Pure Mathematics

Second Form Secondary

Student Book — First term

Science Section

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بنك المعرفة المصري

Introduction

بسم الله الرحمن الرحيم

We are pleased to introduce this book to show the philosophy on which the academic content has been prepared. This philosophy aims at:

1 Developing and integrating the knowledgeable unit in Math, combining the concepts and relating all the school mathematical curricula to each other.

2 Providing learners with the data, concepts, and plans to solve problems.

3 Consolidate the national criteria and the educational levels in Egypt through:

A) Determining what the learner should learn and why.

B) Determining the learning outcomes accurately. Outcomes have seriously focused on the following: learning Math remains an endless objective that the learners do their best to learn it all their lifetime. Learners should like to learn Math. Learners are to be able to work individually or in teamwork. Learners should be active, patient, assiduous and innovative. Learners should finally be able to communicate mathematically.

4 Suggesting new methodologies for teaching through (teacher guide).

5 Suggesting various activities that suit the content to help the learner choose the most proper activities for him/her.

6 Considering Math and the human contributions internationally and nationally and identifying the contributions of the achievements of Arab, Muslim and foreign scientists.

In the light of what previously mentioned, the following details have been considered:

★ This book contains three domains: algebra, relations and functions, calculus and trigonometry. The book has been divided into related and integrated units. Each unit has an introduction illustrating the learning outcomes, the unit planning guide, and the related key terms. In addition, the unit is divided into lessons where each lesson shows the objective of learning it through the title You will learn. Each lesson starts with the main idea of the lesson content. It is taken into consideration to introduce the content gradually from easy to hard. The lesson includes some activities, which relate Math to other school subjects and the practical life. These activities suit the students' different abilities, consider the individual differences throughout Discover the error to correct some common mistakes of the students, confirm the principle of working together and integrate with the topic. Furthermore, this book contains some issues related to the surrounding environment and how to deal with.

★ Each lesson contains examples starting gradually from easy to hard and containing various levels of thoughts accompanied with some exercises titled Try to solve. Each lesson ends in Exercises that contain various problems related to the concepts and skills that the students learned through the lesson.

Last but not least. We wish we had done our best to accomplish this work for the benefits of our dear youngsters and our dearest Egypt.

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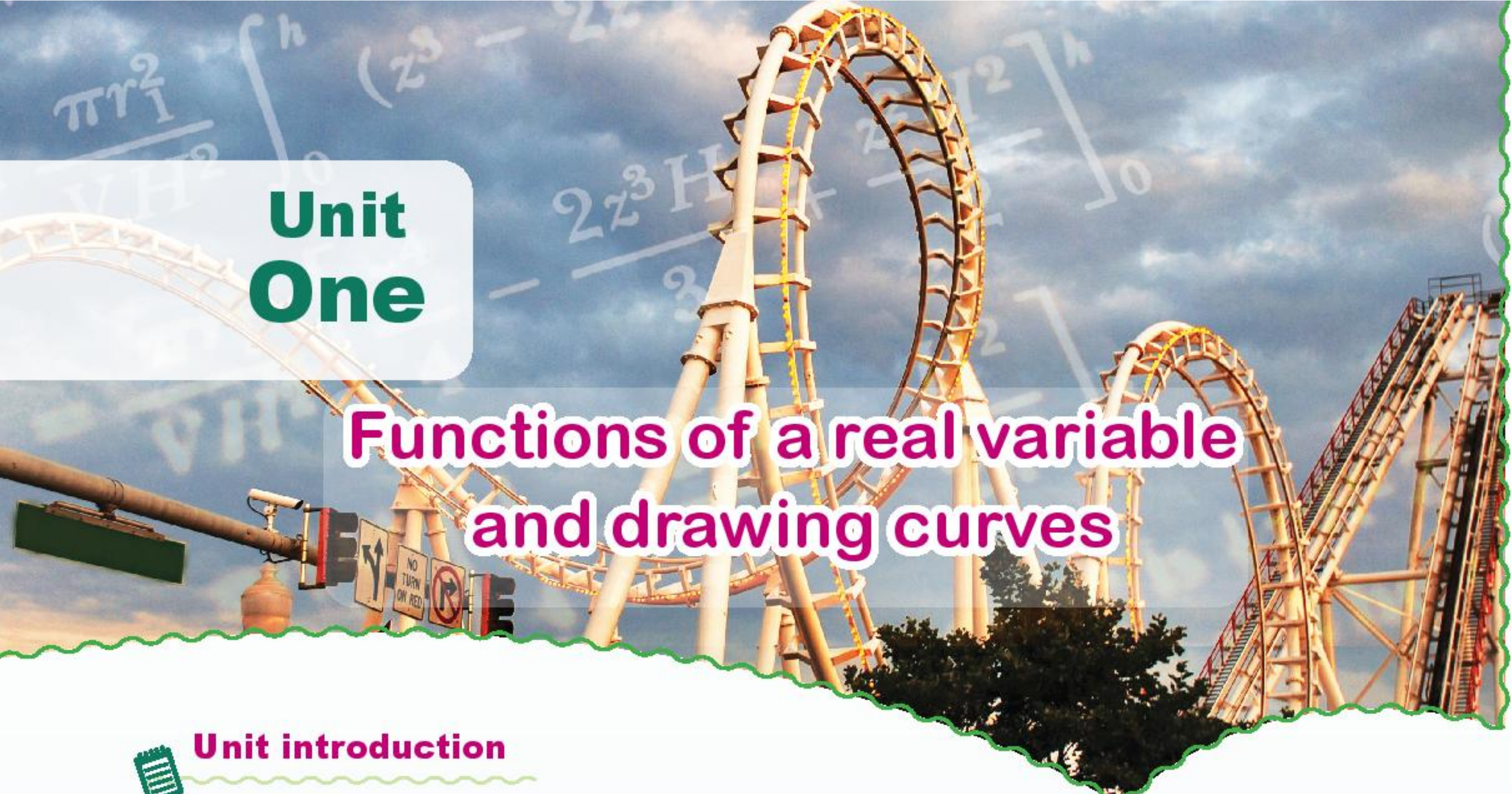


Unit One

Functions of a real variable and drawing curves

Unit introduction

The swiss scientist Leonard Euler (1707 - 1783) is considered one of the most prominent of the eighteenth century in mathematics and physics. He had been credited with using the symbol $y = f(x)$ to express the function. He had considered that the function is a correlation between the elements of two sets with a relation that allows to calculate a variable value of dependent variable Y for another independent X which we choose freely. In such a way, he identified the function but not the curve. This contributed in converting the geometry into arithmetic relations. He had converted all the trigonometric ratios which ancient Egyptians, Babylonians and Arabs had excelled into trigonometric functions. Leonard Euler had inserted the constant number $e \simeq 2.71828$ (Euler's number) as the base of the natural logarithm. Furthermore, he discovered the mathematical relation $e^{i\pi} + 1 = 0$ relating among the most important five constants in Mathematics. He had also related among the trigonometric functions, exponential functions and the composite numbers. In this unit, you are going to learn different forms of the real functions, their behaviour and their graphical representation using the geometrical transformations and graphical programs and to use the real functions in solving life and mathematical problems in different fields.

Unit objectives

By the end of this unit, the student should be able to:

- Identify the concept of the real function.
- Determine the domain, co-domain and range of the real functions.
- Identify a simplified idea about the operations on the real functions (compositions of functions).
- Identify some properties of the real functions.
- Identify the even and odd functions and differentiate between them.
- Identify the one-to-one function.
- Deduce the monotony of the real functions(increasing, decreasing and constant functions).
- Identify polynomial functions.
- Graph the curves of (quadratic function - modulus functions - cubic function - rational function) and deduce the properties of each.
- Deduce the effect of the following transformations: $f(x \pm a) \pm b$ and $a\, f(x \pm b) \pm c$ on the previous functions.
- Apply the previous transformation on graphing the curves of the real functions.
- Solve equations in the form of : $|ax + b| = c$, $|ax + b| = |dx + c|$, $|ax + b| = cx + d$.
- Solve inequalities in the form of: $|ax + b| < c$ and $|ax + b| \leqslant c$, $|ax + b| > c$ and $|ax + b| \geqslant c$
- Use the real functions to solve math and life problems in different fields.
- Relate what they learned about the effect of the previous transformations on the trigonometric functions in the form of activities.
- Investigate the graphical representation of the real functions which have been previously learned and the effect of the previous transformation using the "Geogebra" program.
- Use the graphical calculator to represent some functions that are hard to be represented by the common methods, then learn the properties of these functions.

Key terms

- Real Function
- Domain
- Co-domain
- Range
- Vertical Line
- Piecewise-Defind *Function*
- Composite *Function*
- Even *Function*
- Odd *Function*
- One-to-One *Function*
- Monotony *of* a *Function*
- Increasing *Function*
- Decreasing *Function*
- Constant *Function*
- polynomial *Function*
- Absolute Value *Function*
- Rational *Function*
- Asymptote
- Trans*f*ormation
- Translation
- Reflection
- Stretching
- Graphical Solution

Lessons of the unit

Lesson (1 - 1): The real functions.

Lesson (1 - 2): Some properties of functions.

Lesson (1 - 3): Monotony of function

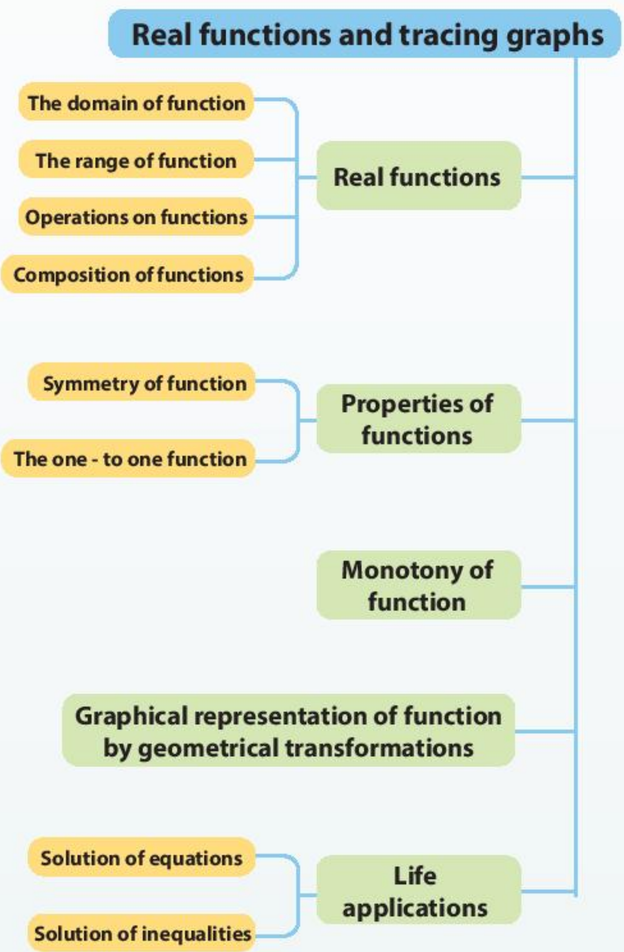
Lesson (1 - 4): Graphical representation of functions and geometrical transformations.

Lesson (1 - 5): Solving Absolute Value Equations and Inequalities.

Materials

Materials Scientific calculator, Computer (Graph, GeoGebra)

Chart of the unit



1 - 1 Real Functions

We will learn
- Concept of real function.
- The vertical line test.
- The function of more than one rule.
- Identify the domain and range of real function.
- Operations on functions.

Key - term
- Function
- Domain
- Co-domain
- Range
- Arrow Diagram
- Cartesian Diagram
- Vertical Line
- Piecewise Function

Matrials
- Computer program for graph
- Scientific calculator.

Remember
If: X ⟶ Y then f =
$\{(x, y): x \in X, y \in Y, y = f(x)\}$

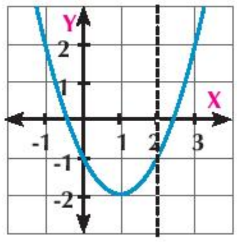
Explore

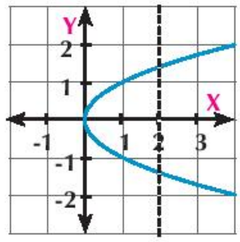
Definition 1

Real Function
A function f is called a real function if each of its domain and co-domain is the set of real numbers $\mathbb{R}$ or subset of it .

Learn

The vertical line test
If the vertical line at each element of the domain passes through only one point of the points representing the relation, then the relation is a function from X ⟶ Y


Function


Not function

Example Identify the Relations Representing a Function

1. In each of the following graphs, show whether y represents a function in x or not.

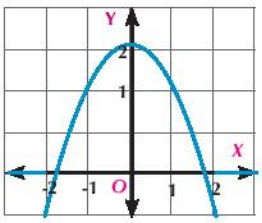


Fig (1)

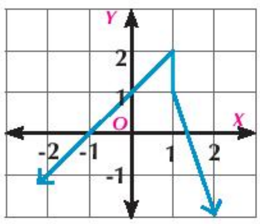


Fig (2)

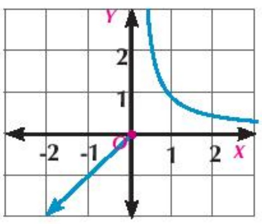


Fig (3)

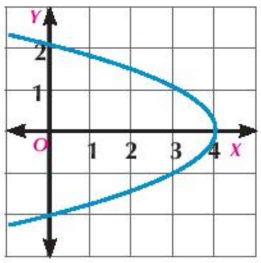


Fig (4)

Solution

Fig (1) represents a function.

Fig (2) doesn't represent a function because the vertical line passing through the point (1, 0) intersects the curve at infinite number of points.

Fig (3) represents a function.

Fig (4) doesn't represent a function because there is a vertical line intersects the curve at more than a point.

Try to solve

1) Show which of the following relations represent a function from X $\longrightarrow$ Y and give reason.

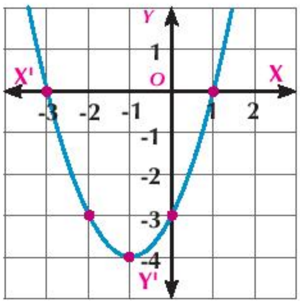


Fig (1)

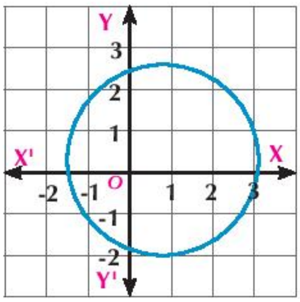


Fig (2)

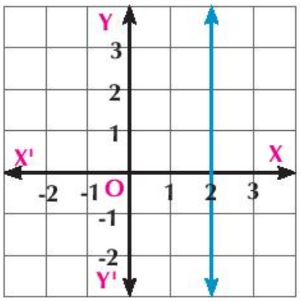


Fig (3)

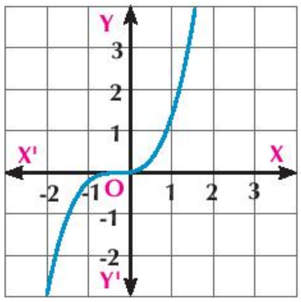


Fig (4)

Example Identifying the domain and the range

2) If $f: [1, 5] \longrightarrow \mathbb{R}$ where $f(x) = x + 1$

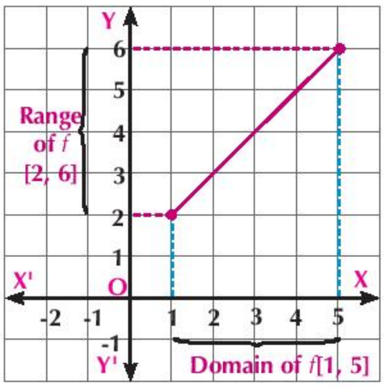
Graph the function f and deduce the range of this function from the graph.

Solution

The function f is linear and its domain is [1, 5]. It is graphically represented by a line segment whose two ends are $(1, f(1))$, $(5, f(5))$. i.e. the two points (1, 2) and (5, 6).

The range of function $f = [2, 6]$

Which is the y-coordinates for all points in the domain of f.



Learn

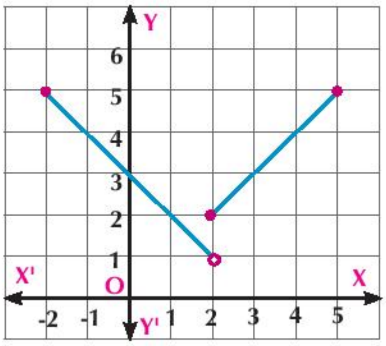
The piecewise - defined function is a real function in which each subset of its domain has a different definition rule.

Graphing the piecewise – defined function:

Example

3 If $f(x) = \begin{cases} 3 - x & \text{when} \quad -2 \leqslant x < 2 \\ x & \text{when} \quad 2 \leqslant x \leqslant 5 \end{cases}$

Graph of the function f and from the graph deduce the domain and range.



Solution

The function f is defined over two intervals and $f(x)$ is defined by two rules:

The first rule:

$f_1(x) = 3 - x$ when $-2 \leqslant x < 2$ (i.e. on interval $[-2, 2[$)

It is a linear function represented by a line segment whose two ends are the points (-2 , 5) and (2 , 1) with open circle at point (2, 1) because $2 \notin [-2, 2[$ as shown in the opposite figure.

The second rule:

$f_2(x) = x$ when $2 \leqslant x \leqslant 5$ (i.e. on interval $[2, 5]$)

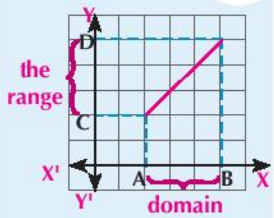
It is a linear function represented by a line segment whose two ends are points (2, 2) and (5, 5). **The domain of the function** $f = [-2, 2[\cup [2, 5] = [-2, 5]$

From the graph, we deduce:

The domain of the function $f = [-2, 5]$

The range of the function $f =]1, 5]$

Notice



In the graph representing the function, the domain of the function = [a , b]
the range of the function = [c , d]

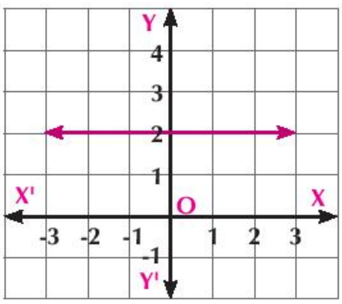
Try to solve

2 If $f(x) = \begin{cases} x - 1 & \text{when} \quad -2 \leqslant x < 0 \\ x + 1 & \text{when} \quad x > 0 \end{cases}$

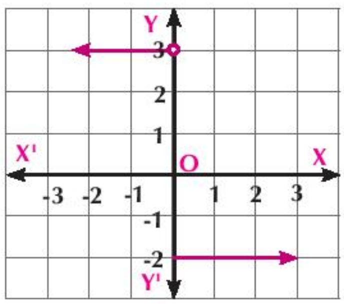
Graph the function f and from the graph, deduce the domain and the range of the function.

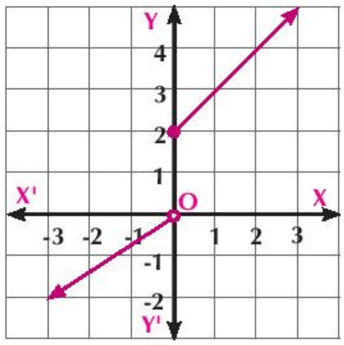
3 For each of the following graphs, deduce the domain and the range of the function.

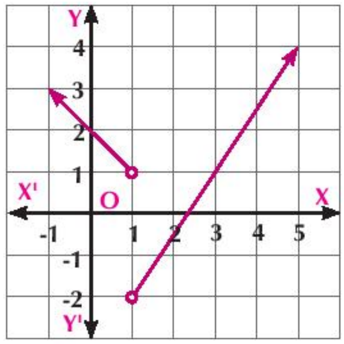
a



b



c


d


Identifying the Domain of the Real Functions and Operations on them

The domain of the function is determined from its graph or from its definition rule.

