground equals $\frac{\pi}{3}$

Critical thinking: A 15 ton rocket is launched and it emits the fuel at a constant rate 200 kg/s. What is the mass of the rocket after 30 second of the launching moment?

Important note: If $x_\circ$ is the initial value of variable x (when t = 0) , $\frac{dx}{dt}$ is the rate of change of x with respect to time and x is the value of the variable after time t , **then :** $x = x_\circ + \frac{dx}{dt} \times t$

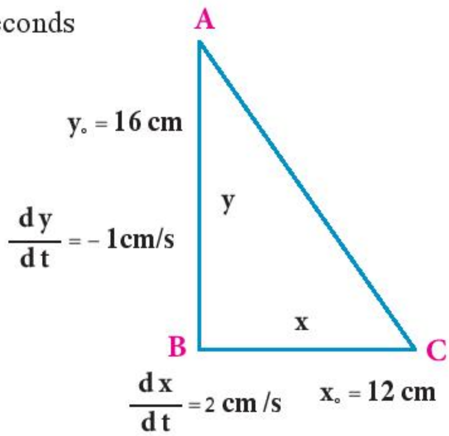
in the previous "**critical thinking**" use the relation $k = k_\circ + \frac{dx}{dt} \times t$ to check your answer.

Example Area

(3) In a right - angled triangle, the lengths of the legs of the right angle are 12cm , 16cm. If the length of the first leg increases at a rate of 2 cm/s and the length of the second leg decreases at a rate of 1 cm/s.

a Find the rate of change of the triangle area after 2 seconds

b When is the triangle isosceles?



Solution

a Let x and y be the lengths of the legs after t second.

the area of the triangle at this time where x, y and a are functions in the time:

$\therefore$ x = 12 + 2 t , y = 16 - t

$A = \frac{1}{2} x \times y = \frac{1}{2} (12 + 2 n) (16 - t)$

A = (6 + t) (16 - t) by differentiating both sides of the equation with respect to time

$\therefore \frac{dA}{dt} = (6 + t) \times -1 + (16 - t) = 10 - 2t$ cm²/s

when t = 2 s $\therefore$ The rate of change of the triangle area = 10 - 2 (2) = 6 cm^2/s

b When x = y then 12 + t = 16 - t $\therefore\ t = \frac{4}{3}$ sec

i.e. after $\frac{4}{3}$ seconds, the right angled - triangle becomes isosceles triangle

Try to solve

3 **Volume:** A metal object in the form of a cuboid whose base is squared shape of side length increases at a rate of 1 cm/min and its height decreases at a rate of 2cm/min. Find the rate of increasing its volume when its base side length is 5 cm and its height is 20 cm. After how many minutes does the volume of the cuboid stop increasing?

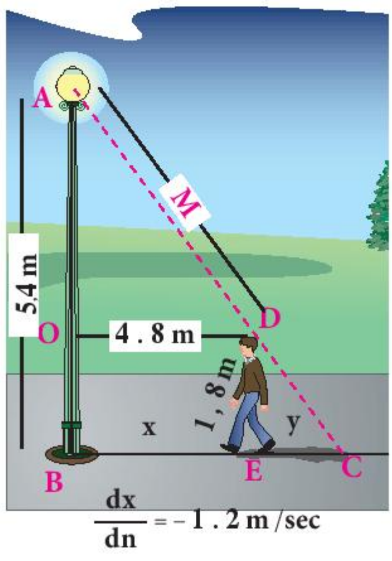
Example Length of shadow

4 A 1.8 meter man walks in a straight line approaching the base of a lamppost at a rate of 1.2 m/s if the height of the lamppost is 5.4 m on the ground , find :

a The rate of change of the length of the man's shadow.

b The rate of change of the man's head from the lamp when the man is 4.8 m from the lamppost.

Solution



Modeling the problem: in the opposite figure $\overline{AB}$ represents the lamppost, point A is the lamp $\overline{DE}$ represents the man and point C is the end of the man's shadow , then :

x = EB the distance between the man and the base of the lamppost.

y = EC the length of the man's shadow.

M = AD the distance between the man's head and the lamp.

First: $\because \triangle ABC \triangle \sim DEC$

$\therefore\ \frac{AB}{DE} = \frac{BC}{EC} = \frac{5.4}{1.8} = \frac{x+y}{y}$

and 2 y = x by differentiating both sides of the equation with respect to time

$\therefore\ 2\frac{dy}{dt} = \frac{dx}{dt}$ i.e $\frac{dy}{dt} = \frac{-1,2}{2} = -0.6$ meter/s

Second: in the right angled - triangle A D O at (O)

$M^2 = x^2 + (3.6)^2$ by differentiating both sides of the equation with respect to time

$2M\frac{dM}{dt} = 2x\frac{dx}{dt}$ at x = 4,8 A   f = 6 m

$6\frac{dM}{dt} = 4.8 \times -1.2$ **i.e.** $\frac{dM}{Dt} = -0.96$ meter/sec

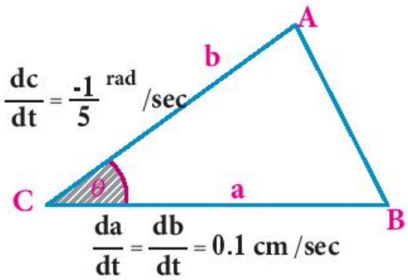
Try to solve

4 **Constructions :** A 5 -meter water pipe with two ends A and B is leaning with its end A on a horizontal ground and with one of its points D against a 3-meter vertical wall. If end A slides away from the wall at a rate of $\frac{5}{4}$ meters/m, find the rate of sliding end B when the pipe reaches the edge of the wall.

Example — Area

(5) Two sides in a triangle, the length of both increases at a rate of 0.1cm/sec and the measure of the included angle increases at a rate of $\frac{1}{5}$ rod/sec. At which rate does the area of the triangle change at the moment the length of each side of the triangle is 10cm?

Solution



Modeling the problem: at a certain moment t, let a side length of the triangle a the other side length b The measure of the included angle C^{rad} and A the area of triangle ABC be differentiable functions at t

where $A = \frac{1}{2}$ a b $\sin\theta$ by differentiating both sides with respect to t

$$\therefore \frac{dA}{dt} = \frac{1}{2} a\, b \frac{d}{dt}[\sin c] + \frac{1}{2}\sin\theta \frac{d}{dt}[a\, b]$$

$$\frac{da}{dt} = \frac{1}{2} a\, b \cos c \frac{dc}{dt} + \frac{1}{2}\sin\theta\left[a\frac{db}{dt} + b\frac{da}{dt}\right] \qquad (1)$$

but $\frac{da}{dt} = \frac{db}{dt} = 0.1$, $\frac{dc}{dt} = \frac{1}{5}$

when the length of each side of the triangle is 10cm, the triangle is equilateral

then $m(\angle\theta) = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$ by substituting in equation (1)

$$\therefore \frac{dA}{dt} = \frac{1}{2}\times 10\times 10\times\frac{1}{2}\times\frac{1}{5} + \frac{1}{2}\times\frac{\sqrt{3}}{2}\left[2\times 10\times\frac{1}{10}\right]$$

$$= 5 + \frac{\sqrt{3}}{2} \simeq 5{,}866 \text{ cm}^2 /\text{s}$$

i.e. the area of the triangle increases at this moment at a rate of 5.866 cm^2 /sec

Try to solve

(5) **Area:** A B C is a right-angled triangle at C and its area is constant and equals $24cm^2$, if the rate of change of b equals 1cm/sec find the rate of change for each of a and m $(\angle A)$ at the moment in which b equals 8 cm.

Critical thinking: If x (the radian measure of an angle) increases at a constant time rate, explain why:

(a) The sine and the tangent increase at the same rate when $x = 0$

(b) The tangent increases at the rate equals 8 times the increase of sine when $x = \frac{\pi}{3}$

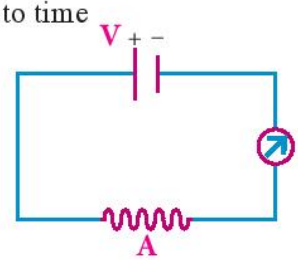
(c) The cosine decreases at a rate of $\frac{3}{8}$ times the increase of the tangent when $x = \frac{\pi}{6}$

Example — Physics

(6) In a closed electric circuit, if v is the potential difference (volt), I is the current intensity (Ampere) R is the resistance (ohm). If the potential difference increases at a rate of 1 volt/s and the current intensity decreases at a rate of $\frac{1}{2}$ Ampere/s, find the rate of the resistance at the moment which v = 12 volt and I = 2 Amperes .

Solution

You know that $V = I \times R$ by differentiating both sides with respect to time



$\therefore \frac{dv}{dt} = I\frac{dR}{dt} + R\frac{dI}{dt}$

$\because \frac{dv}{dt} = 1$ volt / sec , $\frac{dI}{dt} = -\frac{1}{2}$ Amperes /sec

$\therefore$ When v = 12 volts , I = 2 Amperes **then:** $R = \frac{V}{I} = \frac{12}{2} = 6$ ohm

and $1 = 2 \times \frac{dR}{dt} + 6 \times -\frac{1}{2}$ $\therefore \frac{dR}{dt} = 2$ ohm /sec

i.e the rate of the resistance at this moment is 2 ohm /s

Try to solve

6 in the previous example , calculate the rate of the resistance if the current increases at a rate of $\frac{1}{3}$ ampere / Sec.

Exercises 1 - 4

Choose the correct answer :

1) If the radius length of a circle increases at a rate of $\frac{4}{\pi}$ cm/sec, then the circumference increases at this moment at the rate

   a) $\frac{4}{\pi}$ cm/s b) $\frac{\pi}{4}$ cm/s c) $\frac{1}{8}$ cm/s d) 8 cm/s

2) A cube of ice melts preserving its shape at a rate of $1cm^3$/sec, then the rate of change of the cube edge length when its volume is 8 cm^3 is:............................. cm / s

   a) $-\frac{1}{12}$ b) $\frac{1}{12}$ c) $-\frac{1}{6}$ d) $\frac{1}{6}$

3) A body moves on the curve $y^2 = x^3$, if $\frac{dx}{dt} = \frac{1}{2}$ unit / sec when y = - 1, then $\frac{dy}{dx}$ at this moment equals .... unit / s

   a) $-\frac{3}{4}$ b) $-\frac{3}{8}$ c) $\frac{3}{4}$ d) $\frac{3}{2}$

4) If the slope of the tangent to the curve y = f(x) at a point = $\frac{1}{2}$ and the x-coordinate of this point decreases at a rate of 3 units / sec then the rate of change of its y-coordinates equals .... unit(s) / sec

   a) $-\frac{1}{6}$ b) $-\frac{3}{2}$ c) $\frac{1}{6}$ d) $\frac{3}{2}$

Answer the folowing:

5) A point moves on a curve whose equation is $x^2 + y^2 - 4x + 8y - 6 = 0$. If the rate of change of its x-coordinate with respect to time at point (3, 1) equals 4 units / sec, find the rate of change of its y-coordinate with respect to time t .

6) A stone fell in a motionless lake and a circular wave was generated. If its radius length increases at a rate of 4cm/s, find the rate of increasing the surface area of the wave at the end of five seconds.

7) A regular hexagonal - like lamina shrinks by cold. if the rate of change of its side length is 0.1 cm/sec, find the rate of change in the area of the lamina when its side length is 10cm.

8) A known mass of gas with a constant temperature whose volume decreases at a constant rate of $2cm^3$/sec. If the pressure is inversely proportional to the volume and the pressure equals 1000 gm.wt /cm^2 when the volume is $250cm^3$, find the rate of change of the pressure with respect to time when the volume of gas is$100cm^3$.

9) If the gas leaks from a spherical balloon at a rate of $20cm^3$/sec, find the rate of change of the balloon radius length at the moment which the radius length is 10cm and find the rate of change of the balloon external surface area at the same moment.

10) A 5-meter ladder is leaning against a vertical wall by its top and on a horizontal ground by its base. If the base of the ladder slides away from the wall at a rate of 4cm/min when the top of the ladder is 4 meter high of the ground , find the rate of sliding the top of the ladder , then find the rate of change of the measure of the angle between the ladder and the ground at this moment.

11) A ballon rises vertically from point A on the ground surface. An apparatus is placed to follow up the motion of the balloon at point B at the same horizontal plane of point A and distant 200 meters from it. A moment, the apparatus observed the elevation angle of the balloon to find it $\frac{\pi}{4}$, and it increases at a rate of 0.12 rad/min, find the rate of the balloon elevation at this moment.

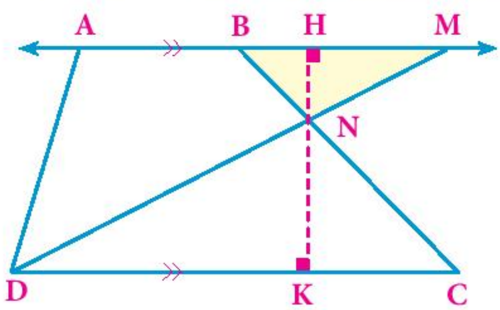
12) A 180 cm man moves far from the base of a 3-meter lamppost at a rate of 1.2m/s, find the rate of change of the length of the man's shadow. If the straight line passing through the highest point of the man's head and the top of the lamp inclines on the ground with an angle of measure θ^{rad} when the man is far from the base of the lamppost for a distance x meter, prove that $x = \frac{6}{5} \cot \theta$, then find the rate of change of θ when the man is 3.6m from the base of the lamppost .

13) An isosceles triangle whose base length is $20\sqrt{3}$ cm . If the length of the two equal legs decreases at a rate of 3 cm/h , find the rate of decreasing the triangle surface area at the moment which the length of the two equal legs is equal to the base length .

14) **Industry:** If the daily production of a factory during a certain period of time t (day) is identified by the relation $y = 400\ (1 - e^{-0.30})^t$ units, find the rate of change in the number of the produced units with respect to time on the tenth day.

15) **Life application:** If the production of a beehive is given by the relation: $y = (n + 100) \ln (t + 5)$ grams in terms of the number of days t, find the rate of change of the beehive production when t = 5 , t = 15, t = 20. Does the production of the beehive increase or decrease?

16) ABCD is a Trapezium in which $\overline{AB} \parallel \overline{DC}$ .Its height equals 3 cm, DC = 5 cm, The point M move on the ray $\overrightarrow{AB}$ with speed 4.8 cm/sec. starting from the point B. Find the rate of change of the area of Triangle MNB at the instant in which MB = 1 cm



Unit Two

Behavior of the Function and Curve Sketching

Introduction

Through reading the graph to a curve of a function , you can determine monotonic intervals [increasing-decreasing -constancy]. You can also know the maximum and minimum values of the function and identify some properties of the function. You can use computer graphics to graph and study the behavior of the function. But it is not usually available. In this unit , you will identify more techniques to graph the curve of the function through differentiation using the function derivatives (first derivative and second derivative) to determine the interval of increasing or decreasing the function. You can also determine the maximum and minimum values related to the values of x (the maximum values and local minimum values) and the absolute maximum and minimum values of a function continuous on a limited interval [a, b]. You can also determine the convexity direction to the curve of the function (upward or downward) you will learn some applications to find the maximum and minimum values to help you model and solve mathematical physical and life problems.

Unit objectives

By the end of the unit and carrying out the involved activities, the students should be able to:

- Use the first derivative to study the increase and decrease of the differentiable function.
- Identify the local minimum and maximum values of the differentiable function
- Identify and find the absolute minimum and absolute maximum values of a function in a closed interval
- Find the critical points، convex upward، convex downward and inflection points of a function.
- Find the relation between the curve of the function and the first derivative
- Learn the behavior of the function in regard to the monotony، maximum values and minimum values throughout the first derivative
- Sketch the curves of the polynomial functions up to the third degree only.

Key terms

- Increasing Function
- Dereasing Functicn
- Maximum and minimum values
- Exterma
- Critical Point
- Local Minimum value
- Local Maximum value
- Absolute Extrema value
- Convexity
- Convex Upward
- Convex Downward
- Inflection Point

Lessons of the unit

Lesson (2 - 1): Increasing and decreasing the functions.

Lesson (2 - 2): maximum and minimum values

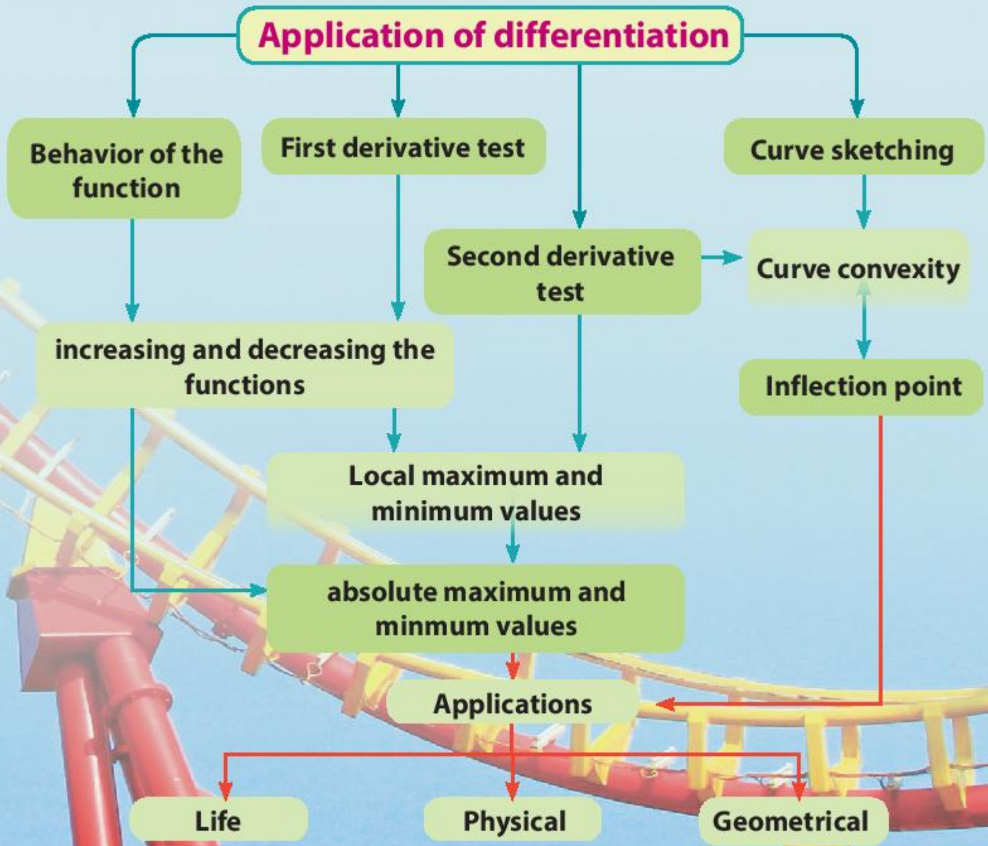
Lesson (2 - 3): Curve sketching

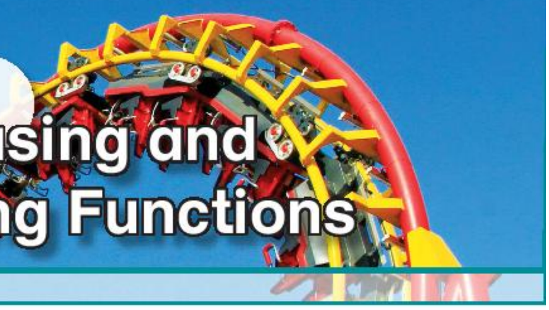
Lesson (2 - 4): Applications on maximum and minimum values

Materials

- Scientific calculator
- Computer graphics

Unit planning guide





2 - 1 Increasing and Decreasing Functions

You will learn

- Using the first derivative to determine the increasing or decreasing intervals of a function.
- Daily life applications on the increasing and decreasing intervals of the functiuons

Key terms

- Increasing Function
- Decreasing Function

Materials

- Scientific calculator
- Computer graphics

Investigating monotony of a function

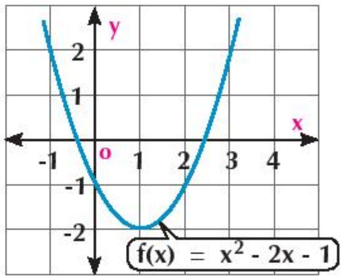
Find f ' (x)
→ solve equation f ' (x) = 0
→ investigate the sign of f ' (x)
→ f ' (x) < 0 → f decreasing
→ f ' (x) > 0 → f increasing

Think and discuss

The opposite figures show the two curves of the functions f , g where

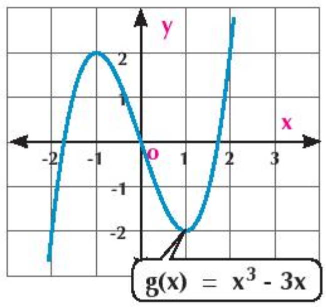
$f(x) = x^2 - 2x - 1$,

$g(x) = x^3 - 3x$



- Determine the increasing and decreasing intervals of the function f
- Find the derivative of function f and investigate the sign of f ' (x) for the different values of x which belong to the increasing interval
- Investigate the sign of f ' (x) for the different values of x which belong to the decreasing interval

Repeat the steps above to determine the sign of g' (x) in the increasing and decreasing intervals of the function g. What do you infer? what kind of angles does the tangent of the curve make at the different values of x in the increasing intervals with the positive direction of x-axis?