Example Identifying Domains of the function

4 Determine the domain of each real functions defined by the following rules:

a) $f_1(x) = \dfrac{x+3}{x^2 - 9}$ b) $f_2(x) = \sqrt{x - 3}$

c) $f_3(x) = \sqrt[3]{x - 5}$

> **Remember**
> **The domain of polynomial function is the set of real numbers unless it is defined on a subset of it.**

Solution

a) The function f_1 is not defined when the denominator = 0, so we put $x^2 - 9 = 0$ i.e. $x = \pm 3$ then the domain of the function f_1 is $\mathbb{R} - \{-3, 3\}$.

b) The domain of the function f_2 is all the values of x which make the quantity under the square root is non - negative, i.e. the values of x which make $x - 3 \geq 0$.

$\therefore x - 3 \geq 0$ $\therefore x \geq 3$ $\therefore$ the domain of $f_2 = [3, \infty[$.

c) The domain of $f_3(x) = \sqrt[3]{x - 5}$ is $\mathbb{R}$ because the index of the root is an odd number.

Notice:

If $f(x) = \sqrt[n]{g(x)}$ $\in Z^+$, $n > 1$ and g (x) is polynomial

First: If n is an odd number, then the domain of the function f is $\mathbb{R}$.

Second: If n is an even number, then the domain of the function f is the values of x which satisfy $g(x) \geq 0$

Try to solve

4 Determine the domain of each of the real functions defined by the following rules:-

a) $f_1(x) = \dfrac{2x+3}{x^2 - 3x + 2}$ b) $f_2(x) = \sqrt{X^2 - 16}$ c) $f_3(x) = \sqrt[3]{x - 5}$

Critical thinking:

If the domain of the function f where $f(x) = \dfrac{2}{x^2 - 6x + k}$ is $\mathbb{R} - \{3\}$, find the value of k.

Learn — Operations on Functions

If f_1 and f_2 are two functions whose domains are D_1 and **D_2** respectively, then **:**

1 $(f_1 \pm f_2)(x) = f_1(x) \pm f_2(x)$, the domain of $(f_1 \pm f_2)$ is $D_1 \cap D_2$

2 $(f_1 . f_2)(x) = f_1(x) . f_2(x)$, the domain of $(f_1 . f_2)$ is $D_1 \cap D_2$

3 $(\frac{f_1}{f_2})(x) = \frac{f_1(x)}{f_2(x)}$, $f_2(x) \neq 0$ the domain of $(\frac{f_1}{f_2})$ is $(D_1 \cap D_2) - Z(f_2)$

where $Z(f_2)$ is the set of zeros of f_2

We notice that, for all previous cases, the domain of the new function equals the intersection of the two domains f_1 and f_2 except the values which make $f_2(x) = 0$ in the division operation.

Example

5 If $f(x) = x^2 - 4x$, $g(x) = \sqrt{x + 2}$, $z(x) = \sqrt{4 - x}$,

first: Find the rule and the domain for each of the following functions:

a $(f + g)$ **b** $(g - z)$

c $(f . z)$ **d** $(\frac{z}{f})$

second: Evaluate the numerical value (if possible) for each of the following :

a $(g - z)(1)$ **b** $(f . z)(5)$ **c** $(\frac{z}{f})(3)$

Solution

first: The domain of $f = D_1 = \mathbb{R}$, the domain of $g = D_2 = [-2, \infty[$

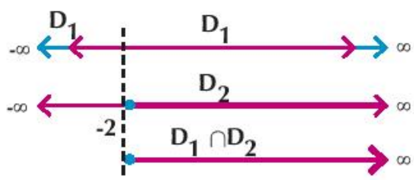
and the domain of $z = D_3 =]-\infty , 4]$

a $(f + g)(x) = f(x) + g(x)$

$= x^2 - 4x + \sqrt{x + 2}$

The domain of the function $(f + g)$ is

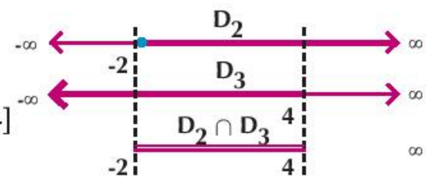
$\mathbb{R} \cap [-2,\infty[= [-2 ,\infty[$



b $(g - z)(x) = g(x) - z(x)$

$= \sqrt{x + 2} - \sqrt{4 - x}$

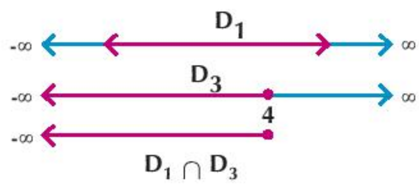
The domain of $(g - z)(x) = [-2 ,\infty[\cap]-\infty, 4] = [-2, 4]$

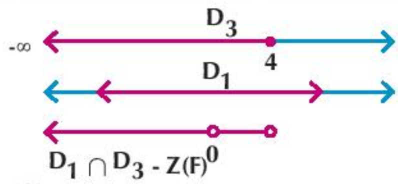


c $(f . z)(x) = f(x).z(x)$

$= (x^2 - 4x)\sqrt{4 - x}$

The domain of $(f . z) = \mathbb{R} \cap]-\infty , 4] =]-\infty , 4]$





d $(\frac{z}{f})(x) = \frac{z(x)}{f(x)} = \frac{\sqrt{4 - x}}{x^2 - 4x}$

The set of zeros of the function f is $\{0, 4\}$

The domain of $(\frac{z}{d}) =]-\infty, 4] \cap R - \{0, 4\} =]-\infty, 4[- \{0\}$

Second : Numerical values:

a $\because (g - z)(x) = \sqrt{x + 2} - \sqrt{4 - x}$ for all $x \in [-2, 4]$

$, 1 \in [-2, 4]$ $\quad\quad \therefore (g - z)(1) = \sqrt{3} - \sqrt{3} = 0$

b $\because (f . z)(x) = (x^2 - 4x)\sqrt{4 - x}$ for all $x \in]-\infty, 4]$

$, 5 \notin]-\infty, 4]$ $\quad\quad \therefore (f . z)(5)$ **not defined**

c $\because (\frac{z}{f})(x) = \frac{\sqrt{4 - x}}{x^2 - 4x}$ for all $x \in]-\infty, 4[- \{0\}$

$, 3 \in]-\infty, 4[- \{0\}$ $\quad\quad \therefore (\frac{z}{f})(3) = \frac{\sqrt{4 - 3}}{9 - 12} = -\frac{1}{3}$

Try to solve

5 If f and g are two real functions, where:
$f(x) = x^2 - 4$, $g(x) = \sqrt{x - 1}$ **find:**

a The domain for each of the functions: $(f + g)$, $(f . g)$, $(\frac{f}{g})$, $(\frac{g}{f})$

b The numerical value for each (if possible):
$(f + g)(5)$, $(f . g)(2)$, $(\frac{f}{g})(3)$, $(\frac{g}{f})(-2)$

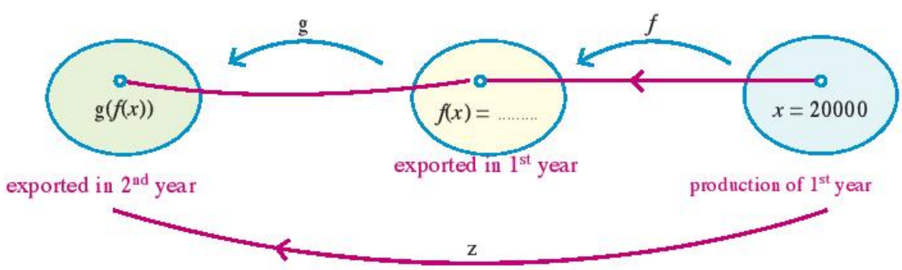
Co-operative learn

Composition of Functions

A factory exports a part of its production. This parts is given by the relation $f(x) = \frac{1}{4}x$ where x is the number of produced units in the first year. and the number of exported units in the next year is given by the relation $g(f) = f + 1500$ where x is the number of exported units in the first year **search (with a classmate)** how many units exported in the second year if the production of the factory in the first year is:

a 20000 units $\quad\quad$ **b** 80000 units

Check your results using the following diagram:



Learn

If the intersection of the range of the function f and the domain of the function g $\neq \varnothing$ then we can get a new function z composed of the two previous functions $z = g \circ f$

it is read as g composed f or g after f where the function f is applied first then the function g.

thus $z(x) = (g \circ f)(x)$

$= g(f(x))$

from the previous diagram, we find :

a) $z(20000) = g[f(20000)]$

$= g(5000)$

$= 5000 + 1500 = 6500$ units

b) $z(80000) =$ ....

Notice

$f(20000) = \frac{1}{4} \times 20000$

$= 5000$

Think: Is the composition of functions a commutative operation?

➢ to search for the answer, find $(f \circ g)(x)$, $(g \circ f)(x)$ Where $f(x) = 4x^2$, $g(x) = 2x$

Try to solve

6) if $f(x) = x^2 + 6$, $g(x) = 3x$

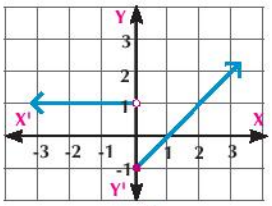
First: find $(f \circ g)(3)$

Second: Determine the values of x which make $(f \circ g)(x) = 42$

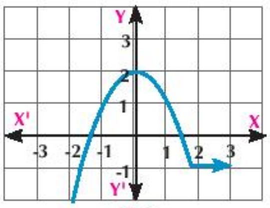
Exercises (1 - 1)

Choose the right answer:

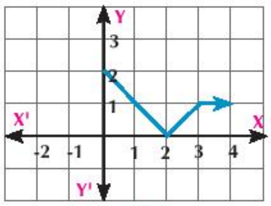
1) The relation shown in the following graphs and doesnot represent a function is:



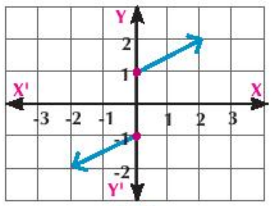
(a)



(b)



(c)



(d)

2) In all the following relations y is a function of x except:

a) $y = 3x + 1$

b) $y = x^2 - 4$

c) $x = y^2 - 2$

d) $y = \sin x$

Answer the following:

3) Determine the domain of the function f where $f(x) = \begin{cases} x - 1 & \text{when } 2 < x \leqslant 4 \\ -1 & \text{when } -2 \leqslant x \leqslant 2 \end{cases}$

then graph the function and deduce its range from the graph.

4) Graph the function f where :

$$f(x) = \begin{cases} x + 3 & \text{when } x > 2 \\ 2x - 1 & \text{when } x < 2 \end{cases}$$ and from the graph, deduce its range.

5) If $f(x) = \begin{cases} 2x + 3 & \text{when } -2 \leqslant x < 0 \\ 1 - x & \text{when } 0 \leqslant x \leqslant 4 \end{cases}$

Graph the function f and deduce its range from the graph.

6) If $f(x) = \begin{cases} -4x + 3 & \text{when } x < 3 \\ -x^3 & \text{when } 3 \leqslant x \leqslant 8 \\ 3x^2 + 1 & \text{when } x > 8 \end{cases}$

Find:

a) $f(2)$ b) $f(3)$ c) $f(10)$

7) **Mechanics:** If the velocity v(t) of a motorcycle is given by

$$v(t) = \begin{cases} 8t & \text{when } 0 \leqslant t \leqslant 10 \\ 80 & \text{when } 10 < t < 200 \\ -4t + 880 & \text{when } 200 \leqslant t \leqslant 220 \end{cases}$$

where t is time in second and v is in cm/sec. Find:

a) v(10) b) v(150) c) v(210)

8) **Trade:** The function f, where:

$$f(x) = \begin{cases} \frac{5}{2}x & \text{when } 0 \leqslant x \leqslant 5000 \\ 2x + 2500 & \text{when } 5000 < x \leqslant 15000 \\ \frac{3}{2}x + 10000 & \text{when } 15000 < x \leqslant 60000 \end{cases}$$

represents the amount of money charged by a company to distribute an electrical appliance in L.E where x represents the number of distributed appliances, find :

a) $f(5000)$ b) $f(10000)$ c) $f(50000)$

9. Determine the domain for each of the real functions defined by the following rules:

a) $f(x) = \frac{x+3}{x^2 - 5x + 6}$

b) $f(x) = \frac{x+1}{x^3 + 1}$

c) $f(x) = \sqrt{x-2}$

d) $f(x) = \sqrt{4 - x^2}$

e) $f(x) = \frac{3x}{\sqrt{2x-1}}$

f) $f(x) = \frac{1}{x} + \frac{1}{x+2}$

10. If f_1: R $\longrightarrow$ R where $f_1(x) = 3x - 1$ and f_2: $[-2, 3] \longrightarrow \mathbb{R}$ where $f_2(x) = 2x + 4$

find : $(f_1 + f_2)(x)$, $(f_1 - f_2)(x)$ and deduce the domain of each function.

11. If $f_1(x) = x + 2$ and the domain of $f_1 = [-3, 4]$, $f_2(x) = x^2 + 2x$ and the domain of $f_2 = [-1, 3]$,

Find : $(f_1 + f_2)(x)$, $(f_2 - f_1)(x)$, $(\frac{f_1}{f_2})(x)$, $(\frac{f_2}{f_1})(x)$ and deduce the domain of each function.

12. If $f(x) = 3x + 1$, $g(x) = x^2 - 5$ and $h(x) = x^3$

Find:

a) $(f \circ g)(2)$

b) $(g \circ f)(-3)$

c) $(g \circ h)(1)$

d) $(h \circ f)(-2)$

13. If $f(x) = \frac{1}{x}$, $g(x) = x + 3$

Find : $(f \circ g)(x)$, $(g \circ f)(x)$ and deduce the domain of each function.

14. If $f(x) = x^2 - 3$, $g(x) = \sqrt{x-2}$

Find: $(f \circ g)(x)$ in the simplest form then find $(f \circ g)(3)$

Some Properties of Functions

Unit one

1 - 2

The graph of the function f where $y = f(x)$ may be characterized by some geometrical properties that can be noticed easily from the graph. These properties can be used in studying the functions and their applications. The most common properties are the symmetry about y-axis or about the origin point.

We will learn

- Symmetry of function curves.
- Even functions
- Odd functions
- One-to- one functions

Key - term

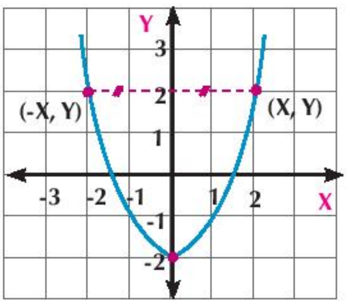
- Symmetry
- Even function
- Odd function
- One - to - one function
- Horizontal line

Matrials

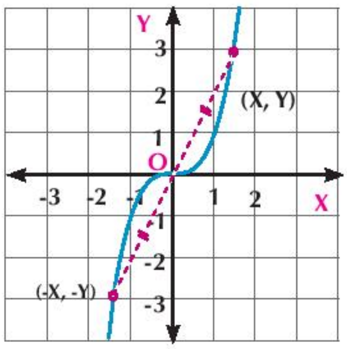
- Scientific calculator
- Computer programs for graph.

Introduction

You have studied the symmetry about a straight line where the curve can be folded about this straight line completely. You have also studied the symmetry about the origin point .



Symmetry about y-axis
figure (1)



Symmetry about origin point
figure (2)

In Figure (1):

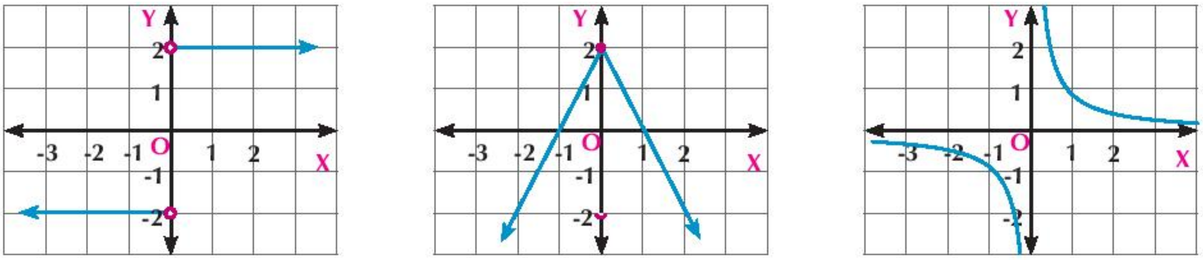
The point $(-x , y)$ lying on the curve is the image of the point (x,y) lying on the curve by reflection in y-axis.

In Figure (2):

The point $(-x , -y)$ lying on the graph of the curve is the image of the point (x, y) lying on the same curve by reflection in origin point.

Try to solve

1. In the following figures, show which curve is symmetric about y-axis and which is symmetric about the origin point.



(A) (B) (C)

Critical thinking:

Are curves of all functions symmetric about y – axis or about the origin point only? Explain.

Even functions and odd functions

Learn

The even function: the function $f: X \longrightarrow Y$ is said to be even if $f(-x) = f(x)$, for all $-x, x \in X$. The curve of the even function is symmetric about y-axis.

The odd function: the function $f: X \longrightarrow Y$ is said to be odd if $f(-x) = -f(x)$ for all $-x, x \in X$. The curve of the odd function is symmetric about the origin point.

Notice : A lot of functions are neither even nor odd

when we investigate whether the function is even or odd, the two elements x, $-x$ must belong to the domain of the function. If this condition is not satisfied, then the function is neither even nor odd without getting $f(-x)$

Example

1) Show the type for each of the following functions (even – odd).

a) $f(x) = x^2$ b) $f(x) = x^3$ c) $f(x) = \sqrt{x+3}$ d) $f(x) = \cos x$

Solution

a) $f(x) = x^2$, domain of $f = \mathbb{R}$

for each x and $-x \in \mathbb{R}$, then $f(-x) = (-x)^2 = x^2$

i.e.: $f(-x) = f(x)$ then f is even function

b) $f(x) = x^3$, domain of $f = \mathbb{R}$

for each x and $-x \in \mathbb{R}$, then: $f(-x) = (-x)^3 = -x^3$

i.e.: $f(-x) = -f(x)$ then f is odd function

Important remark:

The function $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = ax^n$ where $a \neq 0$, $n \in Z^+$ is called the power function. The function is even when n is an even number and it is odd when n is an odd number.

c $f(x) = \sqrt{x+3}$, the domain of $f = [-3, \infty[$

notice $4 \in [-3, \infty[$ while $-4 \notin [-3, \infty[$

then f is neither even nor odd

d $f(x) = \cos x$, the domain of $f = \mathbb{R}$

for each x and $-x \in \mathbb{R}$ then :

$f(-x) = \cos(-x) = \cos x$

then : $f(-x) = f(x)$ then f is an even function

Remember that

$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$
$\tan(-x) = -\tan x$

Try to solve

2) Determine the type for each of the following functions whether even , odd or otherwise.

a $f(x) = \sin x$ **b** $f(x) = x^2 + \cos x$ **c** $f(x) = x^3 - \sin x$

d $f(x) = x^2 \cos x$ **e** $f(x) = x^3 \sin x$ **f** $f(x) = x^3 \cos x$

g $f(x) = x^3 + x^2$ **h** $f(x) = \sin x + \cos x$ **i** $f(x) = \sin x \cos x$

What did you deduce?

important properties :

If each of : f_1 and f_2 is an even function and each of g_1 and g_2 is an odd function, then :

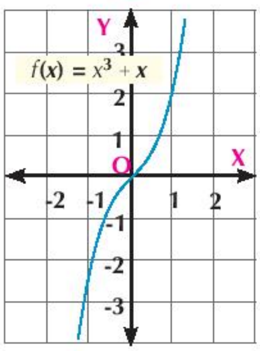
1) $f_1 + f_2$ is an even function
2) $g_1 + g_2$ is an odd function
3) $f_1 \times f_2$ is an even function
4) $g_1 \times g_2$ is an even function
5) $f_1 \times g_2$ is an odd function
6) $f_1 + g_2$ is neither odd nor even

Using these properties, verify your answers in **try to solve** (2)

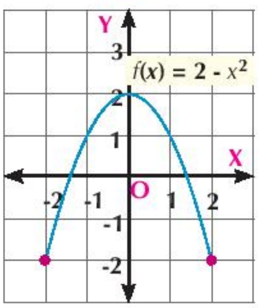
Example

2) Each graph of the following graphs shows the curve of the functions f, Determine which functions is even, odd or otherwise. Verify your answer algebrically.

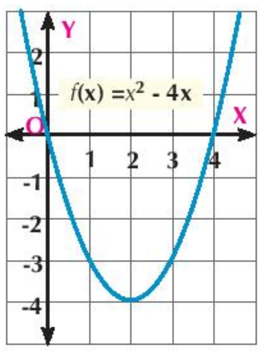
a



b



c



Solution

a From the graph of the function $f(x) = x^3 + x$, we notice that the domain of $f = \mathbb{R}$:
the curve of f is symmetric about the origin point, so the function is odd.

$\because$ for all x , $-x \in \mathbb{R}$ $\therefore f(-x) = (-x)^3 + (-x)$

simplifying : $f(-x) = -x^3 - x$

take off (-1) a common factor $f(-x) = -(x^3 + x)$

$f(-x) = -f(x)$ $\therefore f$ is odd.

b From the graph of the function $f(x) = 2 - x^2$, we notice that the domain of $f = [-2, 2]$
the curve of f is symmetric about y - axis, so the function is even .

$\because$ for all , x , $-x \in [-2, 2]$ $\therefore f(-x) = 2 - (-x)^2$

simplifying $f(-x) = 2 - x^2$

$f(-x) = f(x)$ $\therefore f$ is even

c From the graph of $f(x) = x^2 - 4x$, we notice that the domain of $f = \mathbb{R}$ the curve is neither symmetric about y-axis nor about the origin point, so the function is neither even nor odd:

$\because x , -x \in \mathbb{R}$ $\therefore f(-x) = (-x)^2 - 4(-x)$

simplifying $f(-x) = x^2 + 4x \neq f(x)$ $\therefore f$ is not even

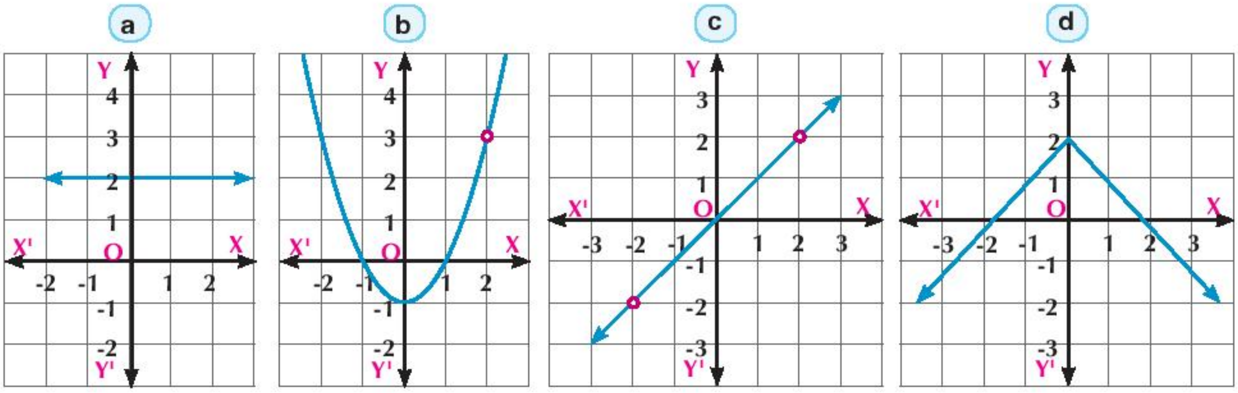
but $-f(x) = -x^2 + 4x$

then $f(-x) \neq -f(x)$ $\therefore f$ is not odd

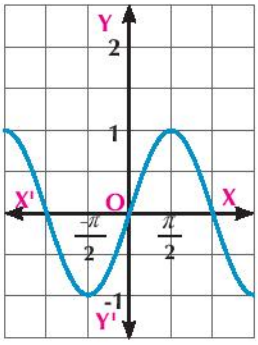
$\therefore$ **i.e.** the function is neither even nor odd.

Try to solve

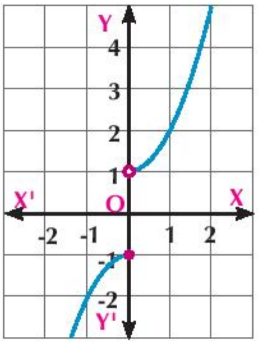
3 Show the type for each of the functions represented by the following graphs (even - odd - otherwise)



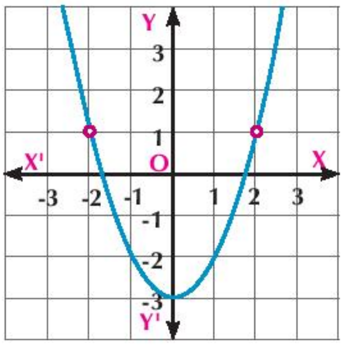
e



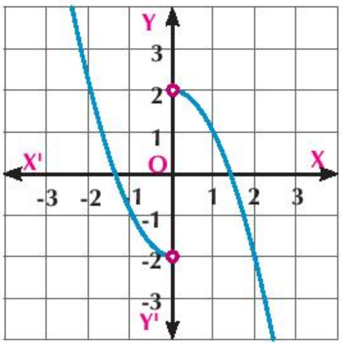
f



g



h



Try to solve

4) Represent graphically the function f where $f(x) = \begin{cases} x + 2 & \text{where } x > -2 \\ -x - 2 & \text{where } x < -2 \end{cases}$

then show whether the function is even, odd or otherwise. Verify your answer algebraically.

One - to - One Function (Injective Function)

definition

the function $f: X \longrightarrow Y$ **is called one- to - one function if :**

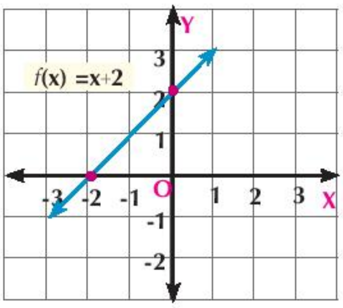
for all a , b $\in$ X , $f(a) = f(b)$ **then** a = b

or for all a $\neq$ b **then** $f(a) \neq f(b)$

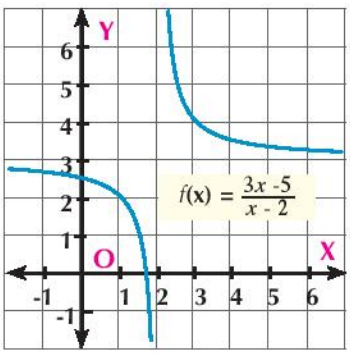
Example

3) Each figure shows the curve of the function $f: X \longrightarrow Y$. Prove that f is one - to - one function.

a



b



Solution

a) $f(x) = x + 2$, the domain of $f = \mathbb{R}$

for all a , b $\in \mathbb{R}$ then $f(a) = a + 2$, $f(b) = b + 2$

let $f(a) = f(b)$ $\therefore a + 2 = b + 2$

eleminate 2 from both sides $\therefore a = b$ **then f is one - to - one function**

b $f(x) = \frac{3x - 5}{x - 2}$, the domain of $f = \mathbb{R} - \{2\}$

for all a , b $\in \mathbb{R} - \{2\}$ then $f(a) = \frac{3a - 5}{a - 2}$, $f(b) = \frac{3b - 5}{b - 2}$

let $f(a) = f(b)$ $\therefore \frac{3a - 5}{a - 2} = \frac{3b - 5}{b - 2}$

By cross multiplying, we get $3ab - 6a - 5b + 10 = 3ab - 6b - 5a + 10$

by eleminating and simplifying $\therefore a = b$ $\therefore$ **f is one- to - one function**

Learn — The horizontal line test

The function $f : X \longrightarrow Y$ is one – to – one function if the horizontal line (parallel to x – axis) at each element of the range elements of the function intersects the curve of the function at one point.

Try to solve

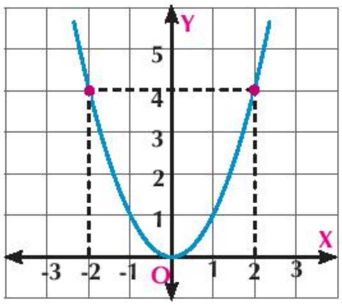
5) In try to solve (3) page (16) show which figures represent one-to one function.

6) Prove that $f : X \longrightarrow Y$ is one-to - one function where:

a $f(x) = 2x - 3$ **b** $g(x) = \frac{3x - 5}{4x + 3}$

Example

4) Show that the function $f : X \longrightarrow Y$ where $f(x) = x^2$ is not one-to-one function.



Solution

$f(2) = 4$, $f(-2) = 4$ $\therefore f(-2) = f(2) = 4$

$\because -2 \neq 2$ then f is not one-to-one

We see that the horizontal line at y = 4 corresponds two unequal values for the variable x which are -2 and 2.

Exercises (1 - 2)

1) Determine the symmetry for each of the following curves (symmetric about x-axis, y-axis or origin point). Explain.

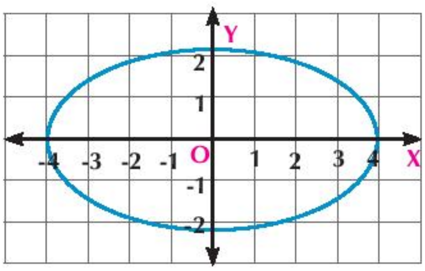


Figure (1)

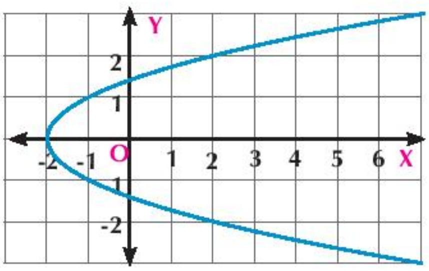


Figure (2)

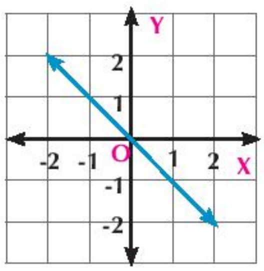


Figure (3)

(2) Find the range for each of the following functions and mention its type (even, odd or otherwise).

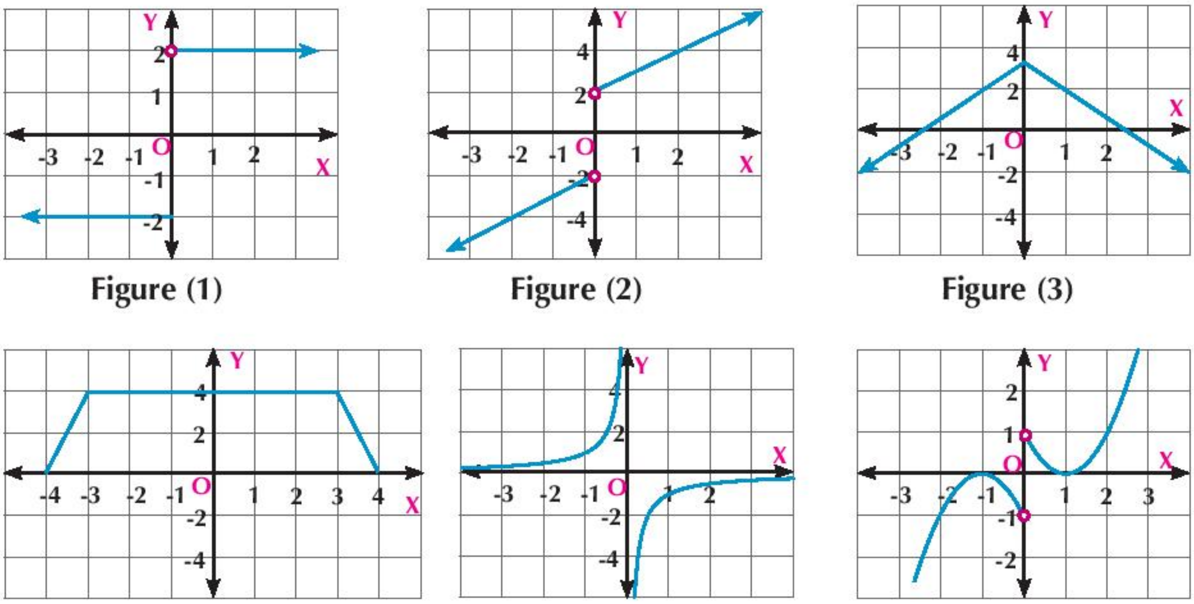


Figure (1) Figure (2) Figure (3)

Figure (4) Figure (5) Figure (6)

(3) Investigate the type for each of the following functions (even - odd - otherwise).

(a) $f(x) = x^4 + x^2 - 1$
(b) $f(x) = 3x - 4x^3$
(c) $f(x) = x^3 - \frac{1}{x}$
(d) $f(x) = x^2 - 3x$
(e) $f(x) = \frac{x^3 + 2}{x - 3}$
(f) $f(x) = x \cos x$
(g) $f(x) = \sqrt{x^2 + 6}$
(h) $f(x) = \frac{x^2}{1 + x}$
(i) $f(x) = (x^2 + 1)^3$

(4) If f_1, f_2 and f_3 are three real functions where $f_1(x) = x^5$, $f_2(x) = \sin x$, $f_3(x) = 5x^2$, then determine which of the following functions is even, odd or otherwise.

(a) $f_1 + f_2$
(b) $f_1 + f_3$
(c) $f_1 \times f_2$
(d) $f_3 \times f_2$

(5) If f and g are two real functions where $f(x) = (3 - x)^2$, $g(x) = (3 + x)^2$ then determine which of the following functions is even, odd or otherwise.

(a) $f + g$
(b) $f - g$
(c) $f \cdot g$
(d) $\frac{f}{g}$

6) Use the following figures to answer the following:

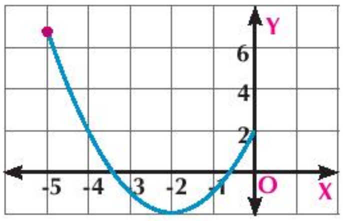


Figure (1)

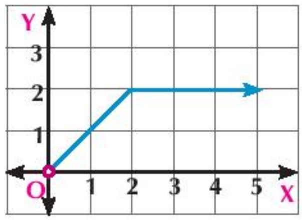


Figure (2)

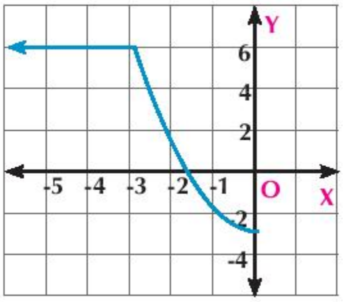


Figure (3)

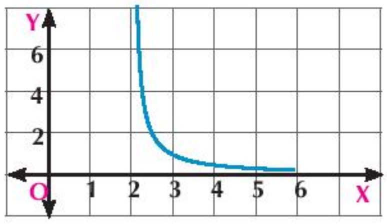


Figure (4)

First: Complete the curve in figures (1) and (3) in your notebook to get an even function over its domain.

Second: Complete the curve in figures(2) and (4) in your notebook to get an odd function over its domain.

Third: Determine the domain and range of the function in each case, then show which graph represents one-to-one function.

7) In each of the following, determine whether the function is one-to-one or not. Give reason.

a) $f(x) = 3x + 1$

b) $f(x) = \dfrac{2x + 1}{x - 2}$

c) $f(x) = x^3 + 1$

d) $f(x) = 2x^2 - x - 3$

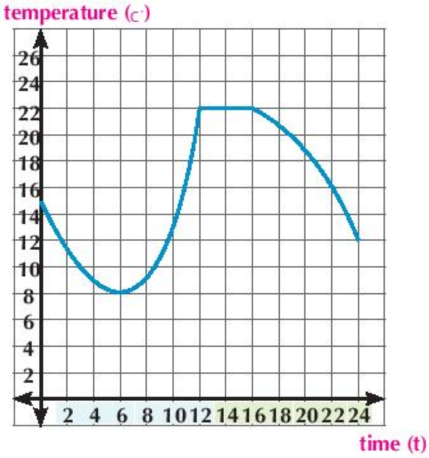
e) $f(x) = x^4 + 2x^2 + 1$

Monotony of Functions

1 - 3

Think and discuss

The opposite graph shows the temperatures recorded in Cairo on a day. Observe the change of temperatures according to time, then find from the graph:



a The periods when the temperature decreases.

b The periods when the temperature increases.

c The periods when the temperature is constant.

The curves help us know the behaviour of the function f and identify the intervals of increasing, intervals of decreasing or intervals of constant which is called monotony of the function.

We will learn

- Monotony of functions.
- Using graphing programs (GeoGebra) to graph the function curve.

Key - term

- Monotony
- Increasing function
- Decreasing function
- Constant function

Learn

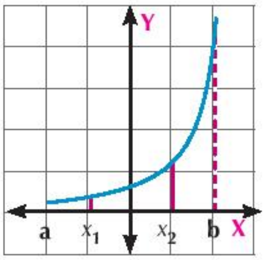
Increasing function

The function f is **said to be increasing on the interval**]a , b[

for all x_1 , $x_2 \in$]a , b[

when: $x_2 > x_1$

then $f(x_2) > f(x_1)$



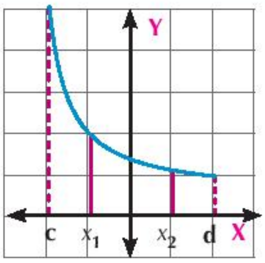
Decreasing function

The function f is **said to be decreasing on the interval**]c , d[

for all x_1 , $x_2 \in$] c ,d [

when: $x_2 > x_1$

then $f(x_2) < f(x_1)$



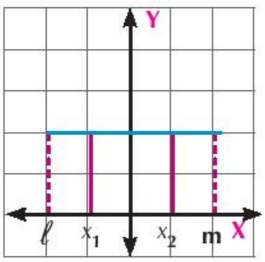
Constant function

the function f is said to be constant on the interval:]ℓ , m[

if : x_1 , $x_2 \in$] ℓ , m[

where $x_2 > x_1$

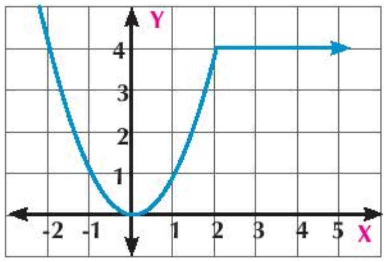
then $f(x_2) = f(x_1)$



Matrials

- Scientific calculator
- Graphic programs

Example

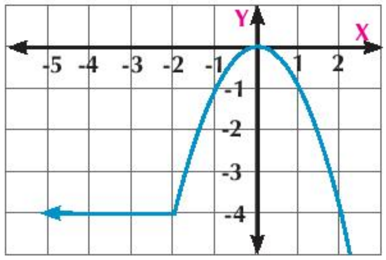


(1) Discuss the monotony of the function represented by the opposite figure.