Learn

First derivative test for the monotonic functions

Theorem

Let f be a differentiable function on the interval] a , b [:

1- If f' (x) > 0 for all the values of x ∈] a , b[
then f is increasing on the interval] a , b [

2- If f' (x) < 0 for all the values of x ∈] a , b [
then f is increasing on the interval] a , b [

Example Determining the increasing and decreasing intervals

(1) Determine the increasing and decreasing intervals of the function f where $f(x) = x^3 - 3x + 2$

Solution

$\because f(x) = x^3 - 3x + 2$ is a continuous and differentiable function on $\mathbb{R}$

$\therefore f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

By putting $f'(x) = 0$ then: $3(x^2 - 1) = 3(x - 1)(x + 1) = 0$

$\therefore f'(x) = 0$ when $x = -1$ and $x = 1$

we investigate the sign of f' (x) in each of these intervals as in the opposite table of changes we find that :

f is increasing on the interval $]-\infty, -1[$

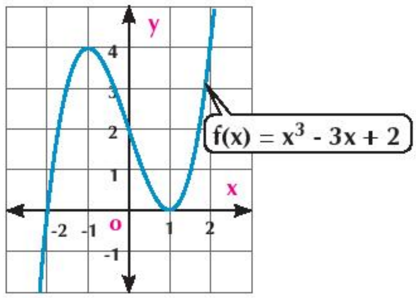
f is decreasing on the interval $]-1, 1[$

f is increasing on the interval $]1, \infty[$

x	$-\infty$		-1		1		∞
sign f' (x)		+	0	−	0	+	
behavior of f (x)		↗		↘		↗	

Note that:

1) When we sketch the curve of the function using a graphical program (opposite figure) we find that the behavior of the function curve is congruent to what we inferred in the table of changes.



2) The tangent to the curve makes an acute angle with the positive direction of x-axis in the increasing intervals and an obtuse angle with the positive direction of x-axis in the decreasing intervals.

3) The values x which separate the increasing and decreasing intervals of the function are the values at which the first derivative of the function equals zero or is not existed

Try to solve

(1) Determine the increasing and decreasing intervals for each of the following :

(a) $f(x) = x^3 - 9x^2 + 15x$ (b) $g(x) = \dfrac{x}{x^2 + 1}$

Example Trigonometric functions

(2) Determine the increasing and decreasing intervals for the functions f where $f(x) = x + 2\sin x$, $0 < x < 2\pi$

Solution

f is continuous and differentiable on $]0, 2\pi[$

$\therefore f'(x) = 1 + 2\cos x$

investigating the sign of f' (x)

when $1 + 2\cos x = 0$ $\therefore \cos x = -\frac{1}{2}$

$\because x \in]0, 2\pi[$ $\therefore x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$

x	$-\infty$		$\frac{2\pi}{3}$		$\frac{4\pi}{3}$		∞
sign of f' (x)		+	0	−	0	+	
behavior of f (x)		↗		↘		↗	

Note that:

when $x = \frac{\pi}{2}$ $\quad f'(x) = 1 > 0$ $\quad \therefore$ f is increasing on $]0, \frac{2\pi}{3}[$

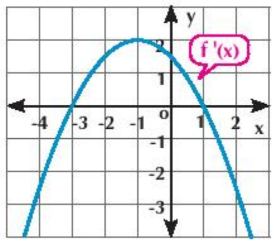
when $x = \pi$ $\quad f'(x) = -1 < 0$ $\quad \therefore$ f is decreasing on $]\frac{2\pi}{3}, \frac{4\pi}{3}[$

when $x = \frac{3\pi}{2}$ $\quad f'(x) = 1 > 0$ $\quad \therefore$ f is increasing on $]\frac{4\pi}{3}, 2\pi[$

Try to solve

(2) Determine the increasing and decreasing intervals of the function f where $f(x) = x - 2\cos x$, $0 < x < 2\pi$

Critical thinking: The opposite figure shows the curve of $f'(x)$ for the function f where f (x) is polynomial.



(a) Determine the increasing and decreasing intervals of the function f

(b) Find the solution set of the inequality $f''(x) > 0$

Example

(3) Determine the increasing and decreasing intervals of the function h where $h(x) = 2\ln x - x^2$

Solution

h (x) is differentiable for each $x \in \mathbb{R}^+$

$$h'(x) = \frac{2}{x} - 2x = \frac{2(1 - x^2)}{x}$$

investigating the sign of $h'(x)$

when $h'(x) = 0$ $\quad \therefore x = 1$ or $x = -1 \notin \mathbb{R}^+$

when $x < 1$ $\quad \therefore h'(x) > 0$ and h is increasing on $]0, 1[$

when $x > 1$ $\quad \therefore h'(x) < 0$ and h is decreasing on $]1, \infty[$

x	0	1	∞
sign f' (x)	+	0	−
behavior of f (x)	↗		↘

Try to solve

(3) Determine the increasing and decreasing intervals of the function f where $f(x) = x - e^x$ using the geogebra program sketch the curve of the function f and check your answer.

Exercises 2 - 1

Determine the increasing and decreasing intervals of the function f for each of the following:

1. $f(x) = x^2 - 4x$
2. $f(x) = (x-3)^2$
3. $f(x) = x^3 - 6x^2 + 5$
4. $f(x) = 9x - x^3$
5. $f(x) = x^4 + 4x$
6. $f(x) = 2 - 3(x-2)^{\frac{4}{3}}$
7. $f(x) = 1 - \frac{1}{x}$
8. $f(x) = \frac{x-2}{x+2}$
9. $f(x) = \frac{\sqrt{x-1}}{x}$
10. $f(x) = x + \ln x$
11. $f(x) = 3 - \ln x^2$
12. $f(x) = 5 - 2e^{2x}$

Answer the following:

13. Prove that the function f where $f(x) = \tan x - x$ is increasing on the interval $]0, \frac{\pi}{4}[$
14. Determine the increasing and decreasing intervals of the function f where:

    $f(x) = 1 - \sin x, \; 0 < x < 2\pi$
15. If f and h are two differentiable functions and $f'(x) < h'(x)$ for each $x \in \mathbb{R}$, prove that the function z where $z(x) = f(x) - h(x)$ is decreasing for each $x \in \mathbb{R}$.

2 - 2 Maxima and Minima (Extrema)

You will learn

- The concept of the critical point
- The concept the local maximum and minimum of a function
- First derivative test for the local maximum and minimum values
- Finding the maximum values of a function on a closed interval

Key terms

- Critical point
- Local Maximum
- Local Minimum
- Absolute Extrema

Materials

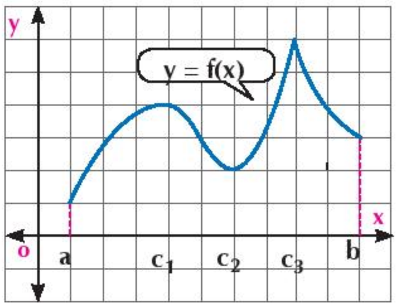
- Scientific calculator
- Computer graphics

Think and discuss

The opposite figure shows the curve of the continuous function f on [a , b]

1- Determine the increasing and decreasing intervals of the function f

2- What is the value of $f'(c_1)$ when $x = c_1$? Describe the change of f on the interval $]a , c_2[$. Is $f(c_1)$ the greatest value of f in this interval?



3- What is the value of $f'(c_2)$ when $x = c_2$? Describe the change of f on the interval $]c_1, c_3[$. Is $f(c_2)$ the minimum value of f in this interval ?

4- Can you find the value of $f'(c_3)$? Explain.
Describe the change of f on the interval $]c_2, b[$. Is $f(c_3)$ the maximum value of f in this interval?

Definition

The Critical Point

The continuous function f on the interval] a , b [has a critical point (c , f (c))

If $c \in]a , b[$ and $f'(c) = 0$ or the function $f'(c)$ is undefined or the function f is not differentiable at x = c when x = c.

In the previous figure, we deduce that:
there is a critical point when $x = c_1$ and $x = c_2$ since $f'(c_1) = f'(c_2) = 0$. It is sometimes called stationary point. there is also another critical point when $x = c_3$ since f is continuous when $x = c_3$ and is not differentiable (the right derivative $\neq$ the left derivative).

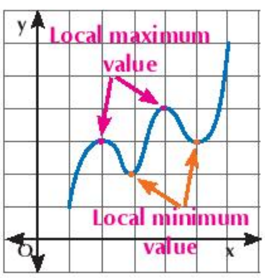
Definition

Local Maximum and Minimum values

If f is a continuous function whose domain is I and $c \in I$, then the function f has:

a local maximum value when x = c if an open interval is found $]a , b[\subset I$ containing c where $f(x) \leqslant f(c)$ for each $x \in]a , b[$

a local minimum value when x = c if an open interval is found $]a , b[\subset I$ containing c where $f(x) \geqslant f(c)$ for each $x \in]a , b[$



Note that :

In think and discuss: there are local maximum values when $x = c_1$ and $x = c_3$ whereas there is a local minimum value when $x = c_2$

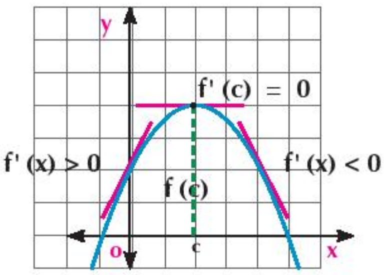
First Derivative Test of the local maximum and minimum values

Learn

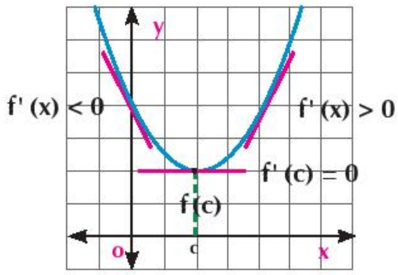
If (c, f(c)) is a critical point of the continuous function f at c and an open interval is found around c where:

1- $f'(x) > 0$ when $x < c$, $f'(x) < 0$ when $x > c$, then f (c) is a local maximum value

2- $f'(x) < 0$ when $x < c$, $f'(x) > 0$ when $x > c$, then f (c) is a local minimum value



f (c) is a local maximum value at c



f (c) is a local minimum value at c

3- If the sign of f ' (x) on the two sides of c doesnot change, then the function f doesont have local maximum or local minimum values at c.

Theorem

If f is a differentiable function on]a , b [and the function has a local maximum value or local minimum value at $c \in$]a , b[
then $f'(c) = 0$ or $f'(c)$ is not existed.

First derivative test

Example

1 If $f(x) = x^3 + 3x^2 - 9x - 7$, find the local maximum values or the local minimum values of the function f

Solution

1) Determining the critical points : f is continuous and differentiable

$\therefore f'(x) = 3x^2 + 6x - 9$

$= 3(x^2 + 2x - 3) = 3(x + 3)(x - 1)$

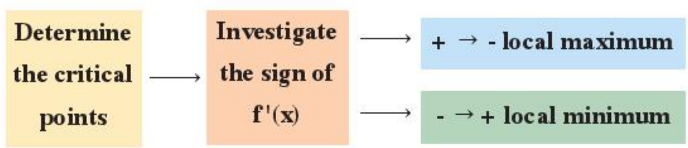
when $f'(x) = 0$

$\therefore x = -3$ or $x = 1$

we have two critical points

$(-3, f(-3)), (1, f(1))$

i.e. the two points : $(-3, 20), (1, -12)$



2) The first derivative test at each critical point and the following table of changes shows it

x	$-\infty$	-3		1	∞
sign of f' (x)	+	0	−	0	+
behavior of f (x)	↗	20	↘	-12	↗

3) In the neighborhood of $x = -3$

the sign of $f'(x)$ changes from positive (before $x = -3$) to negative (after $x = -3$)

$\therefore f(-3) = 20$ is a local maximum value.

in the neighborhood of $x = 1$ the sign of $f'(x)$ changes from negative (before $x = 1$) to positive (after $x = 1$)

$\therefore f(1) = -12$ is a local minimum value.

Try to solve

1) If $f(x) = \frac{1}{3}x^3 - 9x + 3$, find the local maximum and minimum values of the function f

Example

The first derivative is not existed

2) Find the local maximum and minimum values of the function f if $f(x) = x^{\frac{2}{3}}(2x - 5)$

Solution

The domain of the function f is $\mathbb{R}$ and it is continuous for each $x \in \mathbb{R}$

1) Determining the critical points:

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(2x - 5) + 2x^{\frac{2}{3}}$$

$$= \frac{2[2x - 5 + 3x]}{3\sqrt[3]{x}} = \frac{10(x - 1)}{3\sqrt[3]{x}} \quad x \neq 0$$

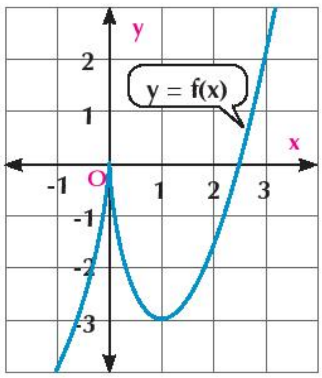
$\because$ f is continuous when $x = 0$ and $f'(0)$ is not existed

$\therefore$ One critical point is found which is $(0, f(0))$ i.e. $(0, 0)$

when $f'(x) = 0$ $\therefore x = 1$ and therefore,

there is a critical point which is

$(1, f(1))$ i.e. $(1, -3)$ as shown in the opposite figure .



2) The first derivative test at each critical point and the following table of changes shows it.

x	$-\infty$	-1		1	∞
sign of f' (x)	+	not existed	−	0	+
behavior of f (x)	↗	-	↘	-3	↗

3) When $x = 0$

there is a local maximum value $= 0$

when $x = 1$ there is a local minimum value $= -3$

Try to solve

2) Prove that the function f where $f(x) = \sqrt[3]{x^2}$ has a local minimum value .

Critical thinking: Does the function f where $f(x) = x^3 + 3x - 4$ have local maximum and minimum values? Explain.

Example Fractional functions

(3) Find the local maximum and minimum values of the function f where $f(x) = x + \frac{4}{x}$ and show its type.

Solution

The domain of f $= \mathbb{R} - \{0\}$

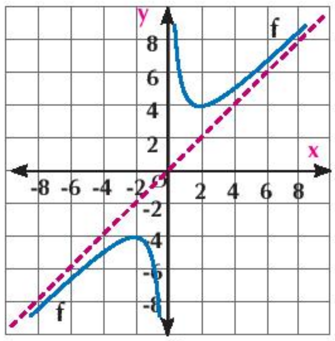
1) Determining the critical points: $f'(x) = 1 - 4x^{-2} = \frac{x^2 - 4}{x^2}$ the function has two critical points which are (2, f(2)) , (-2, f (-2)) i.e. (2 , 4) , (-2 , -4).

2) The first derivative test at each critical point which the opposite table of the function changes shows (note that x = 0 is excluded from the domain of f).

x	-∞		-2		0		2		∞
sign of f' (x)		+		−		−		+	
behavior of f (x)		↗	-4	↘		↘	4	↗	

3) When x = -2 there is a local maximum value = -4

when x = 2 there is a local minimum value = 4



Note that: the local maximum value of the function may be less than the local minimum value

Technology: the opposite figure shows the curve of the function f. Use a graphic program to compare between the table of the function changes and its curve? what do you notice?

Try to solve

(3) Find the local maximum and minimum values of the function f where $f(x) = \frac{x^2}{1 - x}$ and show its type

Learn

The Absalute maximum values of a Function on a Closed Interval

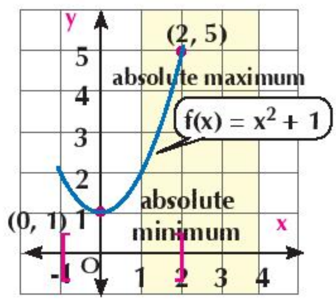
Definition of maximum values: if f function is defined on the closed interval [a , b] and c ∈ [a , b]

1) f (c) is the minimum value on the interval [a ,b] when $f(c) \leqslant f(x)$ for each x ∈ [a , b]

2) f (c) is a maximum value on the interval [a , b] when $f(c) > f(x)$ for each x ∈ [a , b]

➲ The minimum value and maximum value of a function on an interval are called the extrema of thes function on this interval.

➲ The maximum value can be occured at an interior point of the interval or at the boundries of the function once it occurs at the boundries of the interval , it is called the end point extrema

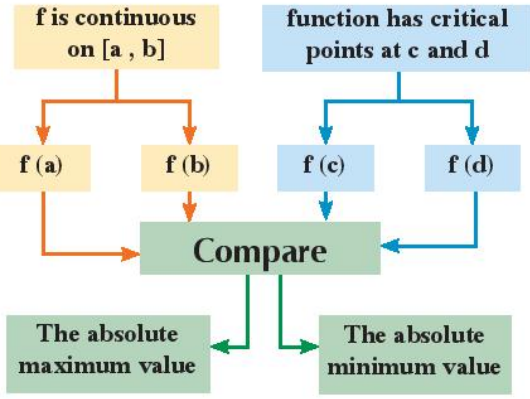


Theorem

If function f is continuous on the interval [a , b], then the function f has an absolute maximum value and absolute minimum value on the interval [a , b].

To find the absolute maximum values of the function f on the closed interval [a , b] we follow up the opposite diagram as follows:

- Calculate f (a) , f (b) , and the value of the function at each critical point.
- Compare between the values above; the maximum value is the absolute maximum value and the minimum value is the absolute minimum value.