Solution

- the function is decreasing on the interval]-∞, 0[
- the function is increasing on the interval]0, 2[
- the function is constant on the interval]2,∞ [

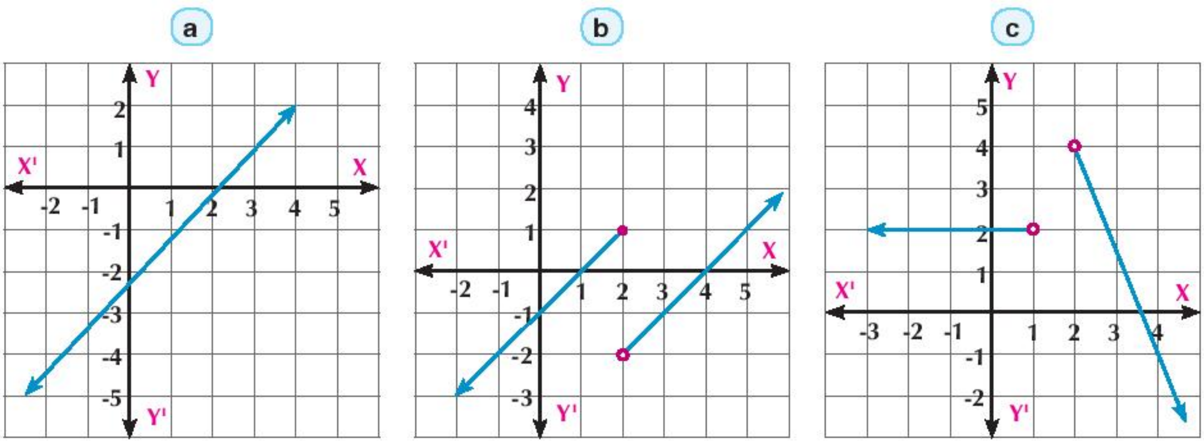
Try to solve



(1) In the opposite graph: Discuss the monotony of the function.

Example

(2) Each of the following figures shows the graph of a function $f: X \longrightarrow Y$ where $Y = f(x)$. Deduce the domain , range and the monotony of the function.



Solution

(a) The domain of $f = \mathbb{R}$ =] -∞ ,∞ [, range of f =] -∞ ,∞ [the function increases on]-∞ , ∞[

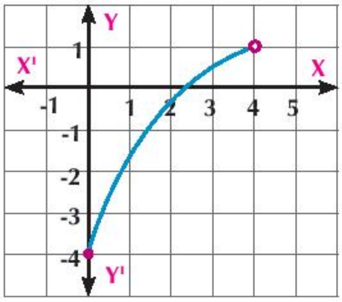
(b) The domain of f =] -∞ , 2] ∪] 2,∞ [=] -∞ ,∞ [, range of $f = \mathbb{R}$
the function increases on]-∞, 2[, and also increases on] 2 ,∞ [

(c) The domain of f =] -∞ , 1[∪] 2,∞ [, range of f =] - ∞, 4[
the function is constant on] -∞ , 1[, and decreases on] 2 ,∞ [

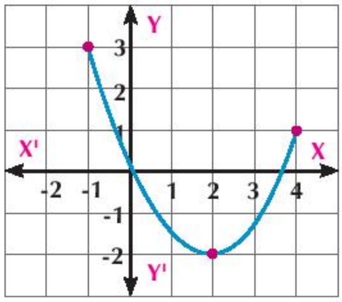
Try to solve

2) In each of the following graphs, deduce the domain , range and monotony of the function:

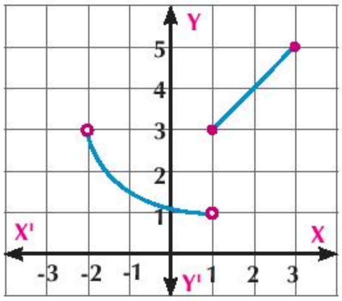
a



b



c



Critical thinking: Which of the previous figures represents one-to-one function? explain your answer.

Using the graphing programs to study the properties of functions

(there are a lot of graphical programs to represent the functions. The most famous is free *GeoGebra* for tablet or computer)

Activity

Use the GeoGebra program in graphing the geometric transformation for functions.

Use *GeoGebra* to represnt graphically the function f where $f(x) = x^3 - 3x + 2$, then find:

a) The domain and the range of the function f.

b) Discuss the monotony and the type (even - odd - otherwise):

1- Open algebraic window , graphing (*GeoGebra*) then press ▾ *Graphics* choose [grid icon] to reach the shown window in **Fig (1).**

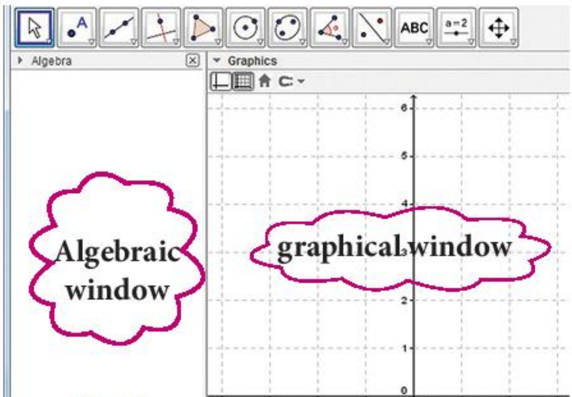


Fig (1)

2- In the algebraic window, write the rule of function $f(x) = x^3 - 3x + 2$ in the input bar as follows:

→ **start** [χ] [^] [3] [-] [3] [χ] [+] [2]

press [↵] the curve of the function appears in graphical window and the rule of the function in algebraic window as shown in **Fig (2)**

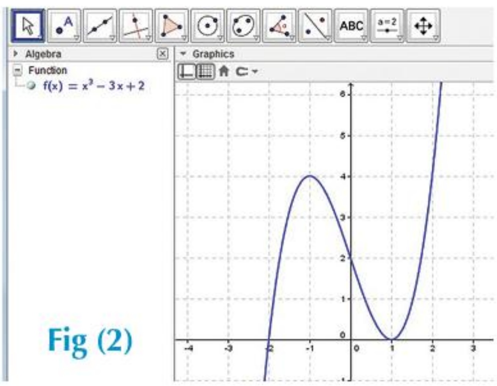
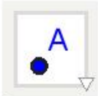


Fig (2)

3- To determine points on the curve of the function, choose  from the tool bar and a new point from the menu. Move the pointer until it reaches the point determined on the curve. Press left click on the mouse so the point will appear on the curve in the graphical window and the coordinate of the point appears in the algebraic window as shown in Fig (3).

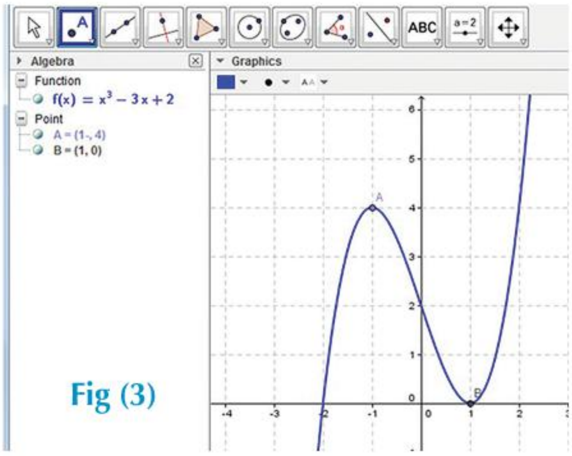


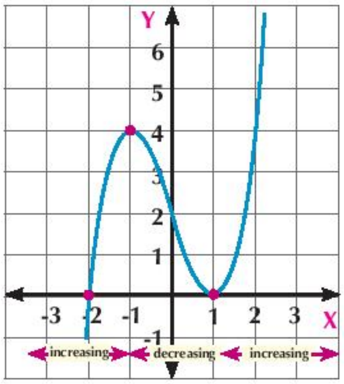
Fig (3)

From the graph:

a The domain of f =] - ∞,∞ [, the range of f =] - ∞, ∞[

b The function is increasing on]- ∞ , -1[, decreasing on]-1 , 1[, increasing on]1 , ∞[

The function is neither even nor odd.



Note:

The point (0, 2) is the point of symmetry of the curve and the function is not one to one.

Drill on the activity

Use *Geogebra* to draw f: $f(x) = 3x - x^3$ and from the graph check the monotony of the function and its type even , odd or otherwise .

Exercises (1 - 3)

1 The following graphs represent the graph of some functions, deduce the range and discuss the monotony from the graph:

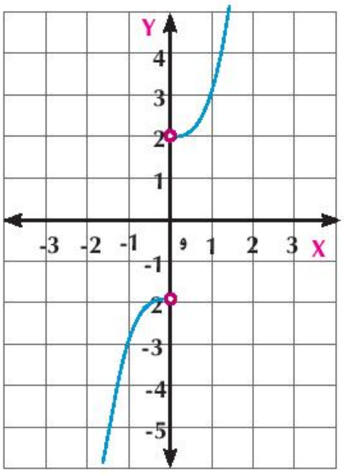


Figure (1)

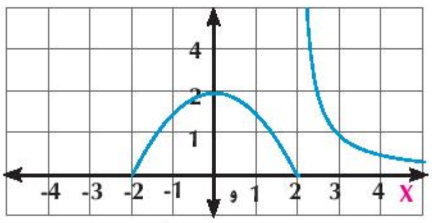


Figure (2)

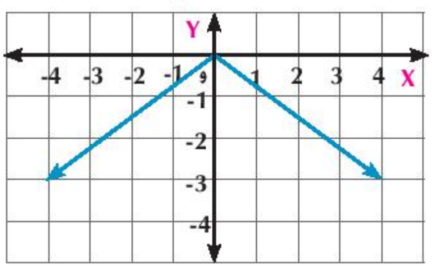


Figure (3)

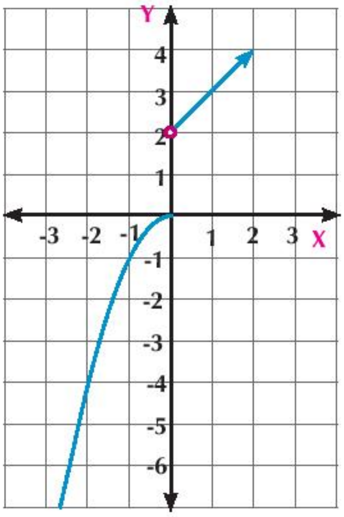
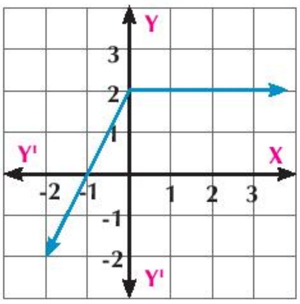


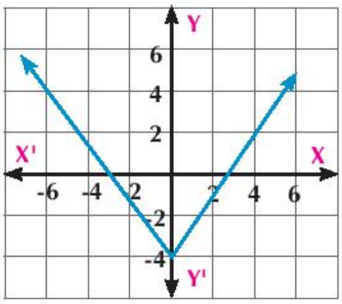
Figure (4)

(2) Using the following graphs, deduce the domain , range and the monotony of each function.

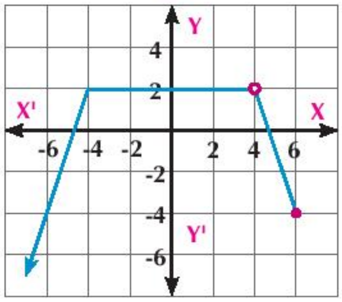
a



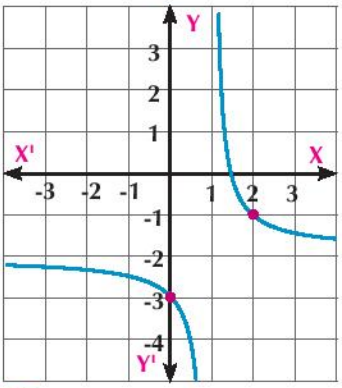
b



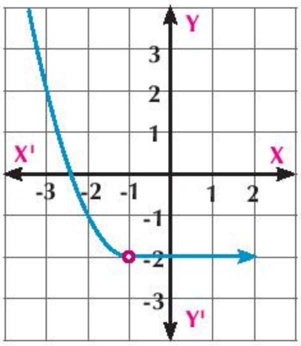
c



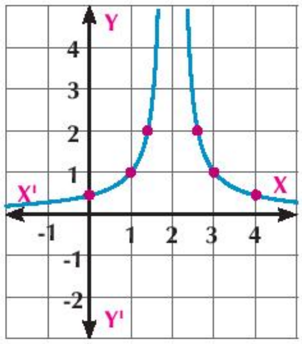
d



e



f



(3) if $f: [-2\ ,\ 6] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 4 - x & \text{when} \quad x < 1 \\ x & \text{when} \quad 1 \leqslant x \leqslant 6 \end{cases}$$

(a) Graph of the function *f* and from the graph, deduce the range of the function and its monotony.

(b) Is the function one -to one . Explain .

(4) **Creative thinking**

Can the function which is increasing or decreasing continuously on its domain be one - to - one? Explain.

Graphical representation of functions and geometrical transformations

We will learn

- The polymial functions (liner - quadratic - cubic).
- Modulus function (absolute value)
- Rational function
- Using the geometric transformation of the function to graph the curves,
  - $y = f(x) + a$
  - $y = f(x + a)$
  - $y = f(x + a) + b$
  - $y = - f(x)$
  - $y = a f(x)$
  - $y = a f(x + b) + c$
- Transformation of some trigonometric functions.

Key - term

- Transformation
- Translation
- Reflection
- Vertical
- Horizontal
- Asymptotes

Matrials

- Scientific calculator
- Graph program

The Polynomial functions

You have studied the polynomial function whose rule is in the form:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots + a_n x^n$$

where: $a_0, a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}, a_n \neq 0, n \in N$

and you knew that the domain and the co-domain are the set of the real numbers R (or a subset of it). As a result, these functions are called polynomial functions of n degree (n is the highest power of the independent variable x).

Notice :

1- If $f(x) = a_0$, $a_0 \neq 0$ then f is called a constant polynomial function.

2- Polynomial functions of the first degree are called linear functions, second degree are called quadratic functions and the third degree are called Cubic functions.

3- Adding or subtracting different power functions and constant, we get a polynomial function.

4- Zeros of the polynomial function are the x –coordinates of the point (s) of intersection of the curve with x –axis.

5- Two polynomial functions f and g are equal if they have the same degree and the coefficients of corresponding power of x are equal.

Example

(1) If f and g are two polynomial functions where $f(x) = (a x + 5)^2$, $g(x) = 9x^2 + 30 x + c - 4$, and if $f(x) = g(x)$, find a , c.

Solution.

$f(x) = (a x + 5)^2 = a^2 x^2 + 10a x + 25$

$\because f(x) = g(x)$ then corresponding coefficients of x are equal

Comparing the coefficients of x $\therefore 10a = 30$ $a = 3$

Comparing the absolute term: $c - 4 = 25$ then $c = 29$

Try to solve

(1) If $f(x) = (a + 2 b) x^3 - c x + 4$, $g(x) = 7x^3 + 5x + (a - b)$ find the values of a, b and c which make $f(x) = g(x)$

Graphing the curves of functions

Polynomial Functions

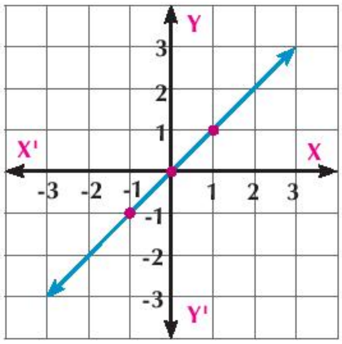
Learn

The following is a graphical representation of some polynomial functions:

1) $f(x) = x$

the function f joins the number by itself and is represented graphically by straight line passing with origin point (0, 0), and its slope = 1

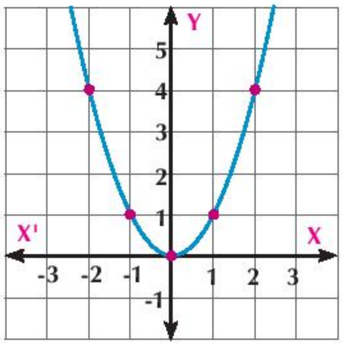
(**check :** its range = $\mathbb{R}$, f is odd and f is increasing on R)



2) $f(x) = x^2$

the function f joins the number by its square and is represented graphically by an upward open curve and symmetrical about y-axis , and its vertex is (0, 0)

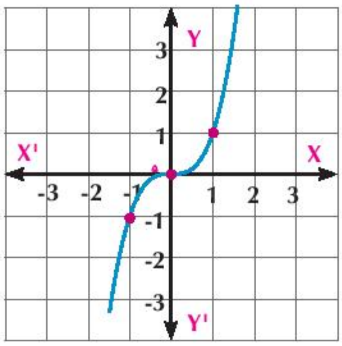
(**check:** its range = $\mathbb{R}$, f is even , and f is decreasing on $]-\infty , 0[$, and increasing on $]0,\infty [$)



3) $f(x) = x^3$

the function f joins the number by its cubic and is represented graphically by a curve its point of symmetry is (0, 0)

(**check :** its range = $\mathbb{R}$, f is odd and increasing on $\mathbb{R}$)



Example

(2) Graph the function f where:

$$f(x) = \begin{cases} x^2 & \text{when} \quad x < 2 \\ 4 & \text{when} \quad x > 2 \end{cases}$$

Solution

1) when $x < 2$, $f(x) = x^2$

we graph $f(x) = x^2$ for each $x \in]-\infty , 2[$

with putting an open circle at point **(2, 4)** as in **Fig (1)**

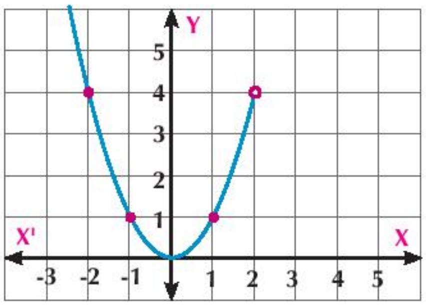


Fig (1)

2) when $x > 2$: $f(x) = 4$

we graph the constant function $f(x) = 4$ for each $x \in]2\ ,\ \infty[$ on the same diagram **fig (2)**

Notice that the domain of $f = \mathbb{R} - \{2\}$, the range of $f = [0\ ,\ \infty[$

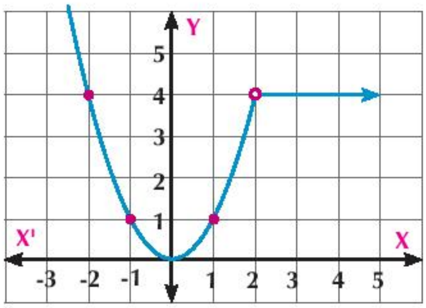


Fig (2)

Try to solve

(2) Graph the function f where:

$$f(x) = \begin{cases} x^2 & \text{when } x < 0 \\ x & \text{when } x > 0 \end{cases}$$

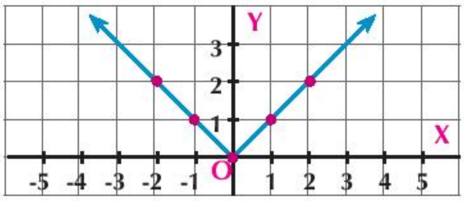
then , deduce the range of the function and its monotony.

Learn — **The Absolute Value Function**

the simplest form for absolute value function is $f(x) = |x|, x \in \mathbb{R}$

and it is defined as follows :

$$f(x) = \begin{cases} x & \text{when } x > 0 \\ -x & \text{when } x < 0 \end{cases}$$



Notice: $|-3| = |3| = 3$, $|0| = 0$, $\sqrt{(-2)^2} = \sqrt{2^2} = 2$

i.e: $|x| > 0$, $|-x| = |x|$, $\sqrt{x^2} = |x|$

The function f is represented graphically by two rays starting from point (0, 0) the slope of one of them = 1 and the slope of the other = -1

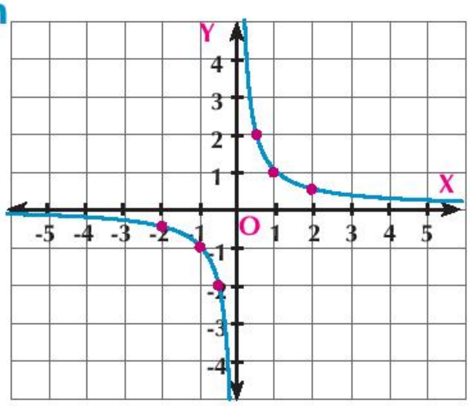
(**check :** its range = $[0\ ,\ \infty[$, f is even , f is increasing on $]0, \infty[$ and f is decreasing on $]-\infty, 0[$)

Learn — **The Rational Function**

the simplest form for the rational function is:

$f(x) = \frac{1}{x}$, $x \in \mathbb{R} - \{0\}$

the function f joins the number by its multiplicative inverse and is represented graphically by a curve whose point of symmetry is (0 , 0). It consists of two parts one of them lies on the first quadrant and the other lies on the third quadrant. Each part approaching to the two axes doesnot intersect them ($x = 0$, $y = 0$ asymptotical line)



(**check :** its range = $\mathbb{R} - \{0\}$, f is odd and is decreasing on, $]-\infty, 0[$, and is decreasing on $]0, \infty[$)

Try to solve

(3) Graph the function f where $f(x) = \begin{cases} |x| & \text{when } x \leqslant 0 \\ \frac{1}{x} & \text{when } x > 0 \end{cases}$

from the graph, find the range of the function and check its monotony.