Example

4 Find the absolute maximum values of the function f where $f(x) = x^3 - 12x + 12$, $x \in [-3 , 3]$

Solution

$\because f(x) = x^3 - 12x + 12 , x \in [-3 , 3]$

$\therefore f(-3) = (-3)^3 - 12(-3) + 12 = 21$ **(1)**

$, f(3) = (3)^3 - 12(3) + 12 = 3$ **(2)**

$f'(x) = 3x^2 - 12 = 3(x - 2)(x - 2)$

to determine the critical point, put $f'(x) = 0$

$\therefore x = 2 \in [-3 , 3]$ or $x = -2 \in [-3 , 3]$

when $x = 2$ there is a critical point and: $f(2) = -4$ **(3)**

when $x = -2$ there is a critical point and: $f(-2) = 28$ **(4)**

By comparing the values of 1 , 2 , 3 and 4 we find that:

The function f has an absolute maximum value = 28 and an absolute minimum value = - 4

Try to solve

4 Find the absolute extrema values of the function f

a $f(x) = 10x e^{-x}$, $x \in [0 , 4]$

b $f(x) = \frac{4x}{x^2 + 1}$, $x \in [-1 , 3]$

Note that:

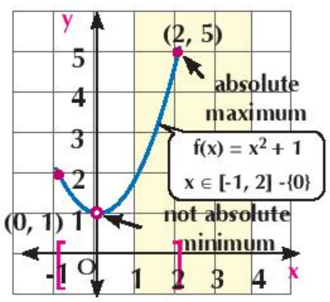


fig (1)

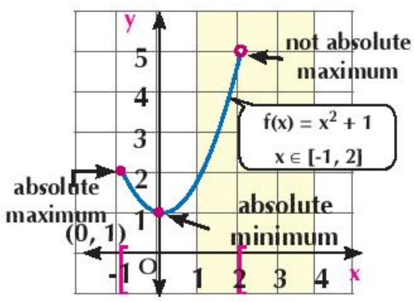
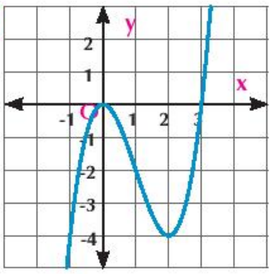
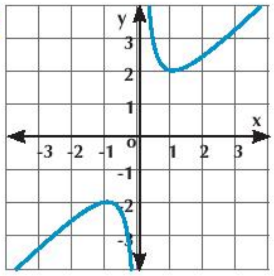


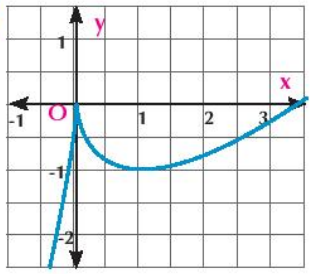
fig (2)

Exercises 2 - 2

Determine the local maximum and minimum values (if existed) of the function f in the following figures and show their types:

1


2


3


Find the local maximum and minimum values (if existed) of the function f for each of the following and show their types:

4) $f(x) = x^3 + 3x^2 + 2$

5) $f(x) = x^4 - 2x^2$

6) $f(x) = 4x - x^3$

7) $f(x) = 3x^5 - 5x^3$

8) $f(x) = 3 - x^{\frac{2}{3}}$

9) $f(x) = (x + 2)^{\frac{2}{3}}$

10) $f(x) = x + \frac{4}{x^2}$

11) $f(x) = x + \frac{4}{x-1}$

12) $f(x) = \frac{3}{x-2}$

13) $f(x) = 4e^{-x^2}$

14) $f(x) = e^x(3 - x)$

15) $f(x) = e^x + e^{-x}$

16) $f(x) = x - \ln x$

17) $f(x) = 8\ln x - x^2$

Find the absolute maximum values of the function f on the given interval:

18) $f(x) = x^3 - 3x + 1, \ x \in [-2, 1]$

19) $f(x) = \sqrt{x-1}, \ x \in [2, 5]$

20) $f(x) = \sin x + \cos x, \ x \in [0, 2\pi]$

21) $f(x) = x e^{-x}, \ x \in [0, 2]$

Answer the following:

22) **Critical thinking:** Find the value of each for a , b , c and d where the curve of $f(x) = ax^3 + bx^2 + cx + d$ satisfies the following conditions together:

a) It passes through the origin point.

b) It has a critical point when $x = 1$

c) The equation of the tangent to the curve at point $(2, f(2))$ on it is $9x + y = 20$.

2 - 3 Study Curves

You will learn

- Determining the convex intervals of the curve of the function upward and downward.
- Finding the inflection points to the curve of the function,
- Using the second derivative test to find the local maximum and minimum values
- Curve sketching

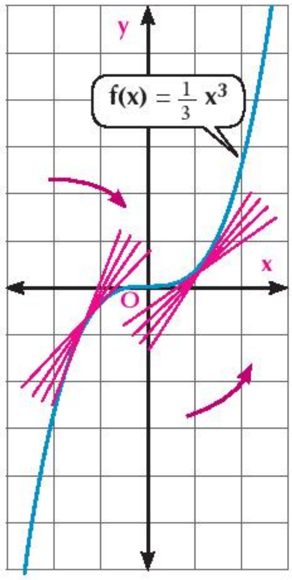
Key terms

- Convexity
- Convex upward
- Convex downward
- Inflection point

Materials

- Scientific calculator.
- Computer graphics

Explore



The opposite figure shows the curve of the function f where:

$f(x) = \frac{1}{3} x^3 , x \in \mathbb{R}$

Notice that the function f is increasing on $\mathbb{R}$. why?

Does the direction of the curve convexity in the interval $]-\infty , 0[$ differ from the direction of its convexity in the interval $] 0 , \infty [$?

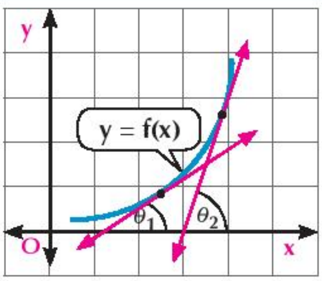
what is the location of the curve of the function in the interval $]-\infty , 0[$ with respect to all its tangents? Does the slope of the tangent $f'(x)$ increase or decrease by the increase of the value of x?

what is the location of the curve of the function in the interval $]0 , \infty[$ with respect to all its tangent? Does the slope of the tangent $f'(x)$ increase or decrease by the increase of the value of x?

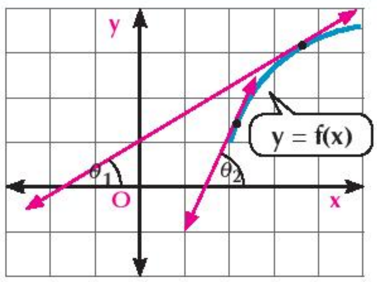
Convexity of a curves

Definition

Let f be a differentiable function on the interval] a , b [, the curve of the function f is convex downward If f' is increasing on this interval and is convex upward if f' is decreasing on this interval.



The curve is convex downward
f' is increasing and its derivative is positive
i.e. $f''(x) > 0$



The curve is convex upward
f' is decreasing and its derivative is negative
i.e $f''(x) < 0$

If the function f has a non-zero second derivative , it can be used to study the increasing and decreasing of the first derivative f ' and to identify the intervals of convex upwards and convex downward to the curve of the function f.

The Second Derivative Test for Convexity of curves

Theorem

Let f be a differentiable function twice on the interval]a , b[

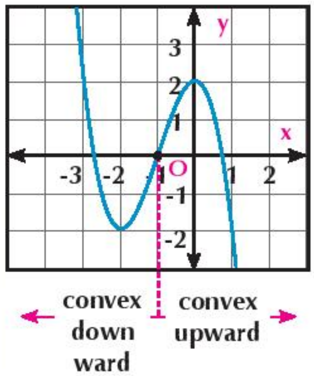
1- If $f''(x) > 0$ for all the values of $x \in]a, b[$, then the curve of f is convex downwards on the interval]a , b[

2- If $f''(x) < 0$ for all the values of $x \in]a, b[$, then the curve of f is convex upwards on the interval]a , b[

Example Determining the intervals of polynomial convexity

(1) If $f(x) = 2 - 3x^2 - x^3$, determine the intervals at which the curve of the function f is convex upwards and the intervals it is convex downwards.

Solution



f is continuous and differentiable for each $x \in \mathbb{R}$ where:

$f'(x) = -6x - 3x^2$, $f''(x) = -6 - 6x = -6(1 + x)$

when $f''(x) = 0 \qquad \therefore x = -1$

The intervals of convexity: the opposite table shows the sign of f''and the intervals of the convexity of the curve of the function f upwards and downwards, **i.e:** The curve of the function is convex downwards in the interval $]-\infty , -1[$ and convex upwards in the interval $]-1 , \infty[$

x	$-\infty$	-1	∞
sign of f''	+	0	−
convexity of curve of f	∪		∩

Try to solve

(1) Determine the intervals of convexity upward and downward for each of the following curves:

(a) $f(x) = x^2 - 4x + 2$ (b) $g(x) = x^4 - 4x^3$

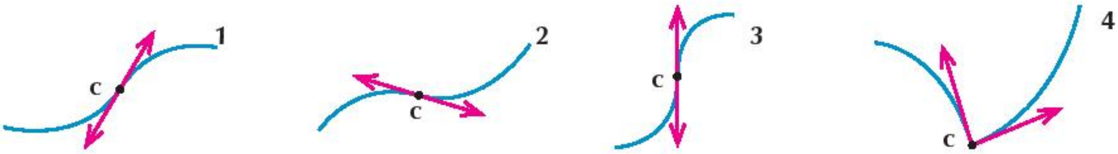
Technology: use a graphic program to sketch the curve of the two function f , h where $h(x) = \sqrt[3]{x}$, $f(x) = x^{\frac{2}{3}}$, then determine the intervals of the convexity upwards and downwards and check your answer using the second derivative test .

Notice that: The direction of the convexity of the curve of the continuous function may change from upwards to downwards or vice versa at a point the second derivative of the function gets vanished or is not existed .

Definition

The inflection point

If f is a continuous function on the open interval]a , b[, c ∈] a , b[and the curve of the function has a tangent at point (c, f (c)), then this point is called the inflection point to the curve of the function f if the convexity of the curve of the function changes at this point from being convex downward to convex upwards or vice versa.



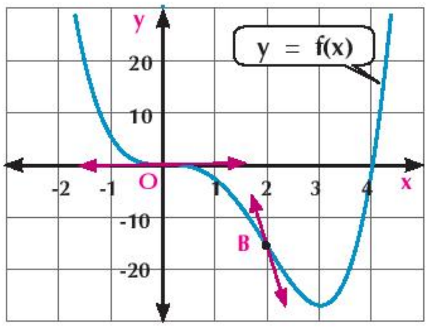
There are inflection points to the curves 1 , 2 and 3 to change the direction of the curve convexity and the existance of a tangent to it at C

There are not inflection points due to the absence of a tangent at C

Notice that:

1- The tangent at the inflection point intersects the curve of the function since the curve at one side of this point lies under the tangent and lies on the tangent in the other side.

2- **In the opposite figure** the curve of the function f has two inflection points at origin point O (0, 0) and the other at point B (2, f(2)).



Example

Convexity and inflection point

(2) If $f(x) = \begin{cases} x^2 - 4 & \text{when } x < -2 \\ x^3 - 3x + 2 & \text{when } x > -2 \end{cases}$

Determine the convexity intervals of the curve of f upwards and downwards and find the inflection points and the equation of the tangent to the function if existed .

Solution

The function f is piece -wise function, its domain is R and is continuous when x = -2 since $f(-2)^- = f(-2)^+ = f(-2) = 0$

$$f'(-2^-) = \lim_{h \to 0^-} \frac{f(-2+h) - f(-2)}{h}$$

$$= \lim_{h \to 0^-} \frac{(-2+h)^2 - 4 - 0}{h} = \lim_{h \to 0^-} (h - 4) = -4$$

$$f'(-2^+) = \lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h}$$

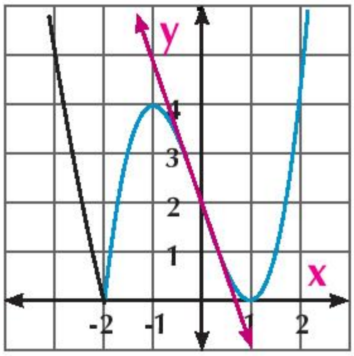
$$= \lim_{h \to o^+} \frac{(-2+h)^3 - 3(-2+h) + 2 - 0}{h} = \lim_{h \to o^+} (h^2 - 6h + 9) = 9$$

$\because f'(-2^-) \neq f'(-2^+)$ $\therefore$ The function is not differentiable when x = -2

$$f'(x) = \begin{cases} 2x & \text{when } x < -2 \\ 3x^2 - 3 & \text{when } x > -2 \\ \text{is not existed when } x = -2 & [f'(-2^-) \neq f'(-2^+)] \end{cases}$$

$$f''(x) = \begin{cases} 2 & \text{when } x < -2 \\ 6x & \text{when } x > -2 \end{cases}$$

The following table shows the sign of f'' and the intervals of convexity of the curve upwards and downward.



x	-∞	-2	0	∞
Sign of f''	+	is not existed	-	+
Convexity of curve of f	◡		◠	◡

Intervals of convexity: the curve of f is convex downwards in the interval]-∞ , -2[and the interval] 0 ,∞ [and it is convex upwards in the interval]-2, 0[

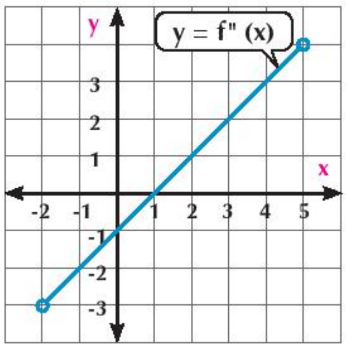
Inflection points

- Point (-2 , f(-2)) i.e. (-2 , 0) is not an inflection point to the curve of f although the changes of the direction of its convexity is changes around it due to the absence of the tangent to the curve of the function at this point (f '(x) is not found)
- Point (0 , f(0)) i.e (0, 2) is the inflection point to the curve of f due to the change of its convexity around it . The tangent to the curve of the function is existed at this point and intersects it at this point, its slope f '(x) = -3 and its equation is: y - 2 = -3 x (as in the graph)

Try to solve

(2) If $f(x) = \begin{cases} (x + 3)^2 & \text{when } x < -1 \\ 3x^2 - x^3 & \text{when } x > -1 \end{cases}$

Determine the intervals of the convexity upwards and downwards to the curve of the function f and find the inflection points and the equation of tangent of the curve at it.



Critical thinking : The opposite figure represents the curve of f'' (x) on the interval] -2, 5[of the continuous function f.

- Find the convexity of the intervals upwards and downwards to the curve of the function f if found .
- Are there inflection points to the curve of function f in this interval? Explain.

Second derivative test of the local maximum or minmum values

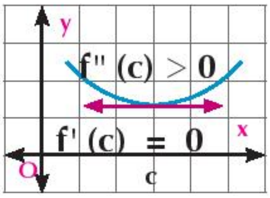
Theorem

Let f be a differentiable function twice on an open interval containing c where f '(c) = 0

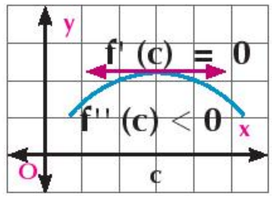
1) If f'' (c) < 0 then f(c) is a local maximum value.

2) If f'' (c) > 0 then f(c) is a local minimum value.

3) If f'' (c) = 0 then the second derivative test cannot determine the type of point(c, f(c)) whether it is local maximum or minimum.



f(c) local minimum value



f(c) local maximum value

Example

3) Use the second derivative test to find the local maximum and minimum values of the function f where : $f(x) = x^4 - 8x^2 + 10$

Solution

f(x) is polynomial, so it is continuous and its domain is $\mathbb{R}$

$f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$, $f''(x) = 12x^2 - 16$

The function has critical points when $f'(x) = 4x(x^2 - 4) = 0$ i.e when $x = 0, x = 2, x = -2$

The second derivative test for existance of the local maximum and minimum values:

when $x = 0$ $f''(0) = -16 < 0$ $\therefore f(0) = 10$ local maximum value

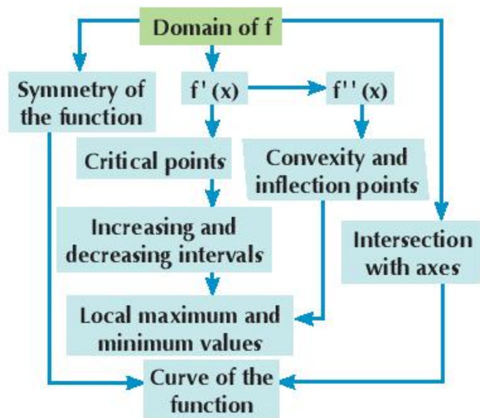
when $x = 2$ $f''(2) = 32 > 0$ $\therefore f(2) = -6$ local minimum value

when $x = -2$ $f''(-2) = 32 > 0$ $\therefore f(-2) = -6$ local minimum value

Try to solve

3) Use the second derivative test to find the local maximum and minimum values of function f where $f(x) = x^3 - 3x^2 - 9x$ and check your answer using the graphical calculator or graphic programs.

Curve Sketching for Polynomials



Differentiation is used for sketching the curves of the functions. It depends on tracing the behavior of f(x) of function f when the value of x changes at a certain interval and representing the ordered pairs (x , y) in the perpendicular coordinate plane where y = f(x). In our study, we will focus on sketching the curves of the functions on the polynomial functions of the third degree and less in the form of $f(x) = ax^3 + bx^2 + cx + d$ to sketch the curve of function f where y = f(x), follow the opposite diagram as follows :

1- If the function is even, its curve is symmetrical about y-axis and symmetrical about the origin point if f is odd.

2- Studying the changes of the function and determining the convexity intervals and the inflection points if found and the local maximum and minimum values if found.

3- Doing the increasing , decreasing and convexity table in order to know the curve sketching and the critical points.

4- Finding the intersection points of the curve of the function with the two axes of coordinates if possible.

5- Sketching the curve of the function and you can use some additional points to improve your sketching.

Example Sketching the curve of the function

(4) Sketch the curve of the function f where $y = f(x) = x^3 - 3x^2 + 4$

Solution

1- The function f is polynomial , its domain is R , and it is neither even nor odd.

2- $f'(x) = 3x^2 - 6x = 3x(x - 2)$, $f''(x) = 6x - 6 = 6(x - 1)$

the function has critical points at $f'(x) = 0$ i.e when $x = 0$, $x = 2$

the function is increasing at the interval $]-\infty , 0[$ and at the interval $]2 , \infty[$ and it is decreasing at the interval $]0, 2[$

$f''(x) = 0$ when $x = 1$

$f''(x) < 0$ at the interval $]-\infty , 1[$ so the curve is convex upwards at this interval,

$f''(x) > 0$ at the interval $]1 , \infty[$ so the curve is convex downwards at this interval

Point (1 , f(1)) i.e. (1 , 2) is an inflection point.

3- Increasing , decreasing and convexity table

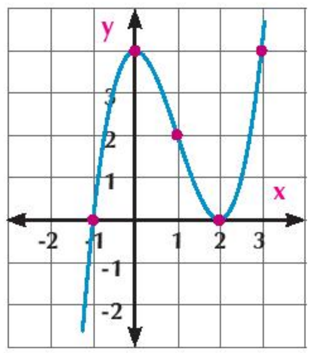
x	$-\infty$		0		1		2		∞
Sign of f'		+	0		-		0	+	
Behavior of f		↗		↘		↘		↗	
Sign of f''			-		0		+		
Convexity of f			⌒				⌣		
y			4		2		0		

Local maximum value — Inflection point — Local minimum value

4- Intersection points with coordinate axes: (0, 4) , (2, 0)

5- Curve sketching of the function f

Additional points : (-1 , f(-1)) i.e. (-1 , 0) (3 , f(3)) i.e. (3 , 4)



Try to solve

(4) Sketch the curve of function f where $y = f(x) = 12x - x^3$

Example Curve sketching of the function

(5) Sketch the curve of the function f where y = f(x) if you know the following:

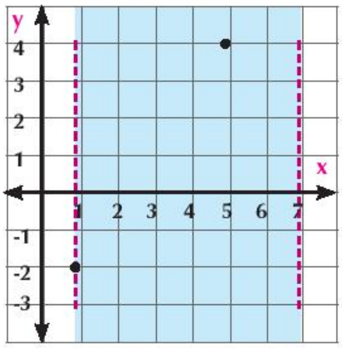
1- f is a continuous function whose domain is [1 , 7] , f(1) = -2 , f(5) = 4

2- $f'(5) = 0$, $f'(x) > 0$ when $x < 5$, $f'(x) < 0$ when $x > 5$

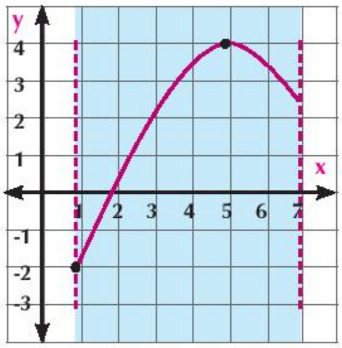
3- $f''(x) < 0$ when $1 < x < 7$

Solution

From (1): we graph the two perpendicular coordinate axes.
the two points (1, -2) and (5, 4) in the domain [1, 7].



from (2): when x = 5 the tangent // x-axis and f is increasing on the interval] 1, 5[and decreasing on the interval] 5, 7[
from (3): the curve is convex upwards on] 1, 7[



Try to solve

(5) Sketch the curve of the function f where y = f(x) if you know the following :

1- f is continuous whose domain is [0, ∞[, f(4) = 3 , f(0) = 1

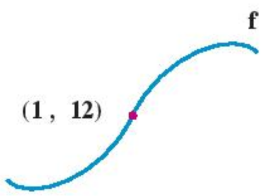
2- f' (x) > 0 when x > 0

3- f'' (x) > 0 when x < 4 , f'' (4) = 0 , f''(x) < 0 when x > 4

Example
Solving the equations

(6) If point (1, 12) is the inflection point to the curve of the function f where $f(x) = ax^3 + b x^2$, find the real values of a and b.

Solution



∵ Point (1 , 12) is an inflection point to the curve of f

∴ f'' (1) = 0 **(1)** , f(1) = 12 **(2)**

$f'(x) = 3 a x^2 + 2 b x$, $f''(x) = 6 a x + 2 b$

from (1): 6 a + 2 b = 0 ∴ b = - 3 a

from (2): a + b = 12 ∴ a - 3 a = 12 and a = -6 , b = 18

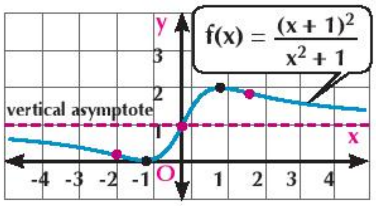
Try to solve

(6) If point (2, 2) is the inflection point to the curve of the function f where $f(x) = x^3 + a x^2 + b x$, find the real values of a and b.

Technology: it is difficult to graph the curves of some functions. You can use geogebra program or any other graphic program to graph the curve of the function and to study its properties.

The opposite figure shows the curve of the function f where

$f(x) = \frac{(x+1)^2}{x^2+1}$



Notice:

1) **The critical points:** the curve has critical points when $x = -1$, $x = 1$

when $x = -1$ $f(-1) = 0$ local minimum value, when $x = 1$ $f(1) = 2$ local maximum value .

2) **Convexity intervals:** upwards: $]-\infty , -\sqrt{3}[$, $]0 , \sqrt{3}[$, and downwards: $]-\sqrt{3} , 0[$, $]\sqrt{3} , \infty[$

3) **Inflection point:** when $x = -\sqrt{3}$, there is a tangent intersects the curve of f and when $x = \sqrt{3}$, there is a tangent intersects the curve of f

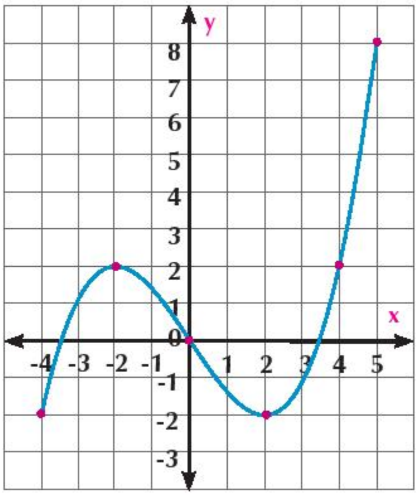
4) **Curve sketching:** the two ends to the curve of the function approach the straight line $y = 1$ and it is called the vertical asymptote to the curve of the function and its equation is $y = a$ where: $a = \lim_{|x| \to \infty} f(x) = \lim_{|x| \to \infty} \frac{(x+1)^2}{x^2+1} = 1$

Application: Graph the two curves of the two functions using a graph program then study the properties for each of them:

$f(x) = \frac{4x^2}{x^2+3}$ $g(x) = \frac{x^2 - 4x}{x^2 - 4x + 3}$

Exercises 2 - 3

1) **The opposite figure shows the curve of the function f where y = f(x), complete:**



a) The domain of f = ...

b) f' (x) = 0 when x ∈ ....

c) f'' (x) > 0 when x ∈ ....

d) The curve is convex upwards when x ∈ ....

e) The inflection point of the curve is ....

f) The function has a local minimum value when x = ...

g) The function has absolute maximum value equals ....

Investigate the convexity intervals of the function f, find the coordinates of the inflection points (if found) for each of the following:

2) $f(x) = 4 - 6x - 3x^2$

3) $f(x) = x^3 - 3x^2 + 1$

4) $f(x) = 15x + 6x^2 - x^3$

5) $f(x) = x^4 - 8x^2 + 16$

6) $f(x) = \frac{6}{x^2 + 3}$

7) $f(x) = \frac{x^2 - 1}{x^2 - 4}$

8) $f(x) = \begin{cases} (x-2)^2 & \text{when } x < 4 \\ 20 - x^2 & \text{when } x > 4 \end{cases}$

9) $f(x) = \begin{cases} x^3 - 3x & \text{when } x < 0 \\ 4x - x^2 & \text{when } x > 0 \end{cases}$

10) Prove that the measure of the angle of inclination of the tangent at the inflection point to the curve of the function f where $f(x) = \frac{x}{1 - x^2}$ equals $\frac{\pi}{4}$

11) If the curve of the function f where $f(x) = x(x-3)^2$ has a local maximum value at x_1 and a local minimum value at x_2, prove that x-coordinate of the inflection point $= \frac{x_1 + x_2}{2}$

12) Find a , b where the curve $x^2 y + a y + b x^2 = 0$ has an inflection point at point (1 , -1).

Sketch the curve of the continuous function f which has the given properties for each of the following:

13) f(0) = 4 f(3) = 4 , f' (x) < 0 for each x < 2 , f' (x) > 0 for each x > 2 , f'' (x) > 0

14) f(1) = f(5) = 0 , f' (x) < 0 for each x < 3 , f' (x) > 0 for each x > 3 , f'' (x) < 0 for each x ≠ 3

15. $f(-1) = 2$ $f(0) = 4$, $f(1) = 0$, $f'(1) = f'(-1) = 1$, $f''(x) < 0$ for each $x < 0$, $f''(x) > 0$ for each $x > 0$

16. $f(3) = 4$, when $x < 3$ then $f'(x) > 0$, $f''(x) > 0$ when $x > 3$ then $f'(x) < 0$, $f''(x) > 0$

Study the changes of the function f and sketch its curve for each of the following:

17. $f(x) = x^2 - 6x + 5$

18. $f(x) = 3 - x^2$

19. $f(x) = x^3 - 3x^2 + 3$

20. $f(x) = \frac{1}{3}x^3 - x + 2$

21. $f(x) = \frac{1}{8}x^3 - \frac{3}{2}x + 1$

22. $f(x) = -x(x - 3)^2$

23. $f(x) = (2 - x)(x + 1)^2$

24. $f(x) = \frac{1}{8}(x + 4)(x - 2)^2$

25. $f(x) = \begin{cases} x^3 - 3x^2 & \text{when } x > 0 \\ x^2 - 2x & \text{when } x \leqslant 0 \end{cases}$

26. $f(x) = x\,|x - 4|$

2 - 4 Applications of Maxima and Minima

You will learn

- Mathematical Modeling

Key terms

- Mathematical Modeling

Materials

- Scientific calculator
- Computer graphics

Mathematical Modeling

The process of taking a scientific decision to solve a problem gets through several stages as follows:

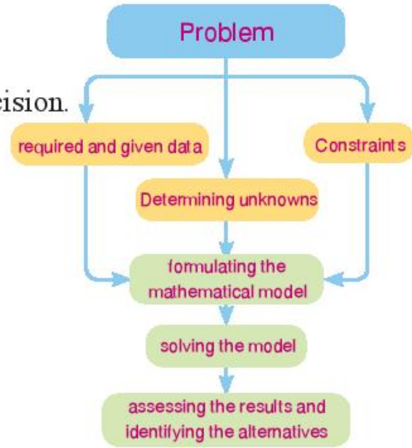
1- Determining the problem (aim and abilities).

2- Putting an ideational model or conceive the dimensions of the problem.

3- Finding a proper scientific model.

4- Solving the model and taking a decision.

mathematical modeling is to model a problem in regard to mathematical relations called the mathematical model this model **can be summarized in the opposite diagram where it includes:**



1- Determining the ends and components of the problem investigated (better profit - less cost- bigger area)

2- Determining the problem unknowns which their values should be found to reach the required end.

3- Showing the relations among unknowns (equations - inequalities).

4- Modeling a mathematical model; representing the problem mathematically to be solved.

5- Solving the mathematical model and interpreting its results in regard to the problem nature .

6- Determining the available alternatives if the problem has more than a solution.

Differentiation helps to solve the mathematical model for most of the practical life problems when the aim is to get the greatest value or least value of a variable in regard to the local maximum values and the absolute extrema value as in the next examples.

Example First derivative test

(1) Find the two dimensions of a rectangle which has the biggest area and it can be drawn inside a triangle whose base length is 16cm and its height is 12 cm such that one of its sides lies on the base of the triangle and the two vertices of the opposite side lie on the two other sides of the triangle .

Solution

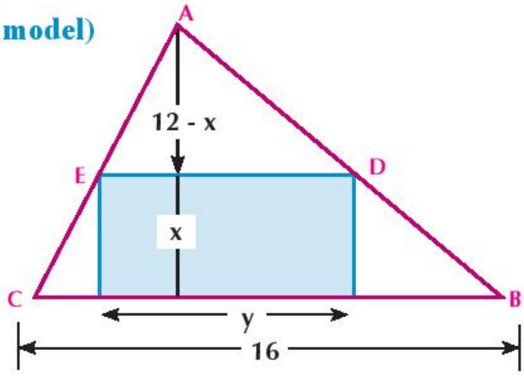
1- To calculate the biggest area, we model the problem according to the given data and constraints.

2- Determine the variables (unknowns) by letting the rectangle width = x cm, its length= y cm and area = A cm^2

3- The relations among the variables **(mathematical model)**

Area of rectangle $A = x \times y$

4- Putting the mathematical model in one variable if possible



$\frac{y}{16} = \frac{AD}{AB} = \frac{12 - x}{12}$ **(from similarity)**

$\therefore \; y = \frac{4}{3}(12 - x), x \in [0, 12]$ **(1)**

Area of rectangle $A = \frac{4}{3} x (12 - x)$

i.e: $A = f(x) = 16x - \frac{4}{3}x^2$ **(2)**

5- **Solving the mathematical model :** by differentiating both sides of relation **(2)** with respect to x.

$\therefore \; f'(x) = 16 - \frac{8}{3}x \, , \, f''(x) = -\frac{8}{3}$

When $f'(x) = 0$ $\quad \therefore x = \frac{16 \times 3}{8} = 6$ and at which $f''(x) < 0$

$\because$ A has an only critical point when x = 6 , and the second derivative is always negative, then this critical point gives the absolute maximum value .

$\therefore$ The function A has an absolute maximum value when $x = 6$ and $y = \frac{8}{3}(12 - 6) = 8$

i.e: the rectangle has the biggest area when its two dimensions are 6cm , 8cm

Try to solve

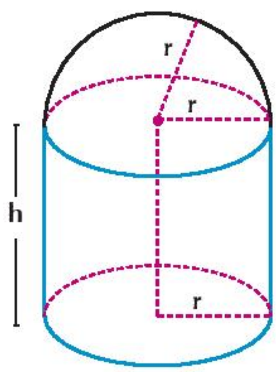
(1) Find the maximum area for an isosceles triangle inscribed in a circle with radius length 12cm.

Example

Calculating the minimum cost

(2) A semi - spherical top vertical cylinder-like silo is needed to be constructed where it holds $108\pi m^3$ of wheat to store **(let the wheat be stored in the cylinderic part only but not the top),** and the cost of the surface unit area from its top is twice the cost of the unit area of the lateral wall. What are the dimensions of the silo which make the cost minimum?

Solution



1- To calculate the minimum cost, graph the problem in regard to the given data and restrictions.

2- **Determining the variables:** let the height of the cylinder = h meter, its base radius length = r meter, the cost of the area unit of the wall = C LE, the cost of the area units of the top= 2C LE and the total cost =K L.E .

3- The relation among the variables (Modeling):

The surface area of the cylinderic surface = perimeter of the base × height = $2\pi rh$ unit area

The area of the semi -spherical surface = $\frac{1}{2}$ area of the sphere = $2\pi r^2$ unit area

the total cost $k = 2\pi rh \times c + 2\pi r^2 \times 2c = 2\pi r c (h + 2r)$

4- Putting the mathematical model in one variable:

$\because$ The volume of the cylindric part = 108π $\therefore \pi r^2 h = 108\pi$ **i.e** $h = \frac{108}{r^2}$

The total cost $k = 2\pi r c (\frac{108}{r^2} + 2r)$

$k = f(r) = 216\pi c r^{-1} + 4\pi c r^2$

5- Solving the model: $f'(r) = -216\pi c r^{-2} + 8\pi c r$

the critical point = when $f'(r) = 0$ $\therefore r^3 = \frac{216}{8}$ **i.e.** $r = 3$(unique point)

Second derivative test:

$\because f''(r) = 432\pi c r^{-3} + 8\pi c$ $\therefore f''(3) > 0$

i.e.: When the vertical cylinder radius length is 3 meters , the silo has the minimum cost and its height is $\frac{108}{9} = 12$ m.

Try to solve

(2) A closed box-like tank whose capacity is 252 cubic meters and base is squared. If we want to coat the interior walls with an insulating material. The cost of the bootm is 50 LE per square meter, the cost of cover is 20 LE per square meter and the cost of the walls is 30 LE per square meter , find the dimensions of the box which makes the cost as minimum as possible.

Example Life applications

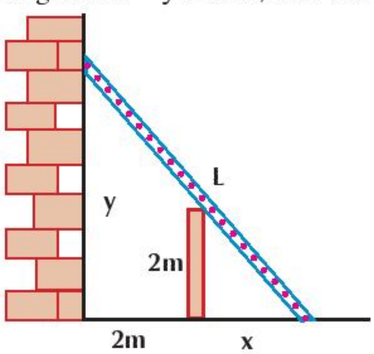
(3) A 2-meter wall and it is 2 meters away from a house, find the minimum length of the ladder that can be used to connect the ground and the house resting on the wall .

Solution

1- To calculate the minimum length of a ladder , we draw the problem in regard to the given data and restrictions .

2- Determining the variables: let the:

ladder length = L meter, the height of the ladder top from the ground = y meter, and the distance between the ladder base and the wall = x meter.



3- Modeling the problem:

From pythagoras : $L^2 = (x + 2)^2 + y^2$ **(1)**

from similarity: $\frac{y}{2} = \frac{x+2}{x}$

$\therefore y = \frac{2x+4}{x} = 2 + 4x^{-1}$ **(2)**

To find the minimum length of the ladder, it is enough to get L^2 the minimum value

4- Solving the model by differentiating both sides of the relation (1) and (2) with respect to x.

$\therefore \frac{d}{dx}(l^2) = 2(x+2) \times 1 + 2y\frac{dy}{dx}$, $\frac{dy}{dx} = \frac{-4}{x^2}$

$\therefore \frac{d}{dx}(l^2) = 2(x+2) + 2\left(\frac{2x+4}{x}\right) \times \frac{-4}{x^2} = 2(x+2)\left(1 - \frac{8}{x^3}\right)$

At the critical point: $\frac{d}{dx}(l^2) = \text{zero}$

$\therefore x = -2$ is refused or $\frac{8}{x^3} = 1$

$\therefore x = 2$

x	2	
sign of $\frac{d}{dx}(\ell^2)$	−	+
ℓ^2	↘	↗

From the first derivative test of increasing and decreasing, we notice that the sign of $\frac{d}{dx}(\ell^2)$ changes from - to +

$\therefore$ When $x = 2$ then ℓ^2 is the most minimum

By substituting in (2) $\therefore y = \frac{2 \times 2 + 4}{2} = 4$

By substituting in (1)

$\therefore \ell^2 = (2+2)^2 + (4)^2 = 32$ $\therefore l = 4\sqrt{2}$

i.e.: The minimum length of the ladder connecting the ground to the house equals $4\sqrt{2}$ meters

Try to solve

(3) In a perpendicular coordinate plane, $\overleftrightarrow{AB}$ is drawn to pass through point C (3 , 2) and intersect the coordinate axes at point A and point B . Prove that the minimum area of triangle AOB equals 12 squared units where O is the origin point (0, 0).

Example Circular sector

(4) A circular sector -like coin whose area is 16 cm^2 . Find the radius length of the sector circle which makes its perimeter as minimum as possible. What is the measure of its angle then?

Solution

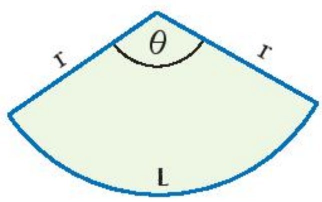
Let the arc length of the sector be l cm , and the radius length of the sector circle = r cm

$\therefore$ Perimeter of the sector $P = 2r + l$ **(1)**

$\because$ Area of the sector $= \frac{1}{2} l r = 16$ $\therefore l = \frac{32}{r}$

By substituting in (1) $\therefore P = 2r + \frac{32}{r}$ **(2)**

by differentiating both sides of the relation **(2)** with respect to r



$\frac{dP}{dr} = 2 - \frac{32}{r^2}$, $\frac{d^2P}{dr^2} = \frac{64}{r^3}$

When $\frac{dP}{dr} = 0$ $r = 4$, $\frac{d^2P}{dr^2} > 0$

$\therefore$ When $r = 4$ the perimeter of the sector is as minimum as possible

$\because$ Area of the sector $= \frac{1}{2} r^2 \theta^{rad}$ $\therefore \theta = \frac{16 \times 2}{4 \times 4} = 2^{rad}$

Try to solve

(4) If the perimeter of a circular sector = 12cm, find the angle measure of the sector which makes its area as maximum as possible.

Exercises 2 - 4

1. The sum of two numbers is 30 and their product is as maximum as possible, find the two numbers.

2. The sum of two positive integer numbers is 5 and the sum of the cube of the smaller number and double the square of the other is as minimum as possible, find the two numbers.

3. Find the positive number which if its multiplicative inverse is added to it, the sum is as minimum as possible.

4. Find the maximum area of a rectangle -like piece of land that can be surrounded by a 120-meter fence.

5. If the perimeter of a circular sector is 30 cm and its area is as maximum as possible , find the radius length of its circle.

6. A cuboid -like box whose base is in the form of square. If the sum of all its edges equals 240cm, find the dimensions of the box that will maximize its volume.

7. If the hypotenuse length of a right-angled triangle equals 10cm, find the lengths of the two legs of the right angle when the area of the triangle is as maximum as possible.

8. An open field is bounded by a straight river from one of its sides. Determine how to place a fence around the other sides of the rectangle -like piece of land to surround as maximum area as possible by a 800 -meter fence. What is the area of the land then?

9. A closed cylindaric -like boxes are made to can beverages. The capacity of each box is k of volume units with minimum amount of the material. Find the ratio of the height (h) of the box to the radius length (r) of its base

10. A rectangle - like playground ends in two semi-circles. If the perimeter of the playground is 420 m, find the maximum area of the playground.

11. A right -angled triangle whose hypotenuse length is 30cm. Find the length of the two legs of the right angle if the length of the altitude drawn from the right angle on the hypotenuse is as maximum as possible .

12. A rectangle -like piece of cardboard paper whose two dimensions are 15cm and 24 cm. Congruent squares of side length □ □□ are cut from its four corners, then the projected parts are folded upward to form a box without a cover, calculate the dimension of the box when its volume is as maximum as possible.