Geometrical transformations of the curves of the functions

First: Vertical Translation of the function's curve

Co-operative learn

Work with a classmate

1) Graph the function $f: f(x) = x^2$

use the program geogebra

2) Put the pointer on the vertex of the curve and drag the curve vertically upwards one unit . Notice the change of the function base to express a new function whose base is $f(x) = x^2 + 1$ as in **Fig (1).**

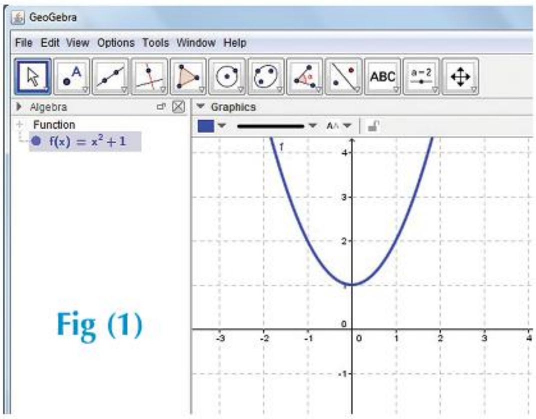


Fig (1)

3) Drag the vertex of the curve to point (0, 2) and (0, 3) then write down your notice each time.

4) Drag the curve of $f(x) = x^2$ vertically downwards 2 units. Notice the change of the function base to express a new function whose base is $f(x) = x^2 - 2$ as in **Fig (2)**

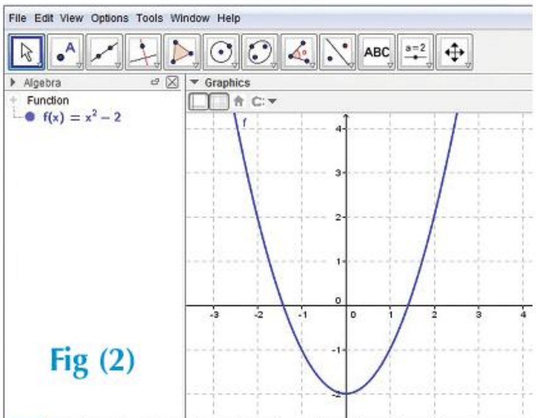


Fig (2)

Think : show how $f(x) = x^2 - 5$ can be graphed using the curve of function $f(x) = x^2$?

we can deduce that:

If $f(x) = x^2$, $g(x) = x^2 + 1$ and $h(x) = x^2 - 2$, then:

1) The curve of $g(x)$ is the same curve of $f(x)$ by translation a unit in the positive direction of y –axis

2) The curve of $h(x)$ is the same curve of $f(x)$ by translation 2 units in the negative direction of y –axis

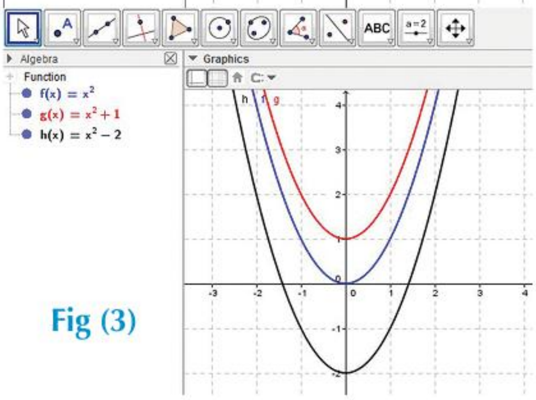


Fig (3)

Critical thinking : using the curve of function $f(x) = x^3$, show how the curves of each can be graphed:

a $g(x) = x^3 + 4$

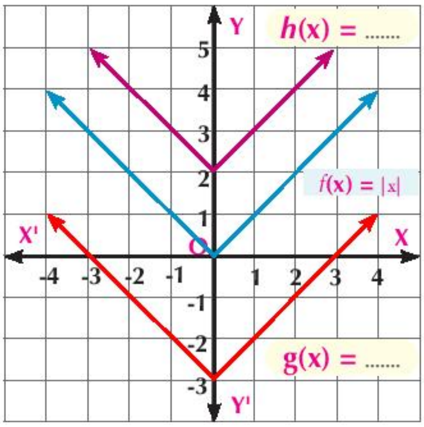
b $h(x) = x^3 - 5$

Learn

Graphing the curve $y = f(x) + a$

For any function f, the curve of $y = f(x) + a$ is the same curve of $y = f(x)$ by translation of a magnitude of a units in the direction of $\overrightarrow{OY}$ when $a > 0$ and in the direction of $\overrightarrow{OY'}$ when $a < 0$

Example



(3) The opposite figure shows the curves of functions *f*, g and *h*. where each of g and h are the image of the function by a vertical translation. Write the rule of the functions of g and h where $f(x) = |x|$

Solution

∵ the curve of the function g is the same curve of the function *f* by translation of a magnitude of 3 units in the direction of $\overrightarrow{OY'}$

then $g(x) = f(x) - 3$

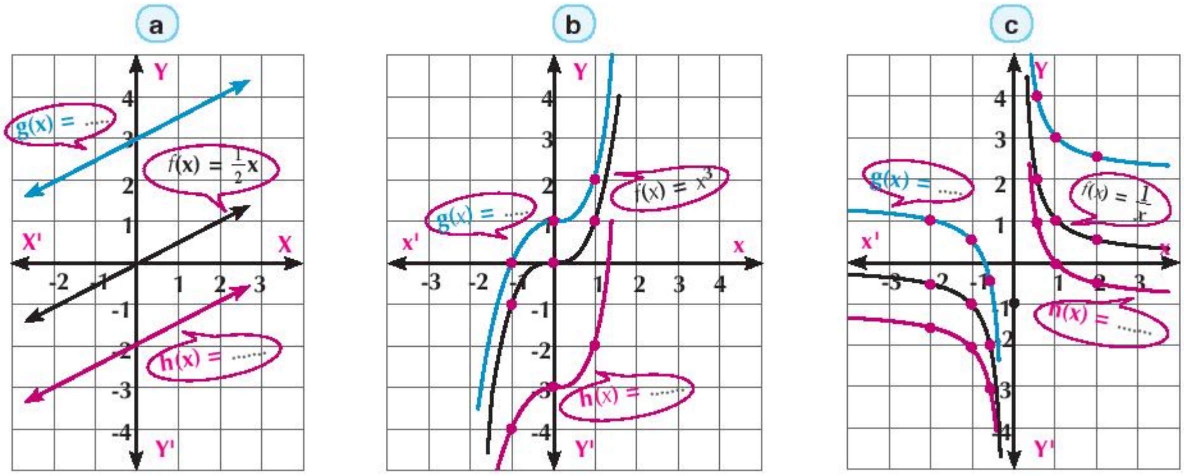
∵ $f(x) = |x|$ then $g(x) = |x| - 3$

∵ the curve of the function h is the same curve of the function *f* by translation of a magnitude of 2 units in the direction of $\overrightarrow{OY}$, then $h(x) = f(x) + 2$

∵ $f(x) = |x|$ then $h(x) = |x| + 2$

Try to solve

(4) The given figures show the curves of the functions *f* , g and h where g and h are the images of the function *f* by a vertical translation. Write the rule for each of g and h in each figure.



Second Horizontal Translation of the function curve

Co-operative learn **Work with a classmate :**

1) Graph the function $f : f(x) = |x|$ using geogebra by writing the rule of the function in the input box as follows: abs(x), then press enter. The curve of the function will appear in the graphical window and its rule $f(x) = |x|$ will appear in the algebraic window as in **Fig (1)**

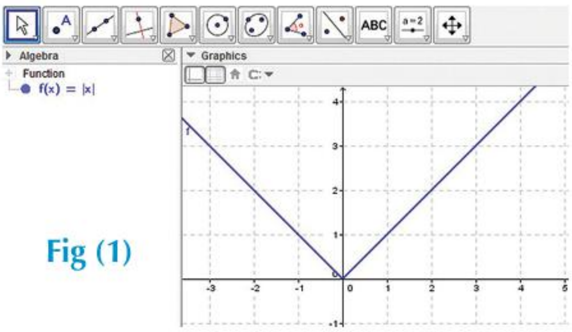


Fig (1)

2) Drag the curve of the function horizontally in the positive direction of x –axis for a number of units. Notice the change of the function base in the algebraic window as in **Fig (2)**

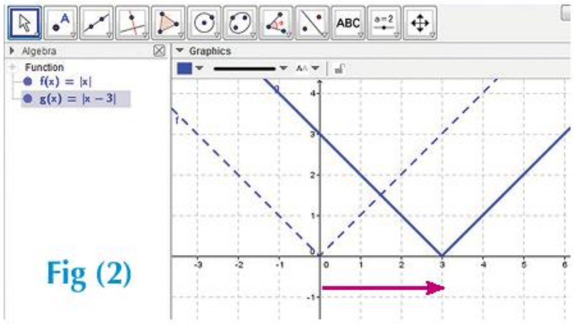


Fig (2)

3) Drag the curve of the function horizontally in the negative direction of x –axis for a number of units. **Fig (3).** What do you notice?

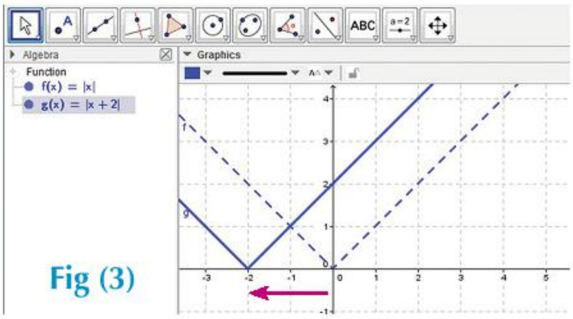


Fig (3)

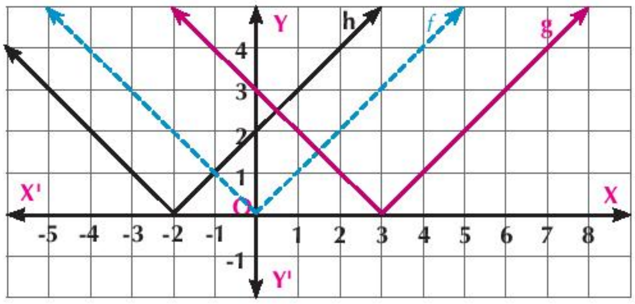
Think : Show how the two curves of the functions g and h can be graphed using the curve of the function f where $f(x) = |x|$, $g(x) = |x - 5|$ and $h(x) = |x + 4|$.

Learn

Graph the curve of y = f (x + a)

For any function f; the curve of $y = f(x + a)$ is the same curve of $f(x)$ by translation of a magnitude of a units in the direction of $\overrightarrow{OX}$ when $a < 0$ and in the direction of $\overrightarrow{OX'}$ when $a > 0$

Notice : In the opposite figure : $f(x) = |x|$:



1) The curve of the function g is the same curve of the function f by translation of a magnitude of 3 units in the direction of $\overrightarrow{OX}$.

$\therefore g(x) = |x - 3|$ and the starting point of the two rays is (3, 0)

2) The curve of the function h is the same curve of the function f by translation of a magnitude of 2 units in the direction of $\overrightarrow{OX'}$

$\therefore h(x) = |x + 2|$ and the starting point of the two rays is (-2 , 0)

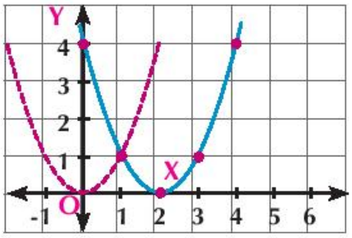
Example

4 Use the curve of the function f where $f(x) = x^2$ to represent each of the two functions g and h where :

a $g(x) = (x - 2)^2$ **b** $h(x) = (x + 3)^2$

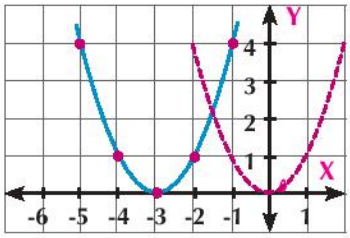
Solution

a



- the curve of $g(x) = (x - 2)^2$ is the same curve of $f(x) = x^2$ by translation 2 units in the positive direction of x –axis and the curve vertex point is (2, 0).

b



- the curve of $h(x) = (x + 3)^2$ is the same curve of $f(x) = x^2$ by translation 3 units in the negative direction of x –axis and the curve vertex point is (-3 , 0).

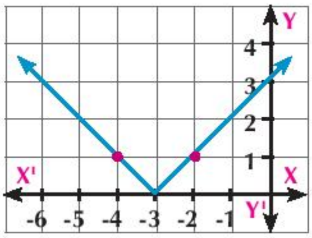
Try to solve

5) Use the curve of the function $f(x) = x^2$ to represent each of the two functions g and h where:

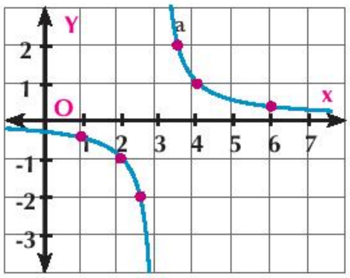
a $g(x) = (x + 4)^2$ **b** $h(x) = (x - 3)^2$

6) Write the rule of the function f represented by each of the following graphs :

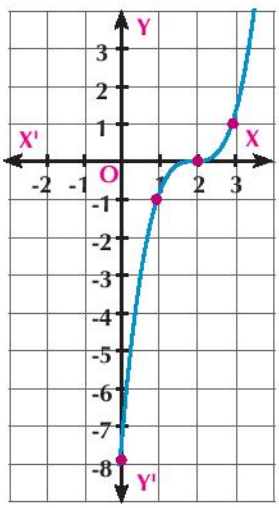
a



b



c



Critical thinking : If $f(x) = x^2$, show how the curve of the function g where $g(x) = (x - 3)^2 + 2$ can be graphed.

Graphing the curve of $y = f(x + a) + b$

From the previous we deduce that: the curve of $y = f(x + a) + b$ is the same curve of $y = f(x)$ by a horizontal translation of a magnitude of a units.

(in the direction of $\overrightarrow{OX}$ when $a < 0$, in the direction of $\overrightarrow{OX'}$ when $a > 0$) , then with a vertical translation of a magnitude of b units (in the direction of $\overrightarrow{OY}$ when $b > 0$ and in the direction of $\overrightarrow{OY'}$ when $b < 0$)

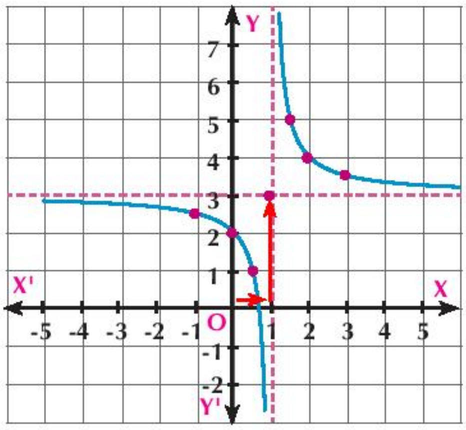
Try to solve

7) Use the curve of function f where $f(x) = x^2$ to represent each of the two functions g and h where :

a) $g(x) = (x + 2)^2 - 4$ b) $h(x) = (3 - x)^2 - 1$

Example

5) Draw the curve of the function g where $g(x) = \frac{1}{x - 1} + 3$ and from the graph, determine the range of the function, and discuss its , monotony:



Solution

the curve of the function g is the same curve of the function f where $f(x) = \frac{1}{x}$ by translation of a magnitude of one unit in the direction of $\overrightarrow{OX}$ ($a = -1 < 0$) , then by translation of a magnitude of 3 units in the direction of $\overrightarrow{OY}$ and the point of symmetry for the curve of the function g is the point (1, 3) , Range of g = $\mathbb{R} - \{3\}$ and the monotony of the function g:
g is decreasing on $]-\infty, 1[$, and is also decreasing on $]1 , \infty[$

Critical thinking : Can it be said that $f(x) = \frac{1}{x - 2} + 3$ is decreasing on its domain? Explain.

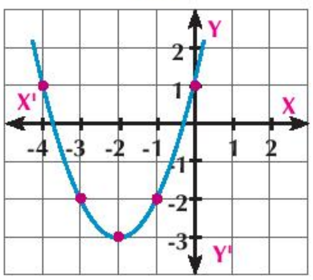
Try to solve

8) Use the curve of the function f where $f(x) = \frac{1}{x}$, $x \neq 0$ to represent each of:

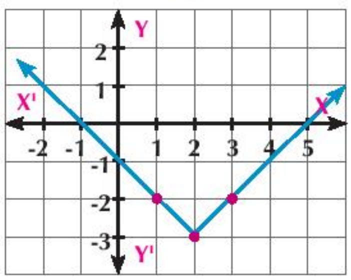
a) $g(x) = \frac{1}{x + 2} + 1$ b) $h(x) = \frac{2x - 3}{x - 2}$

9) Write the rule of the function f represented graphically by each of the following graphs :

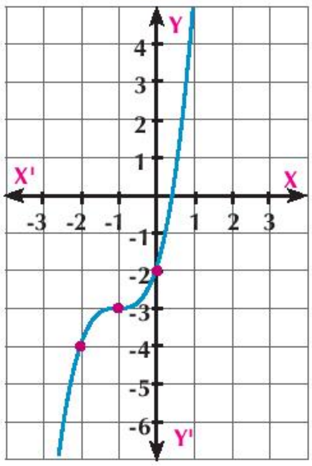
a



b

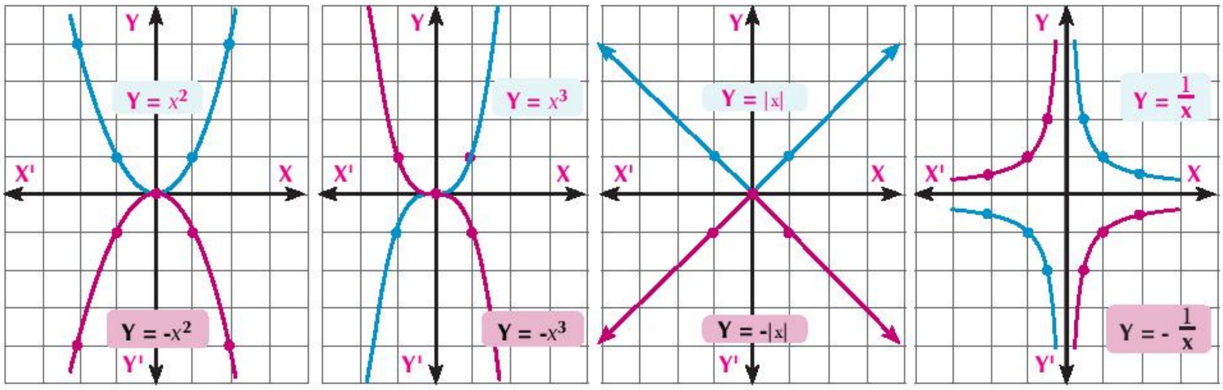


c



Third: Reflection of function's curve in x-axis

The given figures show the reflection of the curves of some standard functions in x-axis.