13. An open tank of squared base, vertical sides and holds a certain amount of water, prove that the cost of coating the tank from interior by a regular insulating material is as minimum as possible if its depth equals half the side length of the base.

14) Find the nearest point to point (0, 5) and lies on the curve $y = \frac{1}{2} x^2 - 4$.

15) Find the shortest distance between the straight line $x - 2y + 10 = 0$ and the curve $y^2 = 4x$.

16) A B C is a triangle where a , b . are constants, find the measure of the angle included between them which makes the area of the triangle as maximum as possible .

17) The current intensity I (Ampere) in a circuit for the alternating current at any moment t (second) is given by the relation $I = 2 \cos t + 2 \sin t$. What is the maximum value of the current in this circuit?

18) The volume of a bacterial culture placed in a food medium grows in regard to the relation $f(n) = 2000 + \frac{5000\,t}{100 + t^2}$ where time t is measured in hours, identify the maximum value of the culture volume.

19) A B C D is a square whose side length is 10cm and $M \in \overline{BC}$ where B M = x cm and $N \in \overline{CD}$ where $CN = \frac{3}{2} x$, find the value of x which makes the area of $\triangle AMN$ as minimum as possible.

20) $\overline{AB}$ is a diameter in a circle of a radius length r, two tangent are drawn to the circle at A and B from point E on the circle . Another tangent is drawn to the circle to intersect the two previous tangents at D and C respectively, prove that the minimum area of the trapezium ABCD equals $2r^2$ squared unit.

Unit Three

The Definite Integral and its Applications

Introduction

Have you ever noticed the basket maker doing a basket? The process of getting the parallel section side by side leads to the integration of his basket. This helped the scientists try to discover general methods to estimate the area of any plane region by dividing any plane area into very tiny areas, then adding the area of such tiny areas to estimate the required area. This had led to discover the science of integration and the symbol of the integration process $\int$ it is the first letter of the word "Sum" which means summation in this unit, you are going to know different methods to calculate the indefinite integration such as integration by substitition and integration by parts to find the set of antiderivatives for a continuous function on a given interval , then identify the definite integration through the fundamental theorem of calculus which links between the definite and indefinite integration and use the definite integration to find the area of a plane region or the volume of revolution solid. You will also identify some economic applications for the definite integration and use the mathematical modling to solve the mathematical and life problems .

Unit objectives

By the end of the unit and carrying out the involved activities, the students should be able to:

- Identify some methods of integration such as integration by substitution and by parts $\int x e^x dx$
- Identify the definite integration (fundamental theorem of calculus) and deduce some of its properties.
  - $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
  - $\int_a^a f(x)\,dx = 0$
  - $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
  - $\int_a^b [f(x) \pm g(x)]\,dx$ $= \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
  - $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
  - $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ where f is an even function.
  - $\int_{-a}^a f(x)\,dx = 0$ where f is an odd function.
- Use the definite integration to solve problem involving finding the area.
- Find the area of the plane region under the curve and on the x-axis for all the values of x in the domain using the definite integration .
- Use the definite integration to solve problems involving finding the volume of a revolution surface around x - axis .
- Identify integration of exponential function
- Find $\int \frac{f(x)}{f(x)}dx$

Key terms

- Antiderivative
- Indefinite Integral
- Differential
- Integration by Substitution
- Integration by Parts
- Rule
- Exponential and logarithmic Functions.
- Definite Integral
- Fundamental theorem of calculus
- Areas in the plane
- Volumes of Revolution solids

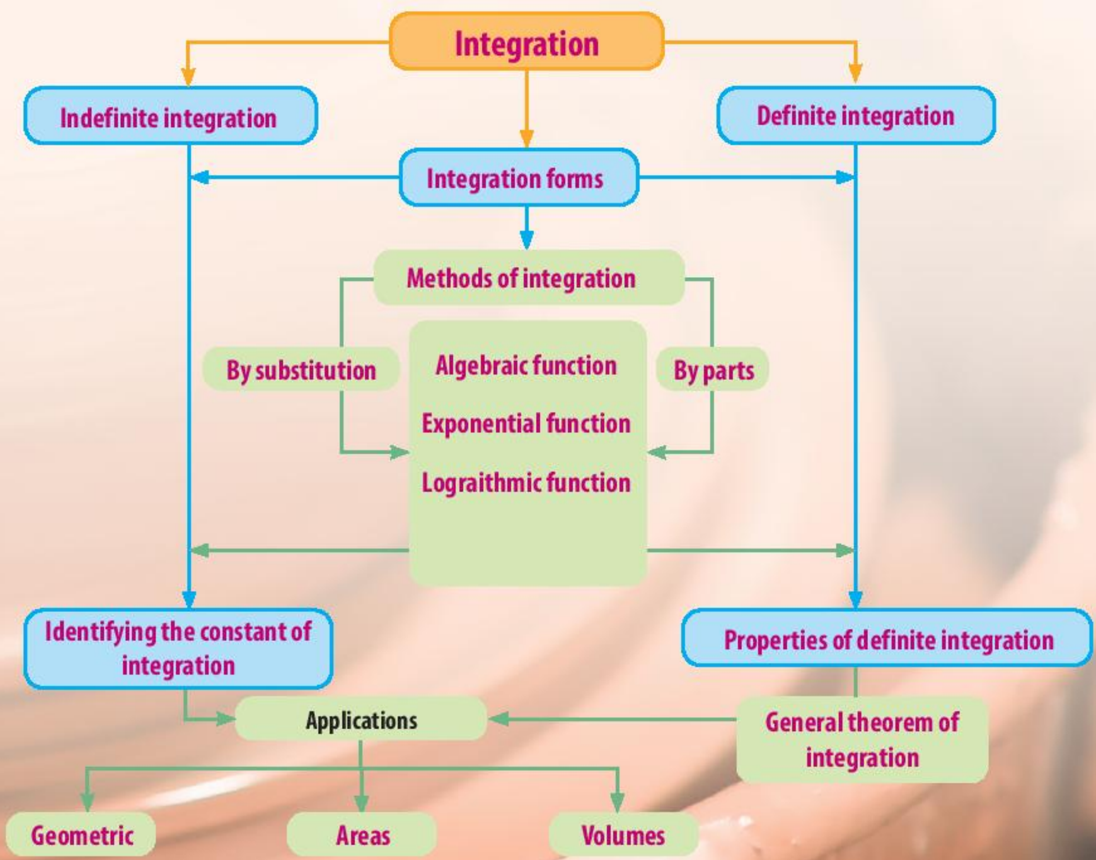
Lessons of the unit

Lesson (3 - 1): Integrals of Exponential and logarithmic function.
Lesson (3 - 2): Methods of integration.
Lesson (3 - 3): Definite integration.
Lesson (3 - 4): Applications on definite integral.

Materials

- Scientific calculator - Computer graphics.
- Internet).

Chart of the unit



3 - 1 Integrals of Exponential and Logarithmic Function

You will learn

- Integration of exponential and logarithmic functions.
- Geometric applications.
- Physical applications.

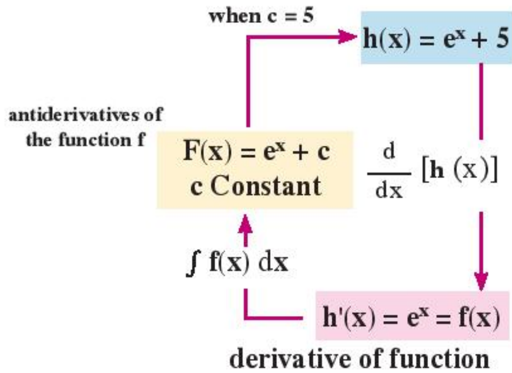
Key terms

- Antiderivative
- Integration
- Indefinite integral
- Arbitrary constant

Materials

- Scientific calculator.
- Computer graphics.

Explore



From your previous learning to differentiation, you knew that the derivative of the function h with respect to x where $h(x) = e^x + 5$ is $h'(x) = e^x$

If we denote the function f '(x) by the symbol f(x), then we can use an inverse operation (indefinite integration) to find an indefinite number of other functions(F(x) + c) the derivative of each equals f(x) and called the set of antiderivatives of the function f one of them equals h(x) where:

$\int f(x)\, dx = F(x) + c$ **where c is an arbitrary**

Explore the set of the antiderivatives for each of:

$f(x) = 5e^{5x}$, $g(x) = 8e^{x^4}$, $g(x) = \frac{1}{x}$

Learn

Indefinite integrals of exponential function

If K is a real number where $k \neq 0$

Then: $\int e^x\, dx = e^x + c$

, $\int e^{kx}\, dx = \frac{1}{k} e^{kx} + c$ **where c is an arbitrary constant**

Example

1 Find:

a $\int e^{7x}\, dx$ b $\int e^{-\frac{3}{4}y}\, dy$ c $\int 8\, e^{2z}\, dz$

Solution

a $\int e^{7x}\, dx = \frac{1}{7} e^{7x} + c$

b $\int e^{-\frac{3}{4}y}\, dy = \frac{1}{-\frac{3}{4}} e^{-\frac{3}{4}y} + c = \frac{-4}{3} e^{-\frac{3}{4}y} + c$

c $\int 8e^{2z}\, dz = 8\int e^{2z}\, dz = \frac{8}{2} e^{2z} + c = 4e^{2z} + c$

Remember

$\int a\, f(x)\, dx = a \int f(x)\, dx$

Try to solve

1) Find

a $\int \pi e^{x}\, dx$ **b** $\int -e^{-5z}\, dz$ **c** $\int -6e^{0.2y}\, dy$ **d** $\int 3\sqrt{2}\, e^{-\sqrt{2}n}\, dn$

Example

2) Find each of the following integrations:

a $\int \frac{e^{x} + e^{-x}}{2}\, dx$ **b** $\int \frac{3e^{x} - 2e^{2x}}{2e^{x}}\, dx$

Solution

a $\int \frac{e^{x} + e^{-x}}{2}\, dx = \frac{1}{2}\int (e^{x} + e^{-x})\, dx$

$= \frac{1}{2}[\int e^{x}\, dx + \int e^{-x}\, dx]$

$= \frac{1}{2}(e^{x} - e^{-x}) + c$

b $\int \frac{3e^{x} - 2e^{2x}}{2e^{x}}\, dx = \int \frac{3}{2}\, dx - \int e^{x}\, dx$

$= \frac{3}{2}x - e^{x} + c$

Remember

$\int [f(x) \pm g(x)]\, dx$

$= \int f(x)\, dx \pm \int g(x)\, dx$

Try to solve

2) Find:

a $\int \frac{e^{x} + e^{-x}}{e^{x}}\, dx$ **b** $\int (x^{2} + 2e^{x})\, dx$ **c** $\int (x^{2e} + e^{3x})\, dx$

Notice that: **If** f(x) is a differentiable function, then:

$$\int e^{f(x)} \cdot f'(x)\, dx = e^{f(x)} + c$$

Example

3) **a** $\int \sin x\, e^{\cos x}\, dx$ **b** $\int 4x\, e^{x^{2}+1}\, dx$

Solution

a By putting $f(x) = \cos x$ $\therefore f'(x) = -\sin x$

$\int \sin x\, e^{\cos x}\, dx = -\int e^{\cos x}(-\sin x)\, dx = -e^{\cos x} + c$

b By putting $f(x) = x^2 + 1$ $\therefore f'(x) = 2x$

$\int 4x\, e^{x^2+1}\, dx = 2\int e^{x^2+1}(2x)\, dx = 2e^{x^2+1} + c$

Try to solve

3) Find the following integrations:

a $\int (\cos x\, e^{\sin x} + 3x^2)\, dx$

b $\int (x - 3)\, e^{x^2 - 6x + 5}\, dx$

Indefinite integral of logarithmic functions

You know that $\frac{d}{dx}(\ln x) = \frac{1}{x}$ where $x > 0$, $\frac{d}{dx}(\ln - x) = \frac{1}{x}$ where $x < 0$

In general $\frac{d}{dx} \ln |x| = \frac{1}{x}$ where $x \neq 0$

I.e the function $\ln |x|$ where $x \neq 0$ is one of the antiderivatives of the function $\frac{1}{x}$

So: $\int \frac{1}{x}\, dx = \ln |x| + c$ where $x \neq 0$

Multiples of the function

Example

4) Find each of the following integrations:

a $\int \frac{2}{x}\, dx$ **b** $\int \frac{7}{x \ln 3}\, dx$

Solution

a $\int \frac{2}{x}\, dx = 2\int \frac{1}{x}\, dx = 2 \ln |x| + c$ where $x \neq 0$

b $\int \frac{7}{x \ln 3}\, dx = \frac{7}{\ln 3}\int \frac{1}{x}\, dx = \frac{7}{\ln 3} \ln |x| + c$ where $x \neq 0$

Try to solve

4) Find:

a $\int \frac{\ln 3}{x}\, dx$ **b** $\int \frac{4}{3x \ln 5}\, dx$ **c** $\int \frac{\ln x^2}{x \ln x^3}\, dx$

Example

5 Find each of the following integrations:

a $\int \left(3x^2 + \frac{5}{x}\right) dx$ b $\int \left(\frac{2e}{x} + \frac{x}{e}\right) dx$ c $\int \frac{(3x-1)^2}{3x} dx$

Solution

a $\int \left(3x^2 + \frac{5}{x}\right) dx = \int 3x^2\, dx + \int \frac{5}{x} dx = x^3 + 5 \ln|x| + c$

b $\int \left(\frac{2e}{x} - \frac{x}{e}\right) dx = 2e \int \frac{1}{x} dx - \frac{1}{e} \int x\, dx = 2e \ln|x| - \frac{x^2}{2e} + c$

c $\int \frac{(3x-1)^2}{3x} dx = \int \frac{9x^2 - 6x + 1}{3x} dx = \int (3x - 2 + \frac{1}{3x}) dx$

$= \frac{3x^2}{2} - 2x + \frac{1}{3} \ln|x| + c$ where $x \neq 0$

Try to solve

5 Find each of the following integrations:

a $\int \frac{6x^2 - 5}{3x} dx$ b $\int \frac{x^2 - 4}{x^2 - 2x} dx$ c $\left(\sqrt{x} - \frac{3}{\sqrt{x}}\right)^2 dx$

Notice that: If the function is differentiable and , $f(x) \neq 0$, then

$$\int \frac{1}{f(x)} \cdot f'(x)\, dx = \ln|f(x)| + c$$

Example

6 Find each of the following integrations:

a $\int \frac{4}{1+2x} dx$ b $\int \frac{2x+3}{x^2+3x-2} dx$ c $\int \tan x\, dx$

Solution

a $\because (1 + 2x)' = 2 \therefore \int \frac{4}{1+2x} dx = 2 \int \frac{(1+2x)'}{1+2x} dx = 2 \log_e |1 + 2x| + c$

b $\because (x^2 + 3x - 2)' = 2x + 3$

$\therefore \int \frac{2x+3}{x^2+3x-2} dx = \ln|x^2 + 3x - 2| + c$

c $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + c = \ln|\sec x| + c$

Try to solve

6 Find each of the following integrations:

a $\int \cot x\, dx$ b $\int \frac{x^2 - 4}{x^2 + 2x} dx$ c $\int \frac{(x^2 + 2)\, dx}{x^3 + 6x + 1}$

Example

(7) **Geometric applications:** the slope of the tangent to a curve at any point (x , y) on it equals $\frac{3x+2}{x}$. Find the equation of the curve if known that it passes through point (e , 3e + 5)

Solution

Let the equation of the curve y = f(x)

$\because$ the slope of the tangent at any point $= \frac{dy}{dx} = \frac{3x+2}{x}$

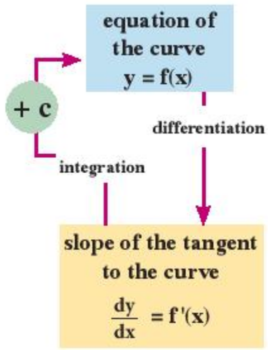
$\therefore y = \int \frac{dy}{dx}\, dx = \int (3 + \frac{2}{x})\, dx$

$\therefore y = 3x + 2 \ln x + c$ where c is an arbitrary constant

$\because$ The curve passes through point (e , 3e + 5), then it satisfies its equation: i.e.

$3e + 5 = 3(e) + 2 \ln e + c \qquad \therefore c = 3$

The equation of the curve is: $y = 3x + 2 \ln x + 3$



Try to solve

(7) The slope of the tangent to the curve of the function f at any point (x , y) on it equals $\frac{1}{2x - e}$ and $f(e) = \frac{1}{2}$. Find f(2e)

Example

(8) **Physical applications:** If the rate of change in the surface area of a lamina A (in square centimeters) with respect to time t (second) is identified by the relation $\frac{dA}{dt} = e^{-0.1t}$ and the area of the lamina at the beginning of the change equals 80 cm^2, find the surface area of the lamina after 10 second.

Solution

Surface area of the lamina $A = \int \frac{dA}{dt}\, dt = \int e^{-0.1t}\, dt$

$\therefore \quad A = -10e^{-0.1t} + c$

At the beginning of change t = 0 , A = 80 $\qquad \therefore c = 90$

The surface area of the lamina at any moment $A = 90 - 10e^{-0.1t}$

After 10 second $\qquad \therefore$ the surface area of the lamina $= 90 - 10e^{-1}\ cm^2$

Try to solve

(8) If the rate of change of the sales in a factory is inversely proportional to time in weeks and the sales of the factory after 2 weeks and 4 weeks were 200 units and 300 units respectively, find the sales of the factory after 8 weeks.

Exercises 3 - 1

Choose the correct answer:

1. If $f''(x) = \frac{1}{2}[e^x + e^{-x}]$, $f(0) = 1$, $f'(0) = 0$, then f(x) equals:
   a) $-f'(x)$ b) $f'(x)$ c) $-f''(x)$ d) $f''(x)$

2. If the slope of the tangent to a curve at any point (x , y) equals $4e^{2x}$, $f(0) = 2$, then $f(-2)$ equals:
   a) 4 b) $4e^{-4}$ c) $2e^{-4}$ d) 2e

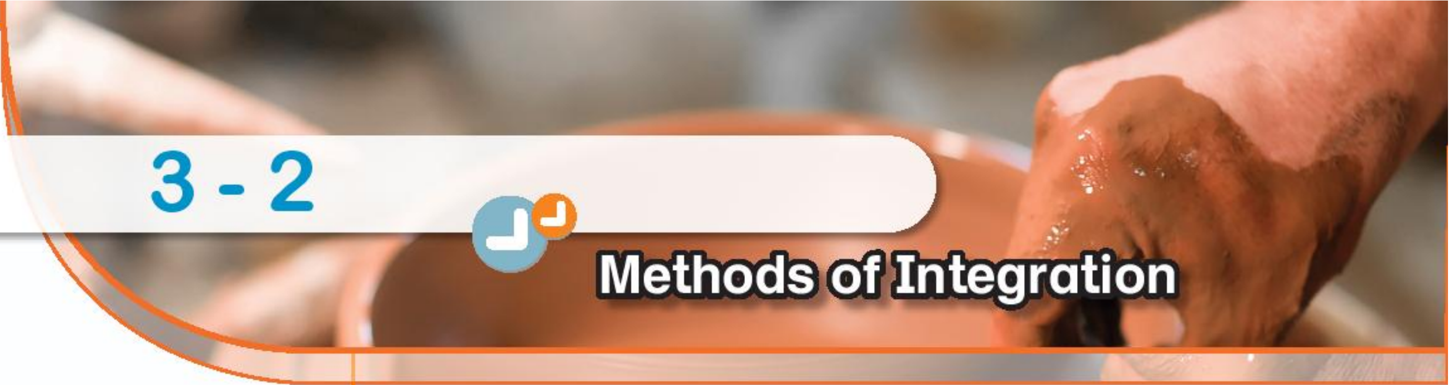
3. $\int \tan\theta \, d\theta$ equals
   a) $-\ln|\cos\theta| + c$ b) $-\ln\cos\theta + c$ c) $\ln\cos\theta + c$ d) $|\ln\cos\theta| + c$

4. $\int 4x\, e^{x^2} \, dx$ equals
   a) $\frac{1}{2}e^{x^2} + c$ b) $e^{x^2} + c$ c) $2e^{x^2} + c$ d) $4e^{x^2} + c$

Find each of the following integrations:

5. $\int e^{4x} \, dx$
6. $\int (3x^2 + 2e^x) \, dx$
7. $\int (\frac{4}{x} - e^{-x}) \, dx$
8. $\int e^{1-3x} \, dx$
9. $\int \frac{7}{3} e^{3x-4} \, dx$
10. $\int 2e^x (e^x + 1)^2 \, dx$
11. $\int \frac{e^{3x} + 2e^{2x} + 4}{e^x} \, dx$
12. $\int x^2 e^{x^3+1} \, dx$
13. $\int \frac{2e^x}{e^x + 1} \, dx$
14. $\int \frac{dx}{4x - 1}$
15. $\int \frac{x}{x^2 + 1} \, dx$
16. $\int \frac{\sec^2 x}{\tan x} \, dx$
17. $\int \frac{\sin x + \cos x}{\sin x - \cos x} \, dx$
18. $\int \frac{\cos x}{1 + \sin x} \, dx$
19. $\int \frac{\sec x \tan x}{\sec x - 1} \, dx$
20. $\int \frac{1}{(x \ln x)} \, dx$
21. $\int \frac{2x}{(x + 1)^2} \, dx$
22. $\int \frac{(\log x)^2}{x} \, dx$
23. $\int \frac{3x^2}{x^3 - 1} \, dx$
24. $\int \frac{3x^2 - 5}{x^3 - 5x + 1} \, dx$
25. $\int \frac{4e^x + x e^{2x}}{x e^x} \, dx$
26. $\int \frac{4}{x \ln 3x} \, dx$
27. $\int \frac{(1 + \ln x)^2}{x} \, dx$

28. **Geometric applications:** If the slope of the tangent to the curve of the function f at any point (x , y) equals $2e^{-\frac{1}{2}x}$ and $f(0) = 1$, find f(3)



3 - 2 Methods of Integration

You will learn

- Finding the original function of a given function
- Finding the differential of a function
- Calculating the integration by substitution
- Calculating the integration by parts

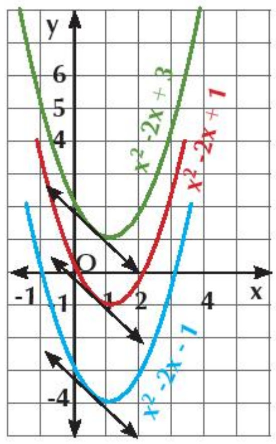
Key terms

- Antiderivative
- Indefinite Integral
- Differential

Materials

- Scientific calculator.
- Computer graphics.

introduction



You have previously identified the anti derivative or the indefinite integration. It is an inverse operation to the process of differentiation . It is said that the function F is an antiderivative to the function f at the interval I if: $\frac{d}{dx}$ F(x) = f (x) for each x ∈ I when we add any constant to the antiderivative F, (**it is known as the arbitrary constant**) then the antiderivative is represented by the set of the curves **y = F(x)** + **c** which are different from each other in the constant C and the slope of the tangent for any of them is equal .As a result these curves are parallel as shown in the opposite figure. The set of antiderivatives has been called the indefinite integration and denoted by the symbol: $\int f(x)\, dx$, then:

$$\int f(x)\, dx = F(x) + c$$

The indefinite integration has the next properties:

If f and g are two antiderivatives at the interval I , then:

1- $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

2- $\int k\, f(x)\, dx = k \int f(x)\, dx$ where K is a constant real number

Notice that:

$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$ where n ≠ -1 and **c** is an arbitrary constant

So

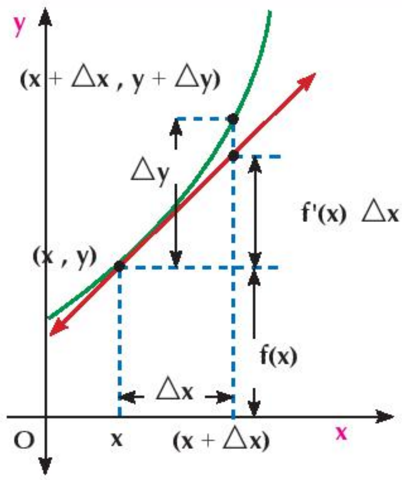
$\int (3x^2 + 4x + 5)\, dx = x^3 + 2x^2 + 5x + c$

➲ The process of finding the antiderivatives requires to know the forms of standard integrations for some functions. But the integrations are needed to be found appear far from the standard integrations. This means that you are to identify other methods for integration such as the integration by substitution and integration by parts depending on the differential of the function.

Differentiations

If the function f is differentiable where y = f (x)

From the definition of the derivative:



$$\frac{dy}{dx} = f'(x) = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = \lim_{\triangle x \to 0} \frac{f(x+\triangle x) - f(x)}{x}$$

$\therefore$ When $\triangle x \longrightarrow 0$, **then**: $\frac{\triangle y}{\triangle x} \longrightarrow f'(x)$

i.e.: $\frac{\triangle y}{\triangle x} \simeq f'(x)$ **when** $\triangle x \simeq 0$, $\triangle x \neq 0$

$\therefore \triangle y \simeq f'(x)\triangle x$ (by multiplying by $\times \triangle x$)

Definition

Let function f be differentiable on an open interval containing x , △ x is denoted for the change in x where △ x ≠ 0 , then

1- The differential of y (is denoted by the symbol d y) = f' (x) △ x

2- The differential of x (is denoted by the symbol d x) = △ x

So:

$dy = f'(x)\, dx$ It is a function in two variables x dx

If $y = x^3$ **Then:** $dy = 3x^2\ dx$

Example Differential of the funciton

1 Find the differential for each of the following:

a $y = \frac{x}{x-1}$ b $V = \frac{4}{3}\pi r^3$

c $y = z\,.\,\ell$ where each of z , ℓ is a function in x

Solution

a $\because dy = y'\ dx$, $y = \frac{x}{x-1} = 1 + \frac{1}{x-1} = 1 + (x-1)^{-1}$

$\therefore dy = \frac{-1}{(x-1)^2}\ dx$

b $\because dV = V'\, dr$ $\therefore dV = \frac{4}{3}\pi \times 3r^2\, dr = 4\pi r^2\, dr$

c $\because dy = (z \bullet \ell)'\ dx$

$= (z \bullet \ell' + \ell \bullet z')\, dx$

$= z \bullet \ell'\ dx + \ell \bullet z'\ dx$

$dy = z\, d\ell + \ell\, dz$

Notice

$d\ell = \ell' dx$

$dz = z' dx$

Try to solve

1 **Find the differential for each of :**

a $y = (2x + 5)^4$ **b** $y = e^{2x - 3}$

c $y = \frac{z}{\ell}$ where z , ℓ are functions in the variable x

Critical thinking: If $x^2 + x^2 = 25$ **find:** d y in terms x , y, d x

The fundamental integrations (standard))

There is not a general method to find the integration of the different functions similar to the methods of finding the derivatives of these functions. Finding the integration of any function f is specified in searching for a function which its derivatives are the function f. This is closely related to your understanding to the fundamental derivatives of the functions which you have previously learned and we can summarize them in the following table:

Table of the fundamental derivatives of the functions and the corresponding standard integrations	
$\frac{d}{dx}(x^n) = nx^{n-1}$ $n \in \mathbb{R}$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ $n \in \mathbb{R} - \{1\}$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x \, dx = \sin x + c$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x \, dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$, $x \neq \frac{2n+1}{2}\pi$, $n \in Z$	$\int \sec^2 x \, dx = \tan x + c$, $x \neq \frac{2n+1}{2}\pi$, $n \in Z$
$\frac{d}{dx}(e^s) = e^x$	$\int e^x \, dx = e^x + c$
$\frac{d}{dx}(a^x) = a^x \ln a$ $a > 0$, $a \neq 1$	$\int a^x \, dx = \frac{1}{\ln a} a^x + c$ $a > 0$, $a \neq 1$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$ $x > 0$	$\int \frac{1}{x} \, dx = \ln \lvert x \rvert + c$ $x \neq 0$

The Integration by Substitntion

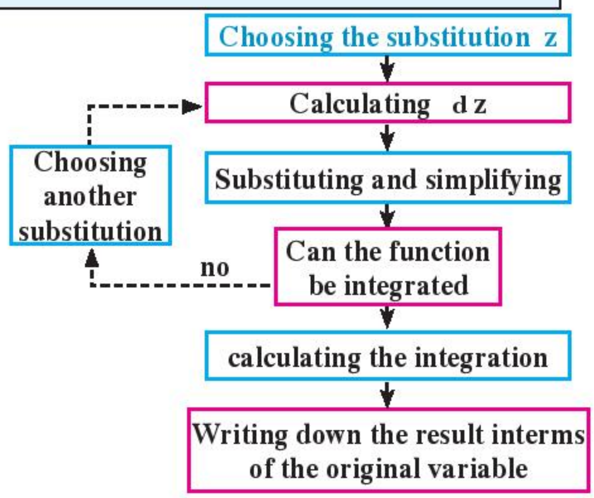
It is one of the most essential methods of integration to find the integration of the product of two functions in the form: $\int f(g(x)) \, g'(x) \, dx$

If $z = g(x)$ is a differentiable function

then $dz = g'(x) \, dx$ and:

$\int f(g(x)) \, g'(x) \, dx = \int f(z) \, dz$

➲ **To do the integration operation by substitution, we follow up the opposite diagram:**



Example Integration by substitution

2 Find

a $\int x^3 (2x^4 - 7)^5 \, dx$

b $\int \frac{x+4}{(x^2+8x)^3} \, dx$

Solution

a By putting $z = 2x^4 - 7$ $\therefore dz = 8x^3 \, dx$

$$\int x^3 (2x^4 - 7)^5 \, dx = \frac{1}{8} \int (\underbrace{2x^4 - 7}_{z})^5 (\underbrace{8x^3 \, dx}_{dz})$$

$$= \frac{1}{8} \int z^5 \, dz = \frac{1}{8 \times 6} z^6 + c$$

$$= \frac{1}{48} (2x^4 - 7)^6 + c$$

b By putting $z = x^2 + 8x$

$\therefore dz = (2x + 8) \, dx = 2(x + 4) \, dx$

$$\int \frac{x+4}{(x^2+8x)^3} \, dx = \frac{1}{2} \int \frac{2(x+4) \, dx}{(x^2+8x)^3} = \frac{1}{2} \int \frac{1}{z^3} \, dz$$

$$= \frac{1}{2} \int z^{-3} \, dz = \frac{1}{2 \times -2} z^{-2} + c$$

$$= \frac{-1}{4(x^2+8x)^2} + c$$

Try to solve

2 Find:

a $\int 3x (x^2 + 3)^4 \, dx$

b $\int \frac{x^2}{(x^3 - 4)^5} \, dx$

Example Integration by substitution

3 Find

a $\int x (x + 4)^7 \, dx$

b $\int (x^2 + 5) \sqrt{x - 1} \, dx$

Solution

a By putting $z = x + 4$ $\therefore x = z - 4, \; dx = dz$

$$\int x (x + 4)^7 \, dx = \int z^7 (z - 4) \, dz = \int (z^8 - 4z^7) \, dz \quad \text{(substitution)}$$

$$= \frac{1}{9} z^9 - \frac{1}{2} z^8 + c \quad \text{(integration)}$$

$$= \frac{1}{18} z^8 (2z - 9) + c \quad \text{(simplification)}$$

$$= \frac{1}{18} (x + 4)^8 (2x - 1) \quad \text{(substitution by z)}$$

b By putting $z^2 = x - 1$ **to simplify the form of integration** $\therefore\ x = z^2 + 1$, $dx = 2z\ dz$

$$\int (x^2 + 5)\sqrt{x - 1}\ dx = \int [(z^2 + 1)^2 + 5] \times z \times 2z\ dz \qquad \textbf{(substitution)}$$

$$= \int [z^4 + 2z^2 + 6] \times 2z^2\ dz$$

$$= 2\int (z^6 + 2z^4 + 6z^2)\ dz \qquad \textbf{(simplification)}$$

$$= 2\left[\frac{1}{7} z^7 + \frac{2}{5} z^5 + 2z^3\right] + c \qquad \textbf{(integration)}$$

$$= \frac{2}{35} z^3 [5z^4 + 14 z^2 + 70] + c \qquad \textbf{(common factor)}$$

$$= \frac{2}{35} (x - 1)^{\frac{3}{2}} [5 (x - 1)^2 + 14 (x - 1) + 70] + c \qquad \textbf{(substitution by z)}$$

$$= \frac{2}{35}\sqrt{(x - 1)^3}\ (5x^2 + 4x + 61) + c$$

Try to solve

3 Find the following integrations:

a $\int x (2x - 3)^4\ dx$ **b** $\int x^2 \sqrt[3]{3x + 1}\ dx$

Example

Integration by sustitution

4 Find :

a $\int \sqrt{\frac{1 + \sqrt{x}}{x}}\ dx$ **b** $\int 6x\ e^{x^2}\ dx$

Solution

a By putting $z = 1 + \sqrt{x}$ $\therefore\ \sqrt{x} = z - 1$, $x = (z - 1)^2$

$$dx = 2 (z - 1)\ dz$$

$$\int \sqrt{\frac{1 + \sqrt{x}}{x}}\ dx = \int \sqrt{\frac{z}{(z - 1)^2}} \times 2 (z - 1)\ dz \qquad \textbf{(substitution)}$$

$$= \int 2z^{\frac{1}{2}}\ dz = 2 \times \frac{2}{3} z^{\frac{3}{2}} + c \qquad \textbf{(integration)}$$

$$= \frac{4}{3} \left(\sqrt{(1 + \sqrt{x})^3}\right) + c \qquad \text{(substitution by z)}$$

b By putting $z = x^2$ $\therefore\ dz = 2x\ dx$

$$\int 6x\ e^{x^2}\ dx = 3\int e^{x^2} (2x\ dx)$$

$$= 3\int e^z\ dz = 3\ e^z + c \qquad \textbf{(substitution and integration)}$$

$$= 3\ e^{x^2} + c \qquad \textbf{(substitution by z)}$$

Try to solve

4 Find :

a $\int \frac{x}{\sqrt{1 - 3x^2}}\ dx$ **b** $\int (3 - x)\ e^{6x - x^2}\ dx$

Example Integration by substitution

5 Find

a $\int \frac{5x}{3x^2 - 1} dx$ b $\int \frac{\sqrt{\ln x}}{x} dx$

Solution

a By putting $z = 3x^2 - 1$ $dz = 6x\, dx$

$$\int \frac{5x}{3x^2 - 1} dx = \frac{5}{6} \int \frac{6\, x dx}{3x^2 - 1} = \frac{5}{6} \int \frac{1}{z} dz \quad \textbf{(substitution)}$$

$$= \frac{5}{6} \ln |z| + c = \frac{5}{6} \ln |3x^2 - 1| + c \quad \textbf{(substitution and integration)}$$

b By putting $z = \ln x$ $\therefore dz = \frac{1}{x} dx$

$$\int \frac{\sqrt{\ln x}}{x} dx = \int \sqrt{\ln x} \left(\frac{1}{x} dx\right) = \int z^{\frac{1}{2}} dz \quad \textbf{(substitution)}$$

$$= \frac{2}{3} z^{\frac{3}{2}} + c$$

$$= \frac{2}{3} [\ln (x)]^{\frac{3}{2}} + c \quad \textbf{(substitution and integration)}$$

Try to solve

5 Find :

a $\int \frac{e^{2x}}{e^{2x} + 3} dx$ b $\int \frac{1}{x(\ln x)^2} dx$

Critical thinking: Use the integration by substitution to prove the correctness of the following integration rules :

1- $\int [f(x)]^n \, f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$ where $n \neq -1$

2- $\int \frac{f'(x)}{f(x)} d(x) = \ln |f(x)| + c$ where $f(x) \neq 0$

Integration by Parts

If y and z are two functions in the variable x and differentiable , then:

$$\frac{d}{dx}(y\, z) = y \frac{dz}{dx} + z \frac{dy}{dx}$$

By integrating both sides with respect to x

$$\int \frac{d}{dx}(y\, z)\, dx = \int y \frac{dz}{dx} dx + \int z \frac{dy}{dx} dx$$

$$y\, z = \int y\, dz + \int z\, dy$$

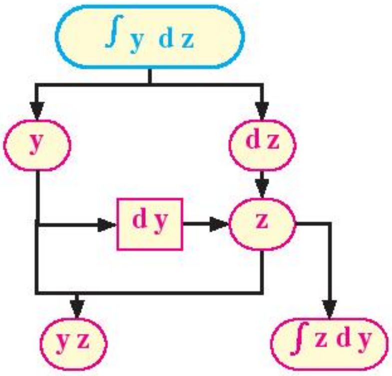
i.e : $\int y\, dz = y\, z - \int z\, dy$

Remember

$dz = \frac{dz}{dx} dx$

$dy = \frac{dy}{dx} dx$

The previous equation is called the rule of integration by parts. It is used to find the integration of the product of two functions each is not a derivative to another by a proper choice for each of y and z such that the integration can be calculated by the left side by a method easier than calculating the integration by the right side. The opposite diagram is to be followed up as shown in the following example :



Example Integration by parts

6 Find:

a $\int x e^x \, dx$

b $\int x^2 e^x \, dx$

Solution

a To find $\int x e^x \, dx$:

Let: $y = x$, $\quad dz = e^x \, dx$

$\therefore dy = dx \quad z = \int e^x \, dx = e^x$

$\because \int y \, dz = yz - \int z \, dy$

$\therefore \int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + c = e^x (x - 1) + c$

Important note: Adding a constant to function z doesnot change the result (prove that)

b To find $\int x^2 e^x \, dx$:

Let: $y = x^2$, $\quad dz = e^x \, dx$

$dy = 2x \, dy \quad z = \int e^x \, dx = e^x$

$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$

$= x^2 e^x - 2\int x e^x \, dx$

$\int x e^x \, dx = e^x (x - 1) + c$ from a

$= x^2 e^x - 2 e^x (x - 1) + c$

$= e^x [x^2 - 2x + 2] + c$

Try to solve

6 Find :

a $\int x e^{-2x} \, dx$

b $\int x^2 e^{x+3} \, dx$

Notice that:

Choosing y and dz is based on:

1- dy is simpler than y

2- dz is easier than integration

Example Integration by parts

7 Find

a $\int \ln x \, dx$ b $\int x \ln x \, dx$

Solution

a **Let :** $y = \ln x$ $dz = dx$

$dy = \frac{1}{x} dx$ $z = \int dx = x$

$\int \ln x \, dx = x\ln x - \int x \times \frac{1}{x} dx$

$= x\ln x - x + c = x(\ln x - 1) + c$

b **Let:** $y = \ln x$ $dz = x \, dx$

$dy = \frac{1}{x} dx$ $z = \int x \, dx = \frac{1}{2}x^2$

$\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \times \frac{1}{x} dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c = \frac{1}{2}x^2(\ln x - 1) + c$

Try to solve

7 Find:

a $\int \ln(x+1) \, dx$ b $\int (\ln x \div \sqrt{x}) \, dx$

Example Integration by parts

8 Find:

a $\int \frac{x e^x}{(x+1)^2} dx$ b $\int \frac{4x}{\sqrt[3]{2x+1}} dx$

Solution

a Notice that $(x+1)^{-2}$ is easier in the integration

By putting $y = x e^x$ $dz = (x+1)^{-2} dx$

$dy = (x e^x + e^x) dx$ $z = \frac{-1}{x+1}$

$\int \frac{x e^x}{(x+1)^2} dx = -\frac{x e^x}{x+1} - \int \frac{-1}{x+1} \times e^x (x+1) \, dx$

$= -\frac{x e^x}{x+1} + \int e^x \, dx$

$= \frac{-x e^x + e^x(x+1)}{x+1} + c = \frac{e^x}{x+1} + c$

b By putting $y = 4x$ $\qquad dz = (2x+1)^{-\frac{1}{3}}\, dx$

$dy = 4\, dx$ $\qquad z = \frac{3}{2 \times 2}\,(2x+1)^{\frac{2}{3}}$

$$\int \frac{4x}{\sqrt[3]{2x+1}}\, dx = 3x\,(2x+1)^{\frac{2}{3}} - \int \frac{3}{4}\,(2x+1)^{\frac{2}{3}} \times 4\, dx$$

$$= 3x\,(2x+1)^{\frac{2}{3}} - \frac{3 \times 3}{5 \times 2}\,(2x+1)^{\frac{5}{3}} + c$$

$$= \frac{3}{10}\,(2x+1)^{\frac{2}{3}}\,[10x - 3\,(2x+1)] + c$$

$$= \frac{3}{10}(2x+1)^{\frac{2}{3}}\,(4x-7) + c$$

Try to solve

8 Find :

a $\int \frac{3x+5}{e^{2x}}\, dx$ $\qquad$ **b** $\int \frac{x}{\sqrt{2x+3}}\, dx$

Critical thinking: can you find $\int \frac{4x}{\sqrt[3]{2x+1}}\, dx$ using the integration by substitution? Explain.