What do you notice? What do you deduce?

Learn

Graphing the curve of y = - f(x)

For any function f, the curve $y = -f(x)$ is the same curve of $y = f(x)$ by reflection in x – axis.

Example Using geometrical transformation in graphing the curve of the functions

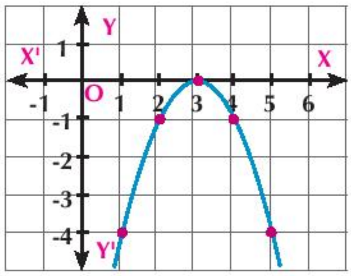
6 Use the curves of the standard functions to graph the curves of the functions g , h and z where:

a $g(x) = -(x - 3)^2$ b $h(x) = 4 - |x + 3|$

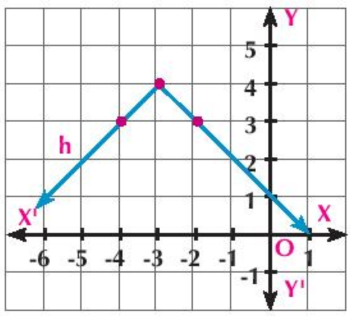
c $z(x) = 2 - \dfrac{1}{x - 3}$

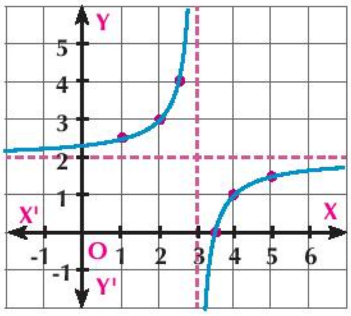
Solution

a The curve of $g(x)$ is the same curve of $f(x) = x^2$ by reflection in x – axis, then horizontal translation of a magnitude of 3 units in the direction $\overrightarrow{OX}$. The curve vertex point is (3, 0) and the curve is open downwards.



b The curve of $h(x)$ is the same curve of $f(x) = |x|$ by reflection in x – axis, then horizontal translation of a magnitude of 3 units in the direction of $\overrightarrow{OX'}$ followed by vertical translation of a magnitude of 4 units in the direction of $\overrightarrow{OY}$ and the starting point of the two rays is (-3, 4) and the curve is open downwards .





(c) The curve of z(x) is the same curve of $f(x) = \frac{1}{x}$ by reflection in x-axis followed by horizontal translation of a magnitude of 3 units in the direction of a magnitude of $\overrightarrow{OY}$ then vertical translation of a magnitude of 2 units in the direction $\overrightarrow{OY}$, and the point of symmetry is (3, 2).

Try to solve

(10) Graph the function g in each of the following where:

(a) $g(x) = 3 - (x+1)^2$

(b) $g(x) = -(x-3)^3$

(c) $g(x) = 3 - |x - 5|$

then check your answer using a graphing program or the graphic calculator.

Example **using the geometrical transformation in graphing the curves of the functions**

(7) Use the suitable transformation to graph the curves of the two functions g and h where $g(x) = 4 - x^2$ and $h(x) = |4 - x^2|$

Solution

First: graph the curve of the function g

the curve of the function g is the same curve of the function f $f(x) = x^2$ by reflection in x – axis, then vertical translation of a magnitude of 4 units in the direction of $\overrightarrow{OY}$ illustrated in fig (1)

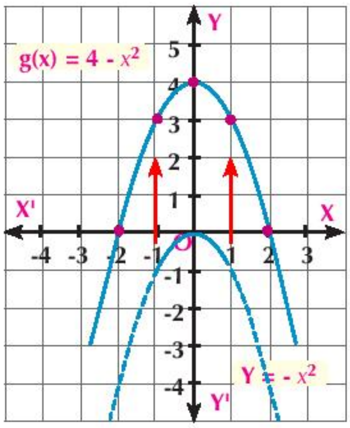


Fig (1)

Second: graph the curve of the function h

$h(x) = |4 - x^2|$ then $h(x) = |g(x)|$

Then y coordinate is positive for all the points of the curve of function where $y = |g(x)|$

$$\therefore y = \begin{cases} g(x) & \text{when } g(x) \geq 0 \\ -g(x) & \text{when } g(x) < 0 \end{cases}$$

i.e the curve of functions h lies in 1st and 2nd quadrants, which means reflection for the curve of the function g for all $g(x) < 0$ in x-axis

as shown in fig **(2).**

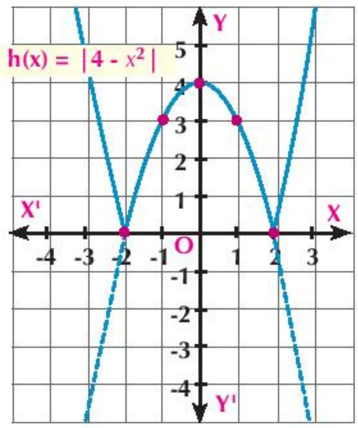
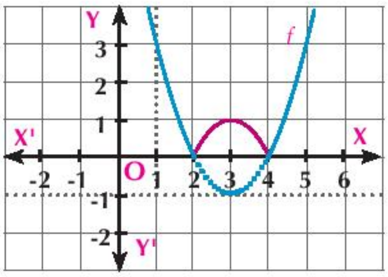


Fig (2)

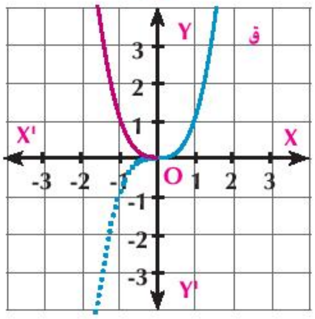
Try to solve

(11) The following figures show the curves of the functions f, g and h. Write the rule of the function in each figure:

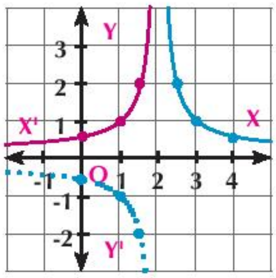
a



b



c



Fourth: stretching of the function curve

Co-operative learn

Graph the curve of g(x) = a f (x)

work with a classmate.

1) Graph the curve of f: $f(x) = x^2$ using *Geogebra* and in the input box, write the rule of function g as follows:

→ start [a] [◀ ▶] [χ] [^] [2] [↵]

A new window will appear **(Fig 1)**

choose *Create sliders*

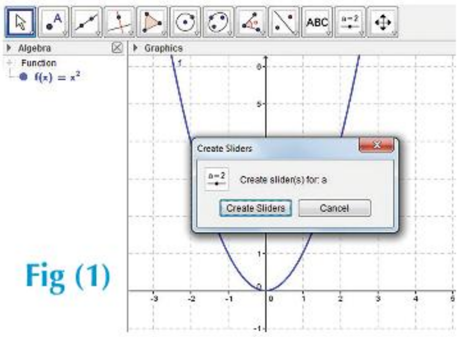


Fig (1)

2) Use the indicator of a to choose other values of a where $1 < a$

Notice the motion of the curve with respect to the curve of the function f for each $x \in \mathbb{R}$ as in **Fig (2)** and when $1 > a$ as in **Fig (3)**. What do you notice? What do you deduce?

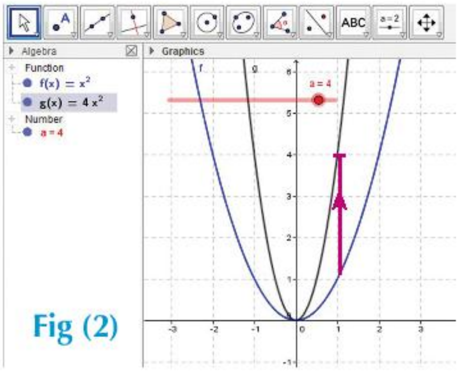


Fig (2)

Learn

Graph the curve of y = a f (x)

for any function f ; the curve of the function y = a $f(x)$ is a vertical stretch for the curve of y = $f(x)$, if a > 1 and a vertical shrinking for the curve of y= f(x) if a < 1

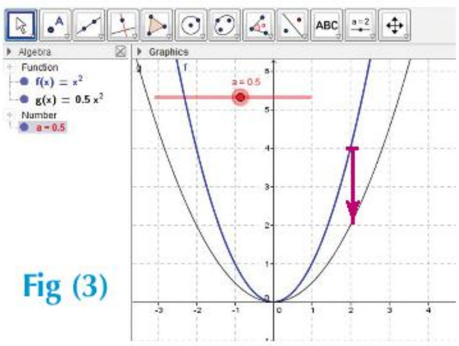


Fig (3)

Graphing the curve of the function g(x) = a f(x + b) + c

Example using the geometrical transformations in graphing the curves of the functions

8) Use the curve of the function f where $f(x) = |x|$ to represent each of the two functions g and h:

a) $g(x) = 2|x|$ b) $h(x) = 2|x - 7| + 2$

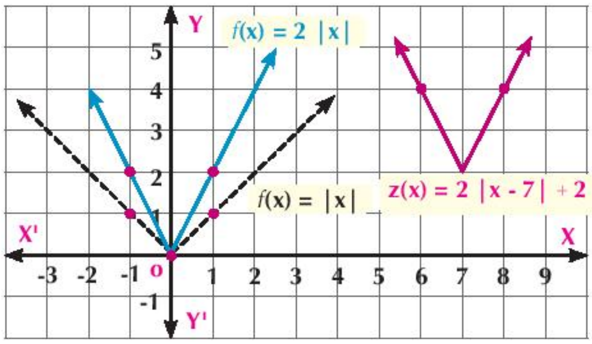
Solution

a) the curve of $g(x)$ is a vertically stretch of the curve of the function $f(x)$ whose coefficient $= 2 > 0$.

then for each $(x, y) \in f$

then $(x, 2y) \in g$

b) the curve $h(x)$ is the same curve of $g(x)$ by a horizontal translation of a magnitude of 7 units in the direction of $\overrightarrow{OX}$, then vertically translation of a magnitude of 2 units in the direction of $\overrightarrow{OY}$



Try to solve

12) Use the curve of function f where $f(x) = x^2$ to represent the two functions g and h :

a) $g(x) = -\frac{1}{2}x^2$ b) $h(x) = 2 - \frac{1}{2}(x - 5)^2$

Check your answer using a graphing program or the graphic calculator, then determine the range of h and its monotony.

Activity

Applying the geometric transformations , which you have learned in the previous algebraic functions on the sine and cosine functions

Trigonometric functions the curve of the sine function

First: Translation on X - axis

1) Use the program (*GeoGebra*) and set the program so that the x-axis scale is in radian by pressing the mouse (right click) and choose the choice in the last line; then choose x-axis and choose the staging system (x-axis) (π).

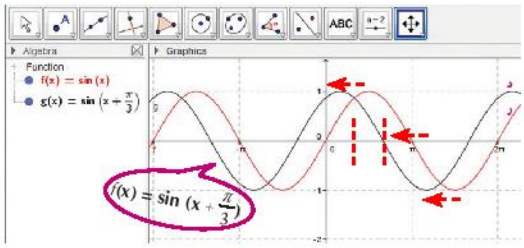
2) At the bottom of the program (input), type the command : **sin (x)** then click (**enter**) to get the red curve of the function. You can control the color and thickness of the curve by pressing the mouse (left-click) and press (object properties), the shown window shows the color, thickness, and ...

3) By the same way, type the command : sin $(x + \frac{\pi}{3})$, then click (**enter**) and color the curve in a different color.

4) Compare between the two curves. What do you notice?

From the graph, we deduce:

The curve of the sine function is translated horizontally to the left by a magnitude of $\frac{\pi}{3}$ units as in the real functions. We notice that the range of the 2nd function is $[-1, 1]$ is the same range of the function **sin x** , also we notice that the function $\sin(x + \frac{\pi}{3})$ is neither even nor odd because its curve is not symmetric about the origin point or y-axis.



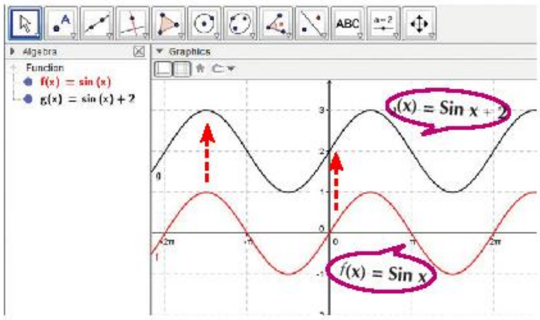
Think:

➢ What do you expect to be the direction of the X-translation if the rule of the second function is : $\sin(x - \frac{\pi}{3})$?

Second: Translation on Y -axis

1) Graph the curve of the function f where $f(x) = \sin x$ as above.

2) Graph the curve of the function g where $g(x) = \sin x + 2$ in different color. **Compare between the two curves. What do you notice?**



From the graph, we deduce:

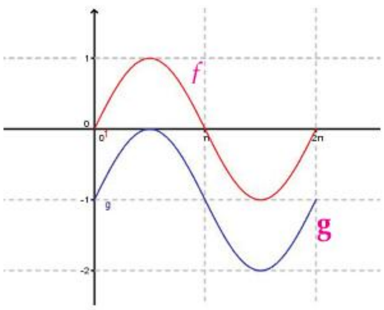
The curve of 2nd function is the same curve of the functions y = **sin x** after translated a magnitude of two units upwards also the range of 2nd function is **[1 , 3]** because it was translated by a magnitude of two units in the direction of y-axis from the first function and the function y = **sin x + 2** is neither even nor odd.

Critical thinking:

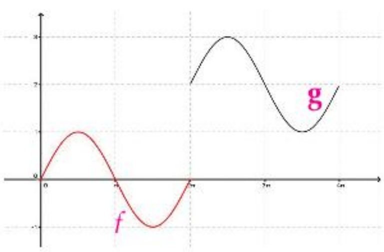
In each of the following figures:

Describe the geometric transformation of the curve of the function f which graphs the curve of the function g then write the rule of function g, its range and its monotony.

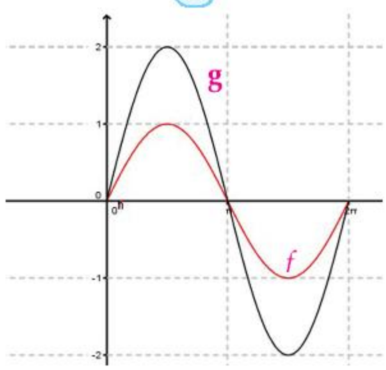
a



b



c



Exercises (1 - 4)

1. Determine the values of a , b and c which make $f(x) = g(x)$ where

$f(x) = (a + b)\,x^3 + 3x - 2$, $g(x) = 5x^3 + (a + c)\,x + b$

2. Graph the curve of the function f, then determine its range and check its monotony from the graph.

a) $f(x) = \begin{cases} |x| & \text{when } x \leqslant 0 \\ x^2 & \text{when } x > 0 \end{cases}$

b) $f(x) = \begin{cases} 4 & \text{when } x < -2 \\ x^2 & \text{when } x \geqslant -2 \end{cases}$

c) $f(x) = \begin{cases} x^3 & \text{when } x < 1 \\ 1 & \text{when } x > 1 \end{cases}$

d) $f(x) = \begin{cases} \frac{1}{x} & \text{when } x < 0 \\ |x| & \text{when } x > 0 \end{cases}$

Choose the correct answer from thoes given:

3. The curve of $g(x) = x^2 + 4$ is the same curve of $f(x) = x^2$ by translation of magnitude 4 units in the direction of:

a) $\overrightarrow{OX}$ b) $\overrightarrow{OX'}$ c) $\overrightarrow{OY}$ d) $\overrightarrow{OY'}$

4. The curve of $g(x) = |x + 3|$ is the same curve of $f(x) = |x|$ by translation of magnitude 3 units in the direction of:

a) $\overrightarrow{OX}$ b) $\overrightarrow{OX'}$ c) $\overrightarrow{OY}$ d) $\overrightarrow{OY'}$

5. The curve vertex point of $f(x) = (2 - x)^2 + 3$ is:

a) (2, 3) b) (2, -3) c) (-2, 3) d) (-2, -3)

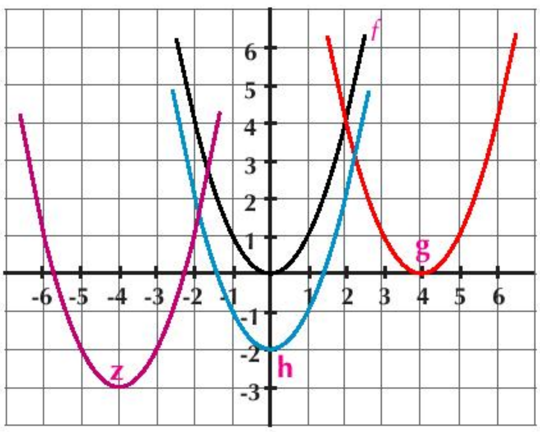
6. point of symmetry of the function f where $f(x) = \frac{1}{x - 3} + 4$ is:

a) (3, -4) b) (-3, -4) c) (3, 4) d) (-3, 4)

7. The curve of the function f where $f(x) = x^2$ is graphed , then translated in the directions of the coordinate axes x, y as in the opposite figure.

Write the rule for each of the following functions:

➢ g, h and z



8 The curve of the function f, where $f(x) = x^3$ is graphed, then translated in the directions of the coordinate axes x, y as in the opposite figure

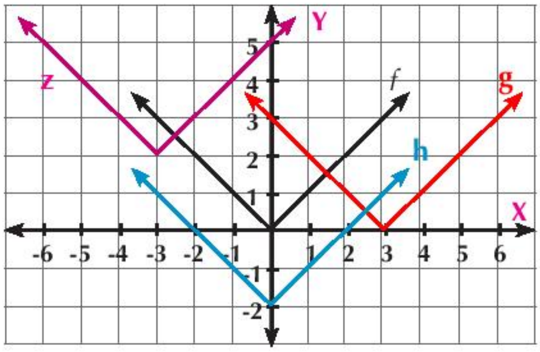
Write the rule for each of the following functions**:**

- g, h, z .

9 The curve of f where $f(x) = |x|$ is graphed then translated in the direction of the coordinate axes x, y as in apposite figure.

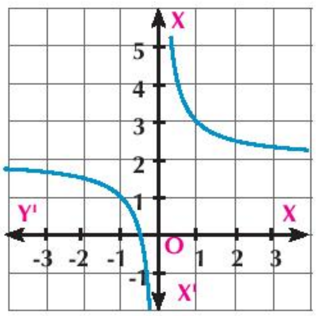
Write the rule for each of the following functions:

- g , h , z

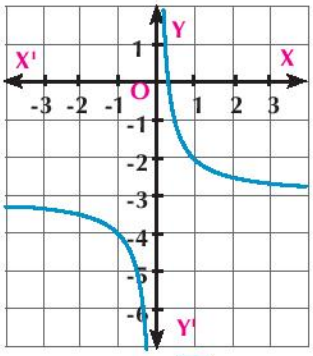


10 The curve of the function f where $f(x) = \dfrac{1}{x}$ is graphed, then translated in the directions of the Coordinate axes x, y. Write the rule of each of the following functions:

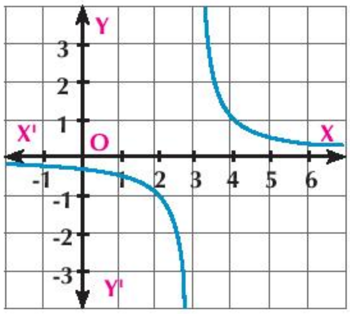
a



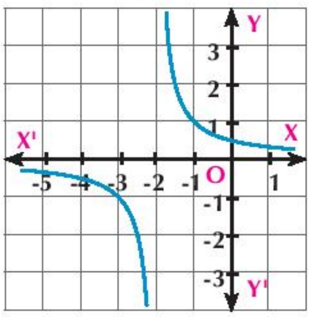
b



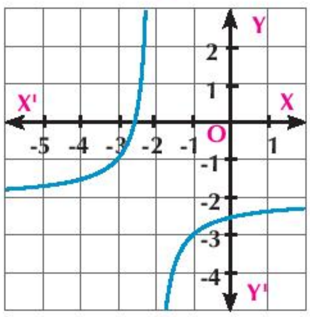
c



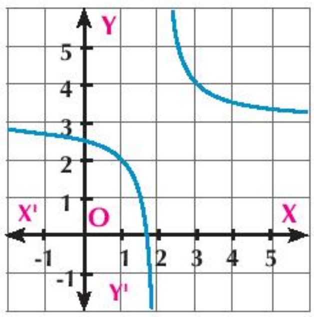
d



e



f



(11) Use the curve of the function f where $f(x) = x^2$ to represent each of the following graphically.

a) $f_1(x) = x^2 - 4$ b) $f_2(x) = x^2 + 1$ c) $f_3(x) = (x + 1)^2$

d) $f_4(x) = (x - 3)^2$ e) $f_5(x) = (x - 1)^2 - 2$ f) $f_6(x) = (x + \frac{3}{2})^2 - \frac{1}{2}$

(12) Use the curve of the function f where $f(x) = |x|$ to represent each of the following graphically:

a) $f_1(x) = |x| + 1$ b) $f_2(x) = |x| - 3$ c) $f_3(x) = |x + 2|$

d) $f_4(x) = |5 - x|$ e) $f_5(x) = |x + 2| + 1$ f) $f_6(x) = |x - 3| - 2$

➢ **Find the coordinates of intersection points of the curves with the two axes.**

(13) Use the curve of the function f where $f(x) = x^3$ to represent each of the following graphically:

a) $f_1(x) = f(x) - 3$ b) $f_2(x) = f(x) + 1$ c) $f_3(x) = f(x - 2)$

d) $f_4(x) = f(x + 3)$ e) $f_5(x) = f(x - 2) - 1$ f) $f_6(x) = f(x + 3) + 2$

➢ **Determine the point of symmetry for each function.**

(14) If the function f where $f(x) = \frac{1}{x}$, graph the function h and determine the point of symmetry of the function curve:

a) $h(x) = f(x + 1)$ b) $h(x) = f(x - 3)$ c) $h(x) = f(x) + 2$

d) $h(x) = f(x) - 4$ e) $h(x) = f(x + 2) - 5$ f) $h(x) = f(x - 2) + 2$

(15) Graph the curve of the function f in each of the following using the suitable transformations, then check its monotony.

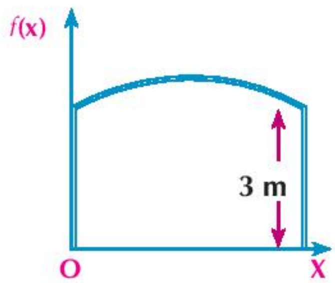
a) $f_1(x) = \begin{cases} x^2 + 2 & \text{when } x > 0 \\ -x^2 - 2 & \text{when } x < 0 \end{cases}$ b) $f_2(x) = \begin{cases} x^2 + 1 & \text{when } -4 \leqslant x < 0 \\ -x^2 - 1 & \text{when } 0 \leqslant x \leqslant 4 \end{cases}$

c) $f_3(x) = x\,|x| - 1$ d) $f_4(x) = \frac{2x}{x + 1}$

(16) Graph the curve of the function f, then determine its range if:

$f(x) = \sqrt{x^2 - 8x + 16}$

(17) **Industry:** An iron gate whose two sides are 3 meters high and its arc is in the form of a part of the curve of the function $f: f(x) = a(x - 2)^2 + 4$ has been designed as shown in the opposite figure, find:



a) Value of a b) Maximal height of the gate

c) Width of the gate

Unit one

1 - 5 Solving Absolute Value Equations and Inequalities

We will learn

- Solve the modulus equations graphically.
- Solve the modulus inequalities algebraically.
- Solve the modulus inequalities graphically.
- Solve the modulus in equalities algebraically.
- Model problems and life applications to solve using the modulus equations and inequalities.

Key - term

- Equation
- Inequality
- Graphical Solution

Matrials

- Graphic calculator.
- Graph paper.
- Graphic programs.

First: Solving equations

Think and discuss

In one figure, represent the two curves of the two functions f and g where f is a modulus function and g is a linear function graphically. Notice the graph , then answer:

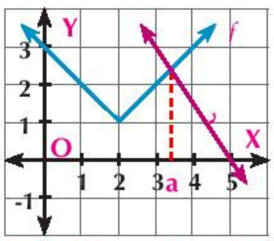
a How many probable intersecting points are there for the two curves of the two functions together ?

b Do the ordered pairs satisfy the rule of each function of both functions if the intersecting points of the two curves are found together?

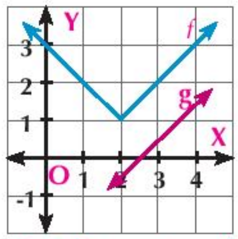
Notice :

1) At the intersecting points (if found) , $f(x) = g(x)$, and vice versa for each x belong to the common domain of both functions.

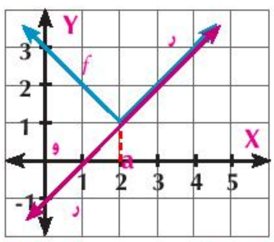
2) For any two functions f and g, the solution set of the equation $f(x) = g(x)$ is the set of x-coordinates of the intersecting points of their two curves as shown in the following figures:



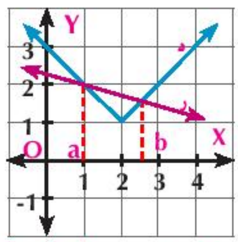
Solution set = {a}



Solution set = ϕ



Solution set = [a,∞[



Solution set = {a, b}

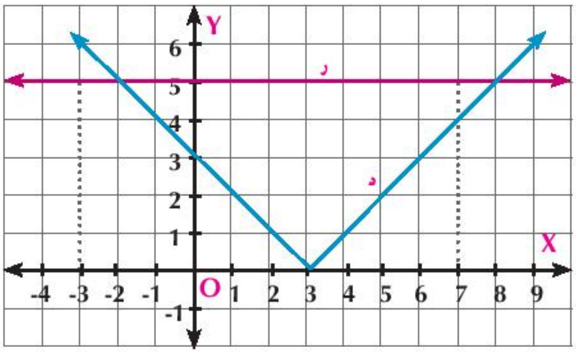
Solve the equation : | a x - b| = c

Example

(1) Solve the equation: $|x - 3| = 5$ graphically and algebraically.

Solution

Let $f(x) = |x - 3|$, $g(x) = 5$



1) Graph curve of the function f : $f(x) = |x - 3|$ by translating the curve of $f(x) = |x|$ 3 units in the direction of $\overrightarrow{OX}$

2) On the same figure, graph $g(x) = 5$ where g is constant function represented by line parallel to x axis and passing through point (0, 5)

$\because$ the two curves intersect at two points (-2, 5) and (8, 5)

$\therefore$ then the solution set of the equation = {-2 , 8}

Algebraic solution:

From the definition of the modulus function: $f(x) = \begin{cases} x - 3 & \text{when } x \geqslant 3 \\ -x + 3 & \text{when } x < 3 \end{cases}$

when $x \geqslant 3$: $x - 3 = 5$ then: $x = 8 \in [3 , \infty[$

when $x < 3$: $-x + 3 = 5$ then: $x = -2 \in]-\infty , 3[$

then the solution set is : **{ - 2 , 8}** . This is coincident with the graphical solution.

Try to solve

1 Solve each of the following equations graphically and algebraically.

a $|x| - 4 = 0$ **b** $|x| + 1 = 0$ **c** $|x - 7| = 5$

Properties of the Absolute Value

Learn

1) $|a\, b| = |a| \times |b|$ for example:

$|2 \times -3| = |-6| = 6$, $|2| \times |-3| = 2 \times 3 = 6$

2) $|a + b| \leqslant |a| + |b|$

The equality holds if **a , b** have the same sign:

$|4 + 5| = |4| + |5| = 9$, $|-4 - 5| = |-4| + |-5| = 9$

Note:

1) If: $|x| = a$ then : $x = a$ **or** $x = -a$ for all $a \in \mathbb{R}^+$

2) If: $|a| = |b|$ if either : $a = b$ **or** $a = -b$ for all $a , b \in \mathbb{R}$

3) $|x|^2 = |x^2| = x^2$

4) If: $|x| = x$, then $x \in [0 , \infty[$

5) If: $|x| = -x$, then $x \in]-\infty , 0]$

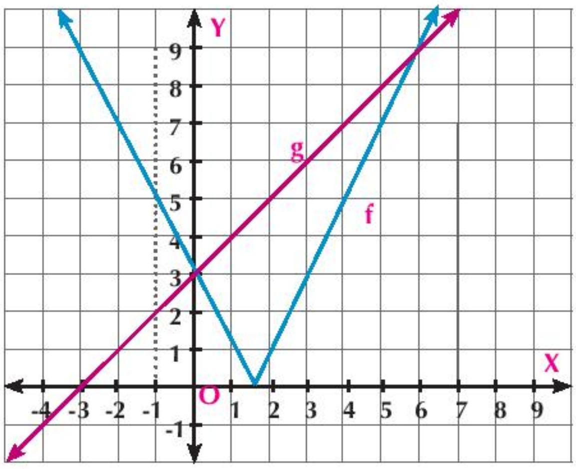
Solve the equation $|a x + b| = c x + d$

Example

(2) Solve the equation : $|2x - 3| = x + 3$ graphically and algebraically.

Solution

let $f(x) = |2x - 3|$, $g(x) = x + 3$



Graphical solution:

$f : f(x) = |2x - 3| = |2(x - \frac{3}{2})|$

$\therefore f(x) = 2\,|x - \frac{3}{2}|$

The curve of f is the same curve of $2|x|$ by horizontal translation of a magnitude of $\frac{3}{2}$ units in the direction of $\overrightarrow{OX}$

g: g(x) = x + 3 represented by a straight line whose slope = 1 and passes through the point (0, 3)

$\because$ intersection points are (0 , 3) and (6, 9)

Then the solution set is : {0 , 6}

The algebraic solution:

$$\because |2x - 3| = \begin{cases} 2x - 3 & \text{when } x \geqslant \frac{3}{2} \\ -2x + 3 & \text{when } x < \frac{3}{2} \end{cases}$$

$\therefore$ when $x \geqslant \frac{3}{2}$ $\quad 2x - 3 = x + 3 \quad$ then $x = 6 \in [\frac{3}{2}, \infty[$

, when $x < \frac{3}{2}$ $\quad -2x + 3 = x + 3 \quad$ then $x = 0 \in]-\infty, \frac{3}{2}[$

$\therefore$ the solution set = {0 , 6}

Try to solve

(2) Solve each of the following equations graphically and algebraically.

(a) $|2x + 4| = 1 - x$ (b) $|2x + 5| = x - 4$ (c) $|x - 3| = 3 - x$

Solve the equation: $|a x + b| = |c x + d|$

Example

(3) Solve the equation $|x - 3| = |2x + 1|$ graphically.

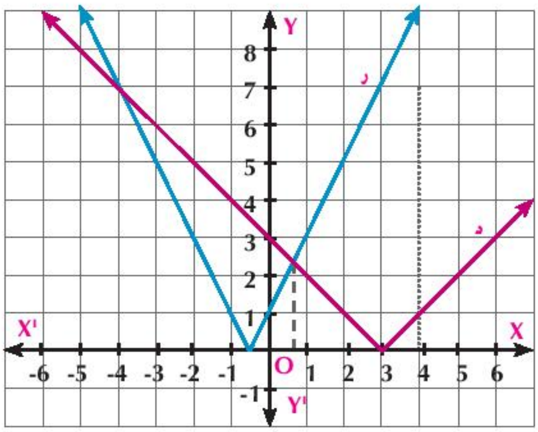
Solution

let $f(x) = |x - 3|$, $g(x) = |2x + 1|$

the curve of f: is same curve of $|x|$ by translation of a magnitude of 3 units in the direction of $\overrightarrow{OX}$ g : $g(x) = 2\,|x + \frac{1}{2}|$

the curve of g is same curve of $2|x|$ by horizontal translation of a magnitude of $\frac{1}{2}$ in $\overrightarrow{OX'}$, the two curves of the functions f and g intersect at $(-4, 7)$ and $(\frac{1}{2}, \frac{5}{2})$

the solution set = $\{-4, \frac{1}{2}\}$



Try to solve

3 Solve each of the following equations graphically.

a $|x + 7| = |2x + 3|$ b $|x - 2| + |x - 1| = \text{zero}$

Example

4 Find the solution set for each algebrically :

a $|x + 7| = |x - 5|$ b $\sqrt{x^2 + 6x + 9} = 9 - 2x$

Solution

a $\because |x + 7| = |x - 5|$ $\therefore x + 7 = \pm(x - 5)$

$\therefore x + 7 = x - 5$ $\therefore 7 = -5$ **(refused).**

or $x + 7 = -x + 5$ **i.e:** $2x = -2$

$\therefore x = -1$ **i.e** solution set = $\{-1\}$

satisfy:

By substituting $x = -1$ in the two sides, we find that:

the right side = left side = 6 **i.e.** the solution set = $\{-1\}$

Remember

if a , b $\in \mathbb{R}$

$|a| = |b|$

then: $a = \pm b$

Think:

Solve the equation above by squaring its two sides, then check your solution.

b $\because \sqrt{x^2 - 6x + 9} = 9 - 2x$

$\therefore \sqrt{(x - 3)^2} = 9 - 2x$ **then:** $|x - 3| = 9 - 2x$

First: when $x \geqslant 3$ **then :** $x - 3 = 9 - 2x$

$\therefore 3x = 12$ **then:** $x = 4 \in [3, \infty[$

Second : when $x < 3$ **then :** $x - 3 = -9 + 2x$

$\therefore x = 6$ **then:** $x = 6 \notin]-\infty, 3[$

$\therefore$ Solution set = **{4}**

Remember

for any real number a:

$\sqrt{a^2} = |a|$

Think: 1) Can you use other methods to solve this equation? Explain.

Try to solve

4) Find the solution set of each of the following equations algebraically:

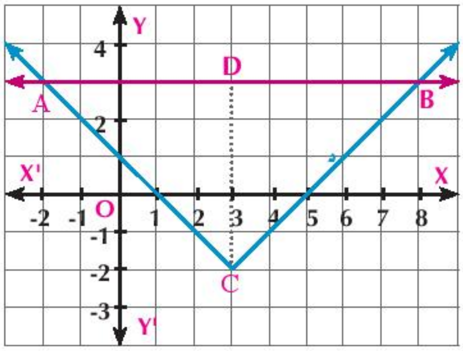
a) $|x - 1| - 2\,|2 - x| = 0$ b) $\sqrt{x^2 - 4x + 4} = 4$

Life applications on solving equations

Example **Planning cities**

5) A piece of land is included between the two curves of the two functions f and g where : $f(x) = |x-3| -2$ and $g(x)=3$. Calculate its area in square units. If the unit length is 8 m, find the area of this land in square meters.

Solution



By graphing the curve of f and g graphically, we find that they intersect at A (-2, 3) and B (8, 3). The land will be in the form of a right angled triangle ABC at C where:

AB= 8-(-2)=10 units

CD = 3 - (-2) = 5 units

$\therefore$ area $\triangle BAC = \frac{1}{2} AB \times CD$

$= \frac{1}{2} \times 10 \times 5 = 25$ square units

Area of land $= 25\,(8 \times 8) = 1600$ square meters.

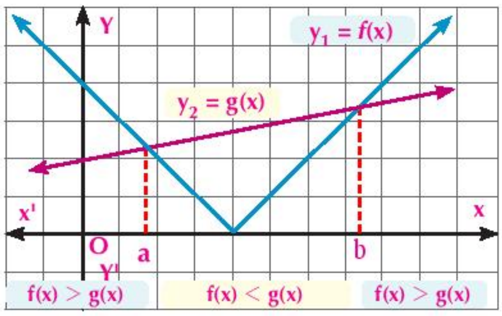
Try to solve

5) Find in square unit the area included between the two curves of the two functions f and g where : $f(x) = |x-2| -1$ and $g(x)=5-|x-2|$

Solving the Inequalities

You have previously learned that the inequality is a mathematical phrase containing one of the symbols: ($<$, $>$, $\leqslant$, $\geqslant$). The solution of the inequality is to find the value (s) of the variable which make the inequality true.

Solving inequalities graphically



The opposite figure shows the curves of the two functions f and g where :

$y_1 = f(x)$, $y_2 = g(x)$ and the solution set of the equation $f(x) = g(x)$ is { a , b}

then: $y_1 = y_2$ when $x = a$ or $x = b$

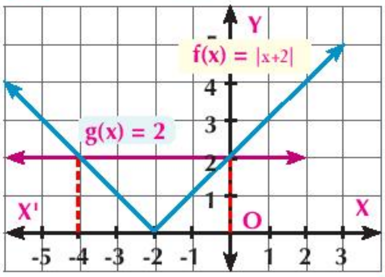
we notice: $y_1 < y_2$ which $f(x) < g(x)$ when $x \in]a, b[$

$y_1 > y_2$ which $f(x) > g(x)$ when $x \in]-\infty, a[\cup]b, \infty[$

Example

6

a

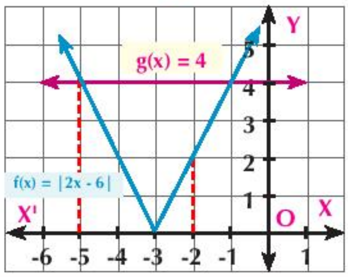


Solution set of inequality

$|x+2| < 2$

is: $]-4, 0[$

b



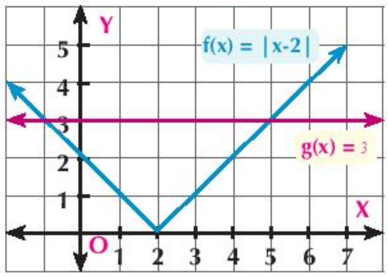
Solution set of inequality

$|2x+6| > 4$

is: $]-\infty, -5] \cup [-1, \infty[$

i.e : $R -]-5, -1[$

c



Solution set of inequality

$|x-2| \leqslant 3$

is : $[-1, 5]$

Try to solve

6 Find the solution set of the following inequalities using the graphs in example (7):

a $|x+2| \leqslant 2$ **b** $|2x+6| \leqslant 4$ **c** $|x-2| > 3$

Solving inequalities algebraically

Learn

first: if $|x| \leqslant a$, $a > 0$ **then** $-a \leqslant x \leqslant a$

second: if $|x| \geqslant a$, $a > 0$ **then** $x \geqslant a$ **or** $x \leqslant -a$

Example

7 Find the solution set of each of the following inequalities in form of an interval:

a $|x-3| < 4$ **b** $\sqrt{x^2 - 2x + 1} > 4$

c $\frac{1}{|2x-3|} > 2$

Solution

a $\because |x-3| < 4$ **then** $-4 < x - 3 < 4$ **adding 3 to inquality**

$\therefore -4 + 3 < x - 3 + 3 < 4 + 3$ **then:** $-1 < x < 7$

$\therefore$ The solution set $=]-1, 7[$

Remember

for all a, b, c

if a < b, b < c

then a < c

if a < b then

a + c < b + c

a c < b c when c > 0

a c > b c when c < 0

if a, b are positive numbers,

a < b then $\frac{1}{a} > \frac{1}{b}$

(b) $\because \sqrt{(x-1)^2} = |x-1|$ **then:** $|x-1| > 4$

$\therefore x - 1 > 4$ i.e $x > 5$ or $x - 1 \leqslant -4$ **then** $x \leqslant -3$

$x \in \mathbb{R} -]-3\ ,\ 5[$ $\therefore$ The solution set $=]-\infty\ ,\ -3] \cup [5\ ,\ \infty[$

(c) $\because \frac{1}{|2x-3|} > 2$ **by taking off the multiplicative inverse of both sides**