Some applications of the indefinite integration

If we know that the function g gives the slope of the tangent (or the function of the profit margin or the rate of change of a function) at any point on the curve of the function f, then we can know the function F from the operation of the indefinite integration of the function g where: $f(x) = \int g(x)\, dx$

it is noticed that the integration does not give only one function since it contains an arbitrary constant which can be determined from the given data.

Example The equation of the curve of the function

9 If the slope of the tangent to the curve of the function f at any point (x , y) lying on it is given by the relation $g(x) = \frac{x e^x}{(x+1)^2}$, find the equation of the curve if it passes through point (1 , 2 e) .

Solution

Let the equation of the curve of the function be $y = f(x)$ $\qquad \therefore f(x) = \int g(x)\, dx$

$\therefore f(x) = \int \frac{x e^x}{(x+1)^2}\, dx = \frac{e^x}{x+1} + c$ $\qquad$ [from the solution of example 9 (a)]

$\because$ The curve of f passes through point (1 , 2 e) it satisfies the equation

$\therefore 2e = \frac{e}{(1+1)^2} + c$

$\therefore c = \frac{7e}{4}$ and $f(x) = \frac{e^x}{x+1} + \frac{7}{4}e$

Try to solve

9) Find the equation of the curve passing through point (0 , 1) and its slope of tangent at point (x , y) lying on it equals $x\sqrt{x^2+1}$

Exercises 3 - 2

Choose the correct answer

1) $\int x(x^2+3)^5\, dx$ equals

a) $\frac{1}{6}(x^2+3)^6+c$ b) $\frac{1}{12}(x^2+3)^6+c$ c) $\frac{1}{4}(x^2+3)^4+c$ d) $\frac{1}{8}(x^2+3)^4+c$

2) If $\int(2x+3)\ln x\, dx = yz - \int z\, dy$, then y z equals

a) $2x\ln x$ b) $(2x+3)\ln x$ c) $\frac{1}{2}(2x+3)\ln x$ d) $x(x+3)\ln x$

3) If $\int (2x-1)e^{2x+3}\, dx = yz - \int z\, dy$, then $\int z\, dy$

a) $e^{2x+3}+c$ b) $\frac{1}{2}e^{2x+3}+c$ c) $-e^{2x+3}+c$ d) $-\frac{1}{2}e^{2x+3}+c$

Use the proper substitution to find the following integrations:

4) $\int x(x-2)^4\, dx$ 5) $\int x^2(x-2)^3\, dx$ 6) $\int x^3(x^2-1)^5\, dx$

7) $\int x\sqrt{x+4}\, dx$ 8) $\int (x^2-1)\sqrt{x+1}\, dx$ 9) $\int x^5(x^2+3)^4\, dx$

10) $\int \frac{x}{3x^2+2}\, dx$ 11) $\int \frac{x}{x+1}\, dx$ 12) $\int \frac{x+1}{\sqrt{x-1}}\, dx$

13) $\int \frac{x^2}{\sqrt{2x-1}}\, dx$ 14) $\int x e^{-x^2}\, dx$ 15) $\int \frac{e^{-x}-1}{e^{-x}+x}\, dx$

16) $\int \ln 5x \frac{dx}{x}$ 17) $\int (\ln x)^3 \frac{dx}{x}$ 18) $\int \frac{1}{x\ln x}\, dx$

Use the proper parts to find the following integrations :

19) $\int 4x e^{2x}\, dx$ 20) $\int x^3 e^{x^2}\, dx$ 21) $\int \frac{x}{e^{2x}}\, dx$

22) $\int x^3 \ln x\, dx$ 23) $\int \ln x^2\, dx$ 24) $\int (\ln x)^2\, dx$

25) $\int \ln x \frac{dx}{x^3}$ 26) $\int (x+1)^2 e^{2x}\, dx$ 27) $\int x(\ln x)^2\, dx$

Answer the following:

28) Find the equation of the curve passing through point A (2 , 3), and the slope of the normal on it at any point (x , y) is 3- x.

29) If the slope of a tangent to a curve at point (x , y) lying on it is $x\sqrt{x+1}$, find the equation of the curve known that the curve passes through point $(0, \frac{11}{15})$

30) Find the equation of the curve y = f (x) if $\frac{d^2y}{dx^2} = ax+b$ where a and b are two constants and the curve has an inflection point at point (0 , 2) and a local minimum value at point (1 , 0) then find the local maximum value to this curve.

3 - 3 The definite integral

You will learn

- The concept of the definite integration.
- Using the fundamental theorem of calculus to find the definite integration
- Some properties of the definite integration

Key terms

- Definite Integral

Materials

- Scientific calculator
- Graphical programs

Think and discuss

If $y = f(x)$, and the slope of the tangent at any point (x, y) on the curve is :

$$\frac{dy}{dx} = f'(x) = 2x + 3$$

Can you determine a definite value for each of $f(3)$, $f(5)$ and $f(5) - f(3)$? Explain .

Notice that

1 From the definition of the indefinite integral:

$$y = \int f'(x)\, dx = f(x) + c$$

where c is an arbitrary constant which independent of x and it is essential to keep it in the integration to be comprehensive for all the functions whose rates of changes are f(x). As a result, the indefinite integral doesnot give a definite result corresponding a definite value of the variable x .

2 If the value of the integration when $x = a$ is $f(a) + c$
and its value when $x = b$ is $f(b) + c$

$\therefore$ The difference between the two values of the integration when $x = a$ and $x = b$

equals $f(b) - f(a)$ it is a definite value (whatever the value of the constant C is) it is denoted by the symbol $\int_a^b f'(x)\, dx$

where:

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

This form is called the definite integral.

Learn

The Foundamental Theorem of Calculus

If the function f is continuous on the interval [a , b] , and F is any anti derivative to the function f on the same interval , then:

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Notes:

1 $\int_a^b f(x)\, dx$ is called the definite integral and read as the definite integral of f(x) with respect to x from a to b, it is a real number whose value is based on:

a The upper and lower terms of the definite integration i.e. on the two numbers a and b respectively

b The rule of the function f

The symbol of the variable x can be replaced by another symbol without affecting the amount of the integration, i.e:

$$\int_a^b f(x)\, dx = \int_a^b f(y)\, dy = \int_a^b f(z)\, dz \ldots\ldots$$

2 F(b) - F(a) is expressed in the form $[F(x)]_a^b$ or $F(x)\,|_a^b$

3 You can get the definite integral by finding the indefinite integral with disregarding the constant of the integration. Why? then , you can substitute the variable by the two terms of the integration.

4 All the rules of the indefinite integrations and the table of the standard integrations are to be applied to find the value of the definite integration of a continuous function , if f and g are two continuous functions on the interval [a, b]

Then:

- $\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$
- $\int_a^b k\, f(x)\, dx = k \int_a^b f(x)\, dx$ where $k \in \mathbb{R}$

Example Calculating the value of the definite integral

1 Find the definite integration to the function f from x = -2 to x = 4 where $f(x) = 3x^2 - 2$

Solution

The function f is a continuous polynomial on $\mathbb{R}$

$$\therefore \int_{-2}^{4} (3x^2 - 2)\, dx = [x^3 - 2x]_{-2}^{4}$$
$$= [(4)^3 - 2(4)] - [(-2)^3 - (-4)]$$
$$= 64 - 8 + 8 - 4 = 60 \in \mathbb{R}$$

Try to solve

1) Find the value for each of the following:

a) $\int_{-1}^{2} (2x + 3)\, dx$ b) $\int_{0}^{5} \frac{3}{\sqrt{n + 4}}\, dn$ c) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta$

Theorem If function f is continuous on the interval [a, b], then it is integrable on this interval.

Critical thinking

What is the difference between the definite and indefinite integration? Explain.

Properties of definite integral

If the function f is continuous on the interval [a, b], c ∈] a, b [, then:

1. $\int_{b}^{a} f(x)\, dx = -\int_{a}^{b} f(x)\, dx$
2. $\int_{a}^{a} f(x)\, dx = 0$
3. $\int_{a}^{b} f(x)\, dx = \int_{a}^{c} f(x)\, dx + \int_{c}^{b} f(x)\, dx$

Example properties of the definite integral

2) If the function f is continuous on $\mathbb{R}$ and $\int_{1}^{3} f(x)\, dx = 6$, $\int_{5}^{3} f(x) = -14$, find $\int_{1}^{5} f(x)\, dx$

Solution

$\because$ f is continuous on $\mathbb{R}$ and x = 3 divides the interval [1, 5]

$$\therefore \int_{1}^{5} f(x)\, dx = \int_{1}^{3} f(x)\, dx + \int_{3}^{5} f(x)\, dx \quad \text{property (3)}$$
$$= \int_{1}^{3} f(x)\, dx - \int_{5}^{3} f(x)\, dx \quad \text{property (1)}$$
$$= 6 - (-14) = 20$$

Try to solve

2) If the function f is continuous on $\mathbb{R}$ and $\int_{-1}^{4} f(x)\, dx = 255$, $\int_{2}^{-1} f(x)\, dx = -15$, find: $\int_{2}^{4} f(x)\, dx$

Example: Calculating the value of the definite integration

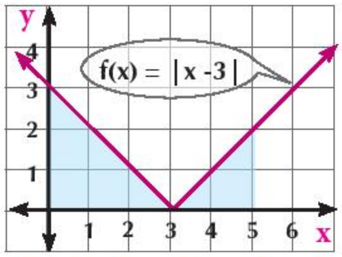
(3) Find $\int_0^5 |x - 3| \, dx$

Solution

From the definition of the modulus function, we find that $|x - 3| = \begin{cases} -(x - 3) & \text{when } x < 3 \\ x - 3 & \text{when } x > 3 \end{cases}$

f is continuous when x = 3

$$\int_0^5 |x - 3| \, dx = \int_0^3 |x - 3| \, dx + \int_3^5 |x - 3| \, dx$$
$$= \int_0^3 (3 - x) \, dx + \int_3^5 (x - 3) \, dx$$
$$= [3x - \frac{x^2}{2}]_0^3 + [\frac{x^2}{2} - 3x]_3^5$$
$$= (9 - \frac{9}{2}) + (\frac{25}{2} - 15 - \frac{9}{2} + 9) = \frac{13}{2}$$



Notice that the colored area equals $\frac{13}{2}$ square units

Try to solve

(3) Find:

(a) $\int_{-4}^2 |x + 1| \, dx$ (b) $\int_{-3}^3 |x^2 - 4| \, dx$

Example: Calculating the value of the definite integration by substitution

(4) Find the value of $\int_0^3 x\sqrt{x^2 + 3} \, dx$

Solution

You can get the definite integration by finding the indefinite integration first, then you can substitute the variable x using the two terms of the integration :

First:

to find $\int x\sqrt{x^2 + 3} \, dx$ put $z = x^2 + 3$ $\therefore dz = 2x \, dx$

$\therefore \int x\sqrt{x^2 + 3} \, dx = \frac{1}{2}\int \sqrt{x^2 + 3} \, (2x \, dx) = \frac{1}{2}\int \sqrt{z} \, dz$ **(substitution)**

$= \frac{1}{2}\int z^{\frac{1}{2}} \, dz = \frac{1}{2} \times \frac{2}{3} z^{\frac{3}{2}} + c$ **(integration)**

$= \frac{1}{3}\sqrt{(x^2 + 3)^3} + c$ **(substitution by z)**

Second:

$\int_0^3 \sqrt{x^2 + 3} \, dx = [\frac{1}{3}\sqrt{(x^2 + 3)^3}]_0^3$

$\frac{1}{3}[\sqrt{(12)^3} - \sqrt{(3)^3}] = 7\sqrt{3}$

Try to solve

4 Find:

a $\int_0^5 x \sqrt{25 - x^2}\, dx$ **b** $\int_{-2}^2 x^3 \sqrt{x^2 + 3}\, dx$

Notice that

1. Example no 4 can be solved directly by finding the values of z corresponding to the values of the two terms of the integration (x = 0 , x = 3)

when x = 0 ∴ z = 3 , when x = 3 z = 12

$$\therefore \int_0^3 x \sqrt{x^2 + 3}\, dx = \frac{1}{2} \int_3^{12} z^{\frac{1}{2}} dx = \left[\frac{1}{3} z^{\frac{3}{2}}\right]_3^{12}$$

$$= \frac{1}{3} \left[\sqrt{(12)^3} - \sqrt{(3)^3}\right] = 7\sqrt{3}$$

2. In **try to solve** 4 b: $f(x) = x^3 \sqrt{x^2 + 3}$ **an odd function**

In **try to solve** 3b: $f(x) = |x^2 - 4|$ **an even function**

The odd and even functions in the definite integration have the following properties:

> **1.** If function f is continuous and odd on the interval [-a , a], then:
>
> $$\int_{-a}^{a} f(x)\, dx = \text{zero}$$
>
> **2.** If function f is continuous and even on the interval [-a , a], then:
>
> $$\int_{-a}^{a} f(x)\, dx = 2 \int_0^a f(x)\, dx$$

Use the properties above to check your answer in **try to solve** 3 and 4

Example The definite integration of odd and even function

5 Find:

a $\int_{-2}^2 \frac{x^3 - 3x}{x^2 + 1} dx$ **b** $\int_{-3}^3 (x^2 - 1)\, dx$

Solution

a f is a continuous function on ℝ

$$\because f(-x) = \frac{(-x)^3 - 3(-x)}{(-x)^2 + 1} = -\frac{x^3 - 3x}{x^2 + 1} = -f(x)$$

∴ **f is an odd function and :** $\int_{-2}^2 \frac{x^3 - 3x}{x^2 + 1} = \text{zero}$

b f is a continuous polynomial function on $\mathbb{R}$

$\because$ $f(-x) = (-x)^2 - 1 = x^2 - 1 = f(x)$

$\therefore$ f is an even function and: $\int_{-3}^{3} (x^2 - 1)\,dx = 2\int_{0}^{3} (x^2 - 1)\,dx$

$= 2\left[\frac{1}{3}x^3 - x\right]_0^3 = 2 \times 6 = 12$

Try to solve

5) Find

a) $\int_{-3}^{3} \frac{x}{1+x^2}\,dx$

b) $\int_{-\pi}^{\pi} (4 + \pi \cos 2x)\,dx$

Critical thinking

1 What is the value of $\int_{-3}^{5} f(x)\,dx$ if the function f is continuous and odd on the interval $[-3, 5]$, $\int_{3}^{5} f(x)\,dx = 9$?

2 what is the value of $\int_{-4}^{-2} f(x)\,dx$ if the function f is continuous and even on the interval $[-4, 4]$, $\int_{-4}^{4} f(x)\,dx = 20$ and $\int_{0}^{2} f(x)\,dx = 6$?

Exercises 3 - 3

Choose the correct answer

1) If $\int_{-2}^{3} f(x)\,dx = 12$, $\int_{-2}^{5} f(x)\,dx = 16$, then $\int_{3}^{5} f(x)\,dx$ equals:

a) -28 b) -4 c) 4 d) 28

2) If $f(x) = |x|$, then $\int_{-2}^{2} f(x)\,dx$ equals:

a) -1 b) zero c) 2 d) 4

Find the value for each of :

3) $\int_{-1}^{3} x^3\,dx$

4) $\int_{1}^{3} (3x^2 - 2)\,dx$

5) $\int_{0}^{4} (2x + 1)^{\frac{3}{2}}\,dx$

6) $\int_{4}^{0} \frac{dt}{\sqrt{8 - t}}$

7) $\int_{0}^{2} x(x^2 - 3)^2\,dx$

8) $\int_{0}^{2} x^2 \sqrt[3]{x^3 + 1}\,dx$

9) $\int_{-1}^{3} |x - 1|\,dx$

10) $\int_{-4}^{0} x(x + 4)^3\,dx$

11) $\int_{10}^{3} x\sqrt{x + 1}\,dx$

12) $\int_{0}^{1} 2\sin \pi z\,dz$

13) $\int_{0}^{\frac{\pi}{4}} \tan z \sec^2 z\,dz$

14) $\int_{0}^{3} (2x - 7e^x)\,dx$

3 - 3 The definite integral

Answer the following :

15. If $\int_1^5 f(x)\,dx = 10$ and $\int_1^5 g(x)\,dx = -2$, calculate the value of

   a. $\int_1^5 [f(x) + g(x)]\,dx$ b. $\int_1^5 [f(x) - g(x)]\,dx$ c. $\int_5^1 3\,g(x)\,dx$

16. If function f is continuous on the interval [-4 , 4] and $\int_0^4 f(x)\,dx = 3$, find the value of

   a. $\int_0^4 [f(x) + 2]\,dx$ b. $\int_{-4}^4 f(x)\,dx$, f odd c. $\int_{-4}^4 f(x)\,dx$, f even

17. If $f(x) = \begin{cases} 2 & \text{when } x < 2 \\ x & \text{when } x \leqslant 2 \end{cases}$, find $\int_0^6 f(x)\,dx$

3 - 4 Application definite integral

Think and discuss

First: Area in a plane

1. Calculate the colored area in each figure of the following geometric figures.

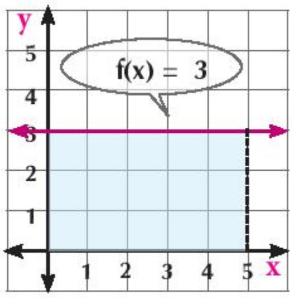


figure (1)

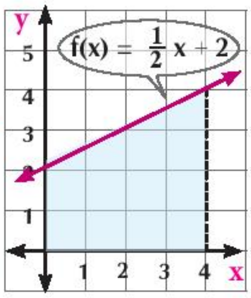


figure (2)

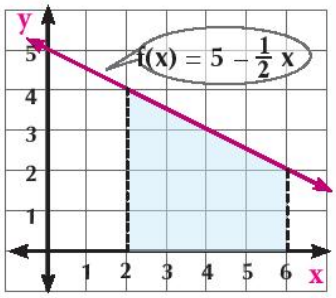


figure (3)

2. In each of the figures above, calculate $\int_a^b f(x)\,dx$ where f(x) is the equation of the curve and the two straight lines x = a , x = b bound the colored area.

3. Compare the area in each figure and the result of its definite integration. What do you infer?

The area of a region bounded by the curve of the function f and x-axis in the interval [a , b]

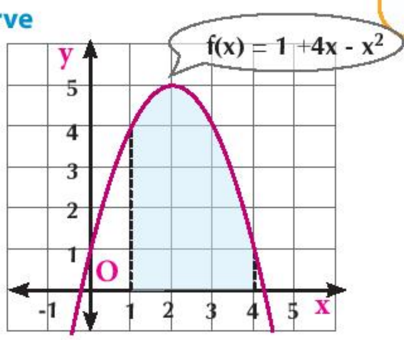
Theorem If function f is continuous on the interval [a, b] and $f(x) > 0$ in this interval, A is the area of the region bounded by the curve of the function f, x-axis and the two straight lines x = a , x = b, then:

$$A = \int_a^b |f(x)|\,dx$$

Example

The area under the curve

(1) The opposite figure shows the curve of the function f where $f(x) = 1 + 4x - x^2$. Find the area of the region bounded by the curve of the function , x-axis and the two straight lines x = 1, x = 4



You will learn

- Identify the area as a definite integration
- Find the area bounded by the curve of a function and x-axis on a closed interval
- Find a revolution volume generated by revolving a region bounded by the curve of the function and the x- axis

Key terms

- Area
- Square unit
- Axis of Revolution
- Solid of Revolution

Materials

- Scientific calculator.
- Computer graphics.

Solution

f is continuous on the interval [1, 4] and f(x) > 0 for each x ∈ [1, 4]

$\therefore A = \int_1^4 f(x)\,dx \qquad = \int_1^4 (1 + 4x - x^2)\,dx$

$= [x + 2x^2 - \frac{1}{3}x^3]_1^4 \qquad = [4 + 32 - \frac{64}{3}] - [1 + 2 - \frac{1}{3}]$

$= 36 - \frac{64}{3} - 3 + \frac{1}{3} = 12$ square unit

Try to solve

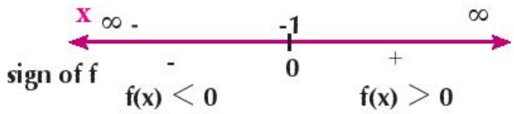
1) Find the area of the region bounded by the curve of function f, x-axis and the two striaight lines x = -1 and x = 2 where $f(x) = 3x^2 + 1$

Example The area under the curve and above e x-axis

2) Find the area of the region bounded by the curve of the function f: $f(x) = \sqrt[3]{2x+2}$ and the straight line x = 3 and on the x-axis.

Solution

Find the zeroes of the function by putting f(x) = 0



$\therefore \sqrt[3]{2x+2} = 0 \qquad$ i.e. : x = -1

$\therefore$ the required area $A = \int_{-1}^3 f(x)\,dx$

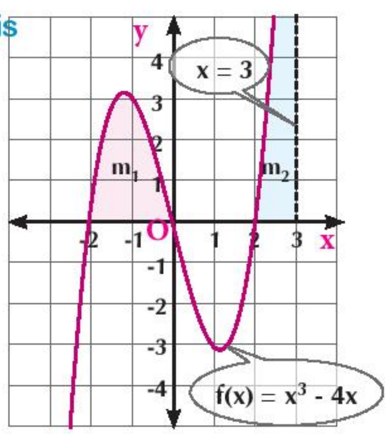
$= \int_{-1}^3 (2x+2)^{\frac{1}{3}}\,dx = [\frac{3}{4 \times 2}(2x+2)^{\frac{4}{3}}]_{-1}^3$

$= \frac{3}{8}[8^{\frac{4}{3}} - 0] = \frac{3}{8}(2^3)^{\frac{4}{3}} = 6$ square units

Try to solve

2) Find the area of the region above x-axis bounded by the curve of the function f: $f(x) = \frac{4x}{x^2+1}$ and the straight line x = 4 .

Example The area bounded by a curve and x-axis



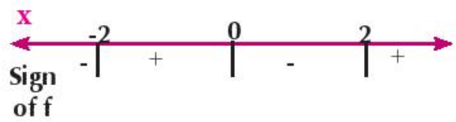
3) If f:]-∞, 3] → R where $f(x) = x^3 - 4x$, find the area of the region above x-axis bounded by the curve of the function and x-axis.

Solution

Find the intersection points of the curve of the function with x-axis (zeroes of the function)

$f(x) = x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2)$

when f(x) = 0 $\therefore$ x = 0 or x = 2 or x = -2



By studying the sign of the function f, we find

f(x) > 0 on the interval [-2, 0] and on the interval [2, 3]

$\therefore$ Area A = $A_1 + A_2$ $= {}_{-2}\int^{0} (x^3 + 4x)\,dx + {}_{2}\int^{3} (x3 - 4x)\,dx$

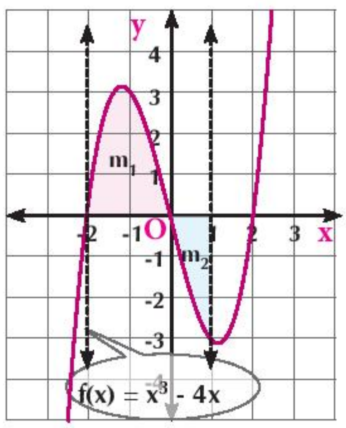
$= \frac{1}{4}x^4 - 2x2]^{0}_{-2} + [\frac{1}{4}x^4 - 2x^2]^{3}_{2}$

$= (0 - (-4)) + (\frac{81}{4} - 18 - (-4))$

$= \frac{121}{4}$ square units

Important note

To identify the area bounded by the curve of the function, x-axis and the two straight lines x = -2 and x = 1 as shown in the graph opposite.



We find that:

f(x) > 0 when x ∈ [-2, 0], f(x) ⩽ 0 when x ∈ [0, 1]

$\therefore$ Area A $= A_1 + A_2$

$= {}_{-2}\int^{0} (x^3 - 4x)\,dx + |\,{}_{0}\int^{1} (x^3 - 4x)\,dx\,|$

$= [\frac{1}{4}x^4 - 2x^2]^{0}_{-2} + |\frac{1}{4}x^4 - 2x^2]^{1}_{0}|$

$= 4 + |(\frac{1}{4} - 2) - 0| = 4 + |-\frac{7}{4}| = \frac{23}{4}$ square units.

Try to solve

(3) Find the area of the plane region bounded by the curve $y = 3 + 2x - x^2$ and x-axis.

Critical thinking

Find the area of the plane region bounded by the curve $y = 3 + 2x - x^2$ and the two straight lines x = -1, x = 4 and y = 0

Example

Architectural applications of the area

(4) An architect has designed an arc -like enteryway of a hotel whose equation $y = -\frac{1}{2}(x - 1)(x - 7)$ where x in meters. How much does the glass cost if this entryway is covered by the glass which costs 1500 LE per square meter?

Solution

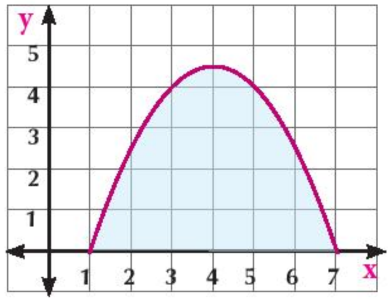
Modeling the problem:

The cost of the glass of the hotel entryway = area of the glass in square meters × cost of a square meter

Let the total cost be k and the area of the glass be A square meter

$\therefore \quad k = 1500\,A$ ①

Finding the area of the glass:



Suppose the horizontal plane is the x-axis whose equation y = 0 and the equation of the arc of the hotel entryway y = f(x) where:

$f(x) = -\frac{1}{2}(x - 1)(x - 7)$

$\therefore$ When f(x) = 0 then: x = 1 or x = 7

Then $f(x) \geq 0$ for each $x \in [1, 7]$

$$\text{Area } A = \int_1^7 -\frac{1}{2}(x - 1)(x - 7)\,dx = \int_1^7 \left(-\frac{1}{2}x^2 + 4x - \frac{7}{2}\right)dx$$

$$= \left[-\frac{1}{6}x^3 + 2x^2 - \frac{7}{2}x\right]_1^7 = \frac{49}{3} - \left(-\frac{5}{3}\right) = 18$$ ② **from ①, ②**

$\therefore k = 1500 \times 18 = 27000$

i.e.: The cost of covering the hotel entryway by glass equals 27000 LE

Try to solve

① If the cost of a squared meter of granite to cover the floor of a hotel corridors is 400 LE and five corridors have been already covered with granite and the area of each is bounded by the curve of the function f and the two straight lines x = 0 and y = 0 where $f(x) = 12 - \frac{1}{3}x^2$, **find the cost of covering the five corridors**.

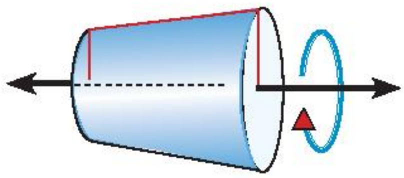
Second: Volumes of revolution solids

Think and discuss

Have you ever watched the pottery maker while he changes the dust into cooking utensils by blending the Aswani mud with water, cutting it and putting it around an axis of revolution. The pottery maker can shape this mud using his fingers and tools to produce very attractive solids. What are these solids named?

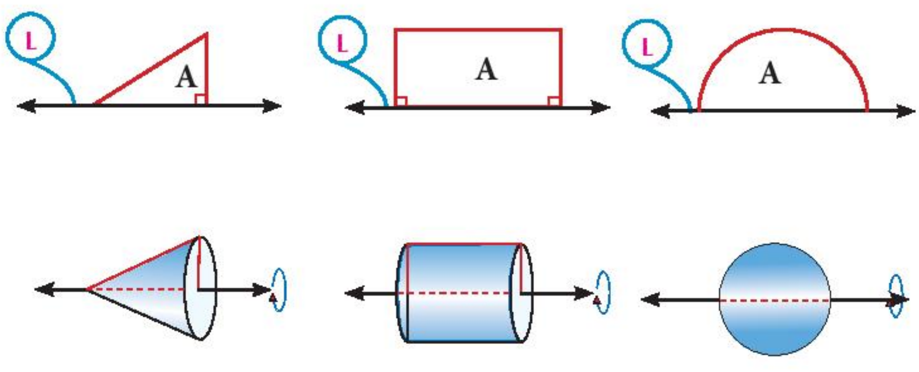
- Plastic containers for packing soft drinks, juice and oil are designed in different volumes and in several capacities. How can their volumes and capacities be calculated as they are being manufactured?

Solid of Revolution



The solid of revolution is generated due to revolving a plane area a complete revolution about a fixed straight line in its plane called «axis of revolution».

The following figures shows some examples for solids of revolution drawn by the area A as they revalve a complete revolution about the straight line **L**



Right cone | Right cylinder | sphere

Volumes of revolution

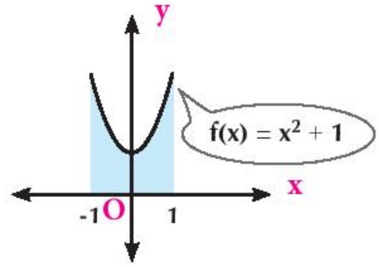
The volume of the solid generated by revolving a plane region about x - axis.

Theorem

If f is a continuous function on the interval [a , b] , $f(x) > 0$ for $x \in [a , b]$, then the solid generated by revolving the area bounded by the curve $y = f(x)$, x-axis and the two straight lines $x = a$ and $x = b$ a complete revolution about x-axis is: $V = \pi \int_a^b [f(x)]^2 \, dx$

Example Revolution about x-axis

5) Find the volume of the solid generated by revolving the plane region bounded by the curve of the function f , x-axis and the two straight lines $x = -1$ and $x = 1$ a complete revolution about x-axis known that $f(x) = x^2 + 1$

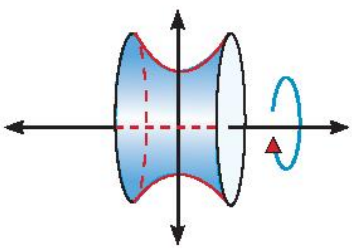


Solution:

The function f is polynomial and continuous on the interval [-1, 1],

$f(x) > 0$ for each $x \in [a, b]$

Let the volume of the solid generated by revolving = V



$$\therefore V = \pi \int_{-1}^{1} (x^2 + 1)^2 \, dx$$

$$= \pi \int_{-1}^{1} (x^4 + 2x^2 + 1) \, dx$$

$$= \pi \left[\frac{1}{5} x^5 + \frac{2}{3} x^3 + x\right]_{-1}^{1} = \frac{56}{15} \pi \text{ cubic unit}$$

Try to solve

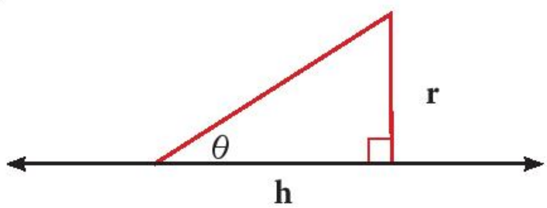
1) Find the volume of the solid generated by revolving the plane region bounded by the curve of function f , x-axis and the two straight line $x = 0$ and $x = 3$ a complete revolution about x-axis given that $f(x) = x$. what is the name of the generated solid? Show how you can check your answer geometrically.

Example Applications of volumes

6) Use the integration to prove that the volume of the right circular cone equals $\frac{\pi}{3} r^2 h$ where r is the radius length of its base and h is its height.

Solution:

The right circular cone is generated by revolving the right - angled triangle such that one leg of the right angle lies on the x-axis a complete revolution about x-axis.

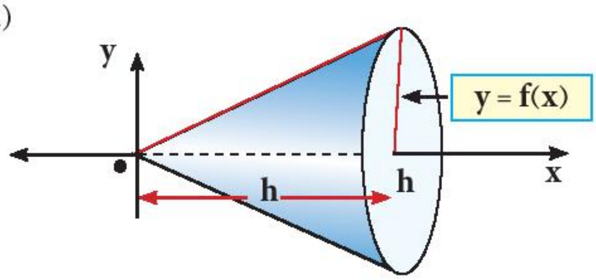


Find the relation between x and y = f(x)

$\tan\theta = \frac{y}{x}$ (1) $\therefore y = x\tan\theta = f(x)$

$\because V = \pi \int_a^b [f(x)^2]\, dx = \pi \int_0^h x^2 \tan^2\theta\, dx$

$= [\frac{\pi}{3} x^3 \tan^2\theta]_0^h = \frac{\pi}{3} h^3 \tan^2\theta$ (2)



from (1) $\tan\theta = \frac{y}{x} = \frac{r}{h}$

$\therefore \tan^2\theta = \frac{r^2}{h^2}$

By substituting in (2) $\therefore V = \frac{\pi}{3} \frac{r^2}{h^2} \times h^3 = \frac{\pi}{3} r^2 h$

Try to solve

(2) Use the integration to prove that:

(a) Volume of sphere = $\frac{4}{3}\pi r^3$
(r is the radius length of the sphere)

(b) Volume of the right circular cylinder = $\pi r^2 h$
(r is the radius length of the cylinder base and h is its height)

Remeber

The equation of the circle whose center is the origin point (0, 0) and its radius length (r) is:
$x^2 + y^2 = r^2$

Example
Revolution about x-axis

(7) Find the volume of the solid generated by revolving the region bounded by the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and x-axis where a and b are two constants a complete revolution about x-axis.

Solution:

$\because$ Revolution around x-axis $\therefore y^2 = b^2(1 - \frac{x^2}{a^2})$

Limits of integration:

$y = 0$ $\therefore x^2 = a^2$ **i.e** $x = -a, x = a$

$V = \pi \int_{-a}^{a} b^2 (1 - \frac{x^2}{a^2})\, dx = 2\pi b^2 \int_0^a (1 - \frac{x^2}{a^2})\, dx$ **(why?)**

$= 2\pi b^2 [x - \frac{x3}{3a^2}]_0^a = 2\pi b^2 [a - \frac{1}{3} a]$

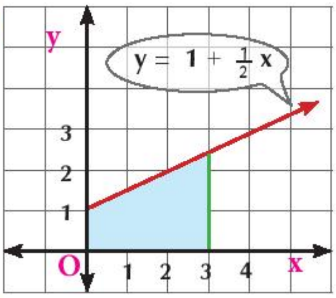
$= \frac{4}{3}\pi b^2 a$ cubic units.

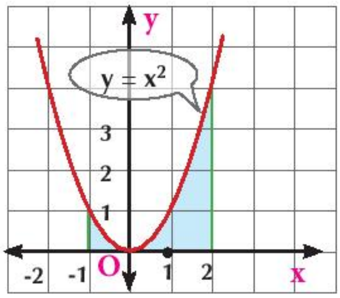
Try to solve

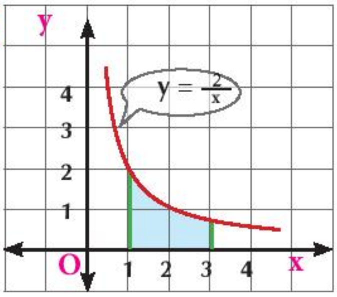
(3) Find the volume of the solid generated by revolving the region bounded by the curve $y = 2x - x^2$ and x-axis a complete revolution about x-axis.

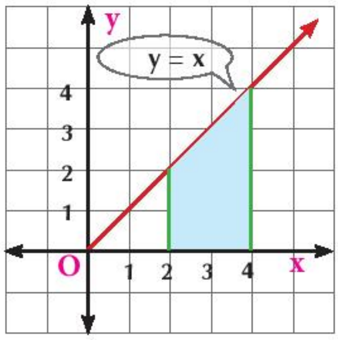
Exercises 3 - 4

Write down the difinite integration which gives the colored area in rach of the following and calculate its value.

1


2


3


4


Choose the correct answer.

5. The area of the region bounded by the straight lines $y = x$, $x = 2$ and $y = 0$: equals:

a) $\frac{1}{2}$ b) 1

c) 2 d) 4

6. The area of the region bounded by the curve $y = x^3$ and the straight lines $y = 0$ and $x = 2$ equals

a) 8 b) 4

c) 2 d) 1

7. The volume of the solid generated by revolving a region bound by the curve $y = 2\sqrt{x}$, $y = 0$ and $x = 1$ a complete revolution about x-axis equals.

a) $-\pi$ b) 0

c) π d) 2π

8. The volume of the solid generated by revolving the region bounded by the curve $y = \sqrt{x+1}$ and the straight lines $y = 0$, $x = -1$ and $x = 1$ equals

a) π b) $\frac{3\pi}{2}$

c) 2π d) $\frac{5\pi}{2}$

For each of the following, calculate the area of the plane region bounded by:

9. The curve $y = 5 - x^2$, x-axis and the two straight lines $x = -2$ and $x = 1$
10. The straight lines: $x + 2y = 9$, $x = 1$, $x = 3$, $y = 0$
11. The curve $y = \sqrt{x+4}$ and the straight lines $x = 0$, $x = 5$, $y = 0$
12. The curve $y = 3 - 2x - x^2$ and x-axis
13. The curve $y = \frac{4}{x^2}$ and the straight lines $x = 1$, $x = 4$, $y = 0$
14. The curve of the function f: $f(x) = (3 - x)(x - 1)^2$ and the two coordinate axes where $f(x) \geqslant 0$
15. The curve of the function f: $f(x) = (x - 1)(x - 2)(x - 3)$ and the two straight lines $x = 4$, $y = 0$ where $f(x) \geqslant 0$

Find the volume of the solid generated by revolving the region bounded by the given curves and straight lines a complete revolution about x-axis for each of the following:

16. $y = x$, $x = 3$, $y = 0$
17. $y = 3 - x$, $x = 0$, $y = 0$
18. $y = \frac{1}{x}$, $x = 1$, $x = 4$, $y = 0$
19. $y = |x|$, $x = -2$, $x = 4$, $y = 0$

Find the volume of the solid generated by revolving the region bounded by the given curves and straight lines a complete revolution about y-axis for each of the following:

20. $y = x$, $y = 1$, $x = 0$
21. $y = x^2$, $x = 0$, $y = 0$, $y = 8$
22. $y = 4 - x^2$, $x = 0$, $y = 0$
23. $y = x^3$, $x = 0$, $y = 0$, $y = 8$
24. $x + 2y = 0$, $x = 0$, $y = 0$, $y = 3$