$\therefore |2x-3| \leqslant \frac{1}{2}$, $x \neq \frac{3}{2}$

$\therefore -\frac{1}{2} \leqslant 2x - 3 \leqslant \frac{1}{2}$ **by adding 3 to inequality**

$\therefore -\frac{1}{2} + 3 \leqslant 2x - 3 + 3 \leqslant \frac{1}{2} + 3$

$\therefore \frac{5}{2} \leqslant 2x \leqslant \frac{7}{2}$ **by dividing by 2**

$\therefore \frac{5}{4} \leqslant x \leqslant \frac{7}{4}$ $\therefore$ **solution set is** $[\frac{5}{4}, \frac{7}{4}] - \{\frac{3}{2}\}$

Try to solve

(7) Find the solution set of each of the following inequalities in form of an interval:

(a) $|x-7| < 11$ (b) $|3x+7| \leqslant 8$ (c) $\sqrt{x^2 - 6x + 9} > 8$ (d) $\frac{1}{|3x|} > 5$

Critical thinking: Write in the form of an absolute value inequality:

(a) $-4 \leqslant x \leqslant 4$ (b) $0 < x < 6$

(c) $x \geqslant 2$ or $x \leqslant -2$ (d) $x \in R - [-2\ ,\ 6]$

Exercise (1 - 5)

Find the solution set of each of the following equations algebraically:

1. $|x - 2| = 3$
2. $|3 - 2x| = 7$
3. $|x + 2| = 3x - 10$
4. $|x + 2| + x - 2 = 0$
5. $x + |x| = 2$
6. $|x - 2| = 3x - 4$
7. $|x - 1| = x - 2$
8. $|2x - 6| = |x - 3|$
9. $\sqrt{x^2 - 6x + 9} + 2x = 9$

Find the solution set of each of the following equations graphically:

10. $|x - 3| = 7$
11. $|x + 2| + x - 2 = 0$
12. $|x - 2| = 3x - 4$
13. $|2x - 4| = |x + 1|$
14. $|x| + x = 0$
15. $|x + 2| = |x - 3|$

Find the solution set of each of the following inequalities graphically:

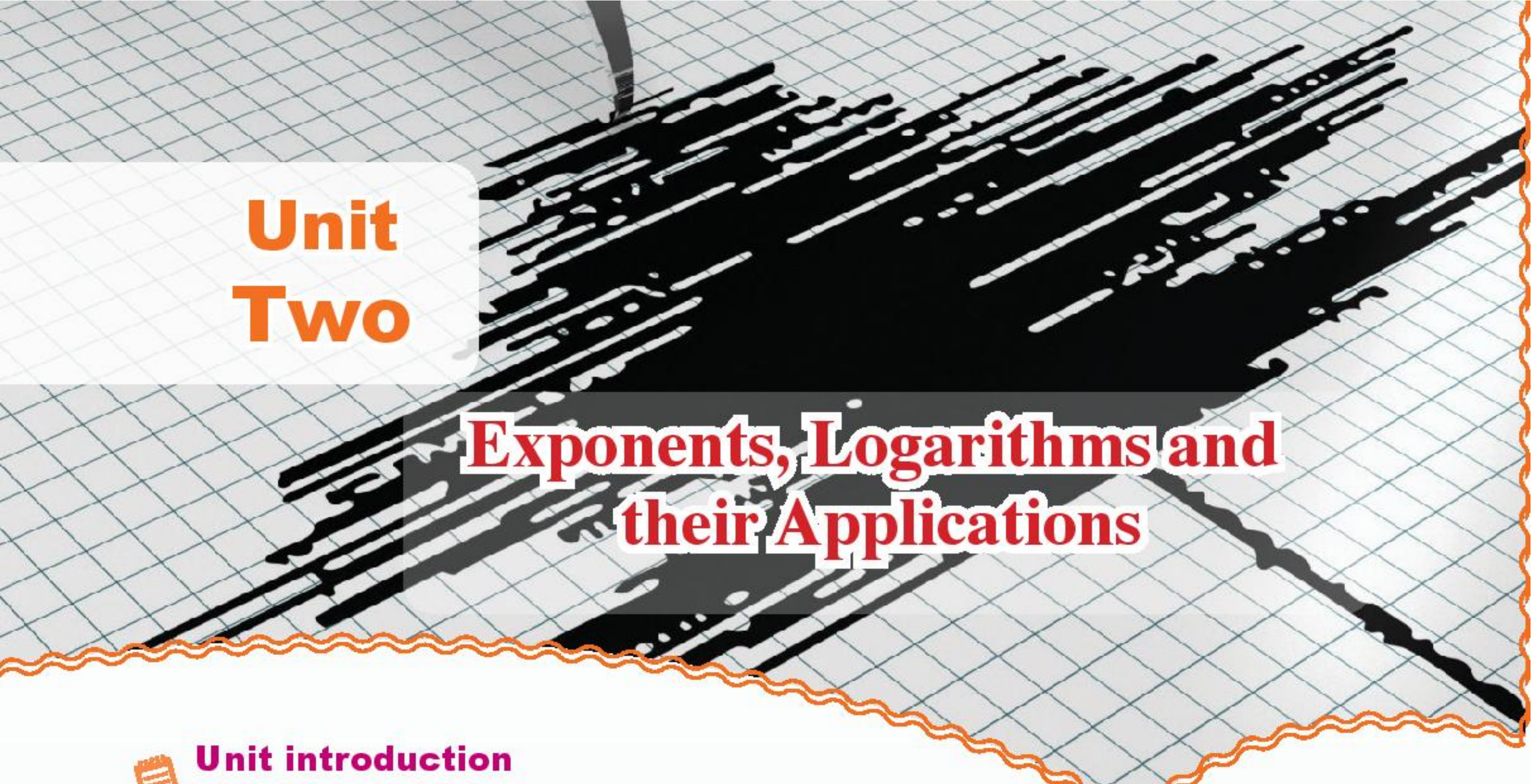
16. $|x - 1| < 2$
17. $|x - 2| < 3$
18. $|5 - x| > 3$
19. $|2x - 3| \geq 7$
20. $|x + 3| > -1$
21. $|2x - 5| \geq 2$

Find the solution set of each of the following inequalities algebraically:

22. $|x - 3| \leq 15$
23. $|3x - 2| < 4$
24. $|3x - 7| \geq 2$
25. $|3x + 2| + 5 < 4$
26. $\sqrt{x^2 - 2x + 1} \geq 4$
27. $\sqrt{4x^2 - 12x + 9} \leq 9$
28. $|2x - 3| + |6 - 4x| < 12$
29. $\frac{1}{|2x - 5|} \geq 3$
30. $\frac{1}{|2x - 3|} > 2$

General Exercises

For more exercises, please visit the website of Ministry of Education.



Unit Two

Exponents, Logarithms and their Applications

Unit introduction

The concept of logarithm was introduced to mathematics at the beginning of the seventeenth century on hand of the scientist Jhon Nabeer as away to simplify calculations . So the navigations, scientists , engineers and the others can easily satisfy their calculations using the tables of logarithms , calculator ruler . they also get use of properties of logarithms to transform the multiplication operations to addition using the property according to the formula $\log_a(xy) = \log_a x + \log_a y$, and thanks to the scientist Leonhard Euler in the eighteenth century to join the concept of the logarithm with the concept of the exponential function so the concept of logarithms was enlarged ,and connected with functions . The logarithmic measure was wildly used in many fields as for example the decibel is a logarithmic unit used to measure the sound intensity , the volt ratio , also the hydrogenous power is (logarithmic measure) used in chemistry to determine the acidic of certain solution .

Unit objectives

By the end of this unit , the student should be able to:

- Recognize the exponential function f: x $\longrightarrow$ a^x where $a \in R^+ - \{1\}$.
- Recognize the graphical representation of the exponential function and deduce its properties.
- recognize the laws of rational exponents.
- Solve exponential equations at the form $a^x = b$.
- Solve applications used exponential equation. $a^x = b$.
- Recognize the logarithmic function $y = \log_a x$ or $f(x) = \log_a x$ where $a \in R^+ - \{1\}$, $x \in R^+$.
- Converting from exponential form to logarithmic form and vice versa.
- Recognize the inverse function and the condition of existence (horizontal line test).
- recognize the graphical representation of the inverse function as an image of the curve of the function under reflection in the straight line y = x Like the graphical representation of the logarithmic function in a bounded interval as inverse function of the exponential function and deduce its properties.
- recognize the relation between the exponential function and logarithmic function graphically.
- recognize some logarithms laws:
  - $\log_a(xy) = \log_a x + \log_a y, x > 0, y > 0$
  - $\log_a(\frac{x}{y}) = \log_a x - \log_a y, x > 0, y > 0$
  - $\log_a x^n = n \log_a x, x > 0, a \in R^+ - \{1\}, n \in R$
  - $\log_a(\frac{1}{x}) = -\log_a x, x > 0, a \in R^+ - \{1\}$
  - $\log_a x = \frac{\log_b x}{\log_b a}, x > 0, a, b \in R^+ - \{1\}$
  - $\log_a b = \frac{1}{\log_b a} \quad a, b \in R^+ - \{1\}$
  - $\log_a a = 1 \quad a \in R^+ - \{1\}$
  - $\log_a 1 = 0 \quad a \in R+ - \{1\}$
- Solve logarithmic equations.
- Solve problems by using the logarithms laws.
- Use the scientific calculator to find logarithms .
- Use the scientific calculator to solve some exponential equations by using logarithms.

Key terms

- Exponent
- Power
- Base
- Radicals
- Rational Exponents
- Square Root
- Cube Root
- n^{th} Root
- Real Root
- Exponential Growth
- Exponential Decay
- Even
- Odd
- Laws of Exponents
- Exponential Function
- Exponential Equation
- Increasing Function
- Decreasing Function
- Symmetry
- Compound Interest
- Inverse Function
- Logarithm
- Exponential Form
- Logarithmic Form
- Common Logartithm
- Natural Logarithm
- Logarithmic Function
- Logarithmic Equation

Lessons of the unit

Lesson (2 - 1): Rational exponents.

Lesson (2 - 2): The exponential function and its applications.

Lesson (2 - 3): The exponential Equations.

Lesson (2 - 4): The inverse function.

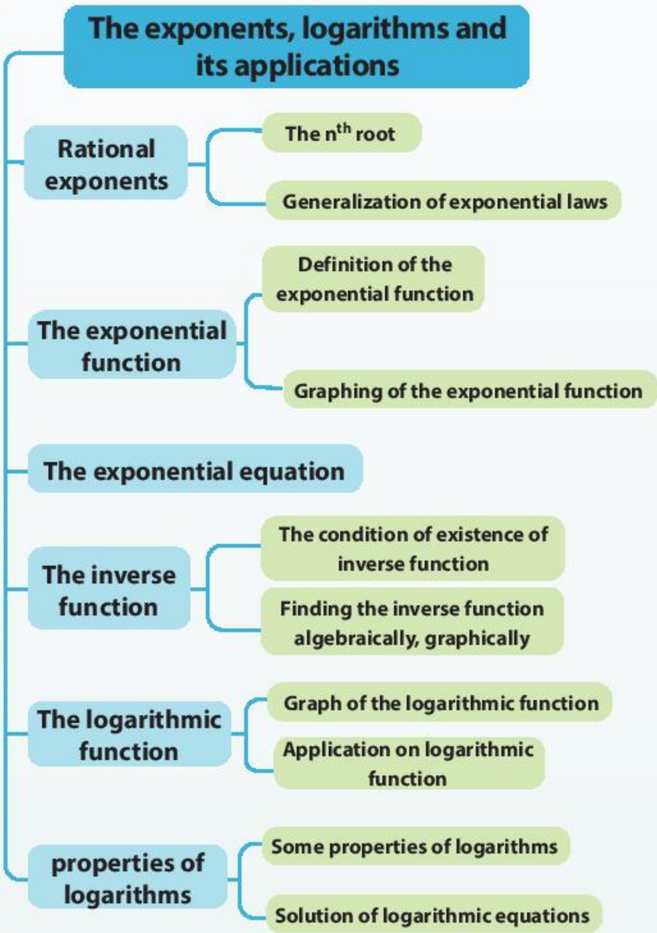
Lesson (2 - 5): The logarithmic function and its graph.

Lesson (2 - 6): Some logarithms properties.

Materials

graph paper - scientific calculator - computer - graphic programs

Chart of the unit



Unit Two

2 - 1 Rational Exponents

We will learn

- Generalization of the laws of the exponents.
- The n^{th} root
- The laws of the rational exponents

Key - term

- The n^{th} Power
- Base
- Exponent
- n^{th} Root
- Rational Exponent

Notice

For any real number

$\sqrt{a^2} = |a|$

Matrials

- Scientific ca.
- Excel program

Introduction

You have studied before the square roots of a real non-negative number and some properties of the square roots and the cubic roots. Also you've studied the integer exponents and some of its properties. In this lesson we will study the rational exponents.

The n^{th} root

You've studied:

The equation $x^2 = 9$ has only two real roots $\sqrt{9} = 3$ and $-\sqrt{9} = -3$

Notice that $3^2 = 9$, $(-3)^2 = 9$

and the equation $x^3 = 8$ has only one real root

$\sqrt[3]{8} = 2$ (The other roots are complex numbers and not real)

$(2)^3 = 8$

In general:

the equation $x^n = a$ **such that** $a \in R$, $n \in Z^+$ **has n roots. We discuss the following cases:**

1) If n is an even number and $a > 0$

then the equation $x^n = a$ has two real roots one is positive and the other is negative (the other roots are complex numbers and not real) and we express these two roots as $\sqrt[n]{a}$, $-\sqrt[n]{a}$, and the n^{th} root of the same sign of a is called the principle n^{th} roots of a.

i.e.: the equation $x^4 = 16$ has two real roots $\sqrt[4]{16} = 2$, $-\sqrt[4]{16} = -2$ **(And the other roots are complex not real).**

Notice that $(2)^4 = 16$, $(-2)^4 = 16$

2) If n is an even number and a is negative , $a < 0$

the equation $x^n = a$ has no real roots (all its roots are complex and not real).

i.e.: the equation $x^2 = -9$ has no real roots (all its roots are complex and not real).

3) If n is an odd number , $a \in R - \{0\}$

then the equation $x^n = a$ has only one real root $\sqrt[n]{a}$ **(and the other roots are complex numbers)**

i.e.: the equation $x^5 = -32$ has only one real root $\sqrt[5]{-32} = -2$ **(notice that $(-2)^5 = -32$)**

4) If $n \in Z^+$, $a = 0$

then the equation $x^n = 0$ has only one solution which is $x = 0$ **(The equation has n of the repeated roots and each one of it = 0 at n > 1).**

Try to solve

1) Find in R the solution set of each of the following equations:

a) $x^4 = 81$ b) $x^5 = 243$ c) $x^4 = -16$ d) $x^3 = -64$

Critical thinking: Explain using a numerical example the difference between the sixth root of a and $\sqrt[6]{a}$

Learn

The Rational Exponents

We know that the square root of the non-negative real number a is the number whose square is a and if a^m represents the principle square roots of a

$\therefore (a^m)^2 = a$ $\therefore a^{2m} = a$ **then** $2m = 1$ $\therefore m = \frac{1}{2}$

which means that. $a^{\frac{1}{2}}$ is the principle square root of a   **Thus** $\sqrt{a} = a^{\frac{1}{2}}$

Similarly $a^{\frac{1}{3}}$ is **the principle cubic root of** a   **Thus** $\sqrt[3]{a} = a^{\frac{1}{3}}$ **in general** $\sqrt[n]{a} = a^{\frac{1}{n}}$

Definition 1

1) (1) For any real number $a \geqslant 0$, $n \in z^+ - \{1\}$ **then** $a^{\frac{1}{n}} = \sqrt[n]{a}$

this statement is also true at $a < 0$, n odd integer number more than 1

2) $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$ where $a \in R$, m, n integer numbers with no common factor between them $n > 1$, $\sqrt[n]{a} \in R$

Generalization of exponents rules

Rational exponents has the same rules of integer exponents

Example

1) Find each of the following (if possible) in R.

a) $(16)^{\frac{1}{4}}$ b) $-(27)^{\frac{1}{3}}$ c) $(-243)^{\frac{1}{5}}$

d) $(-9)^{\frac{1}{2}}$ e) $16^{\frac{3}{2}}$ f) $(27)^{\frac{4}{3}}$

Solution

(a) $(16)^{\frac{1}{4}} = \sqrt[4]{16} = 2$

(b) $-(27)^{\frac{1}{3}} = -\sqrt[3]{27} = -3$

(c) $(-243)^{\frac{1}{5}} = \sqrt[5]{-243} = -3$

(d) $(-9)^{\frac{1}{2}} = \sqrt{-9} \notin R$ **Note** $\sqrt{-9} = 3i$

(e) $16^{\frac{3}{2}} = (\sqrt{16})^3 = 4^3 = 64$

(f) $(27)^{\frac{-4}{3}} = \frac{1}{27^{\frac{4}{3}}} = (\frac{1}{\sqrt[3]{27}})^4 = (\frac{1}{3})^4 = \frac{1}{81}$

Try to solve

(2) **Find (if possible) the value of each of:**

(a) $(125)^{\frac{1}{3}}$ (b) $(-81)^{\frac{3}{4}}$ (c) $(128)^{\frac{-2}{7}}$ (d) $-(343)^{\frac{2}{3}}$

Give reason?

The number $(-8)^{\frac{1}{3}}$ is defined in R, $\sqrt[3]{-8} = -2 \in R$, but the number $(\sqrt[6]{-8})^2$ is undefined in R

Properties of n^th Roots

1) $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

2) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, $b \neq 0$ **such that** $\sqrt[n]{a}$, $\sqrt[n]{b} \in R$

Notice

$\sqrt[n]{a^n} = |a|$ if n is even

$\sqrt[n]{a^n} = a$ if n is odd

Example

(2) Find in the simplest form each of:

(a) $-\sqrt[3]{8a^6 b^9}$ (b) $\sqrt[4]{16 x^4 y^8}$

Solution

(a) $-\sqrt[3]{8 a^6 b^9} = -\sqrt[3]{8} \times \sqrt[3]{a^6} \times \sqrt[3]{b^9} = -2 a^2 b^3$

(b) $\sqrt[4]{16x^4 y^8} = \sqrt[4]{16} \times \sqrt[4]{x^4} \times \sqrt[4]{y^8} = 2 |x| y^2$

Try to solve

(3) Find in the simplest form each of:

(a) $\sqrt[4]{16 a^{12}}$ (b) $\sqrt[6]{(x+2y)^{18}}$

Example

(3) Find in the simplest form each of

(a) $(18)^{\frac{1}{2}} \times (12)^{\frac{3}{2}} \times \frac{1}{(24)^{\frac{1}{2}}}$ (b) $\frac{3^{\frac{1}{2}} \times (147)^{\frac{1}{6}}}{(63)^{\frac{1}{3}}}$

Solution

a The expression $= (2 \times 3^2)^{-\frac{1}{2}} \times (3 \times 2^2)^{\frac{3}{2}} \times (3 \times 2^3)^{-\frac{1}{2}} = 2^{-\frac{1}{2}} \times 3^{-\frac{2}{3}} \times 3^{\frac{3}{2}} \times 2^{\frac{2 \times 3}{2}} \times 3^{-\frac{1}{2}} \times 2^{-\frac{3}{2}}$

$= 2^{\frac{-1+6-3}{2}} \times 3^{\frac{-2+3-1}{2}} = 2^{\frac{2}{2}} \times 3^0 = 2 \times 1 = 2$

b The expression $= \dfrac{3^{\frac{1}{2}} \times (3 \times 7^2)^{\frac{1}{6}}}{(3^2 \times 7)^{\frac{1}{3}}} = \dfrac{3^{\frac{1}{2}} \times 3^{\frac{1}{6}} \times 7^{\frac{1}{3}}}{3^{\frac{2}{3}} \times 7^{\frac{1}{3}}}$

$3^{\frac{1}{2}+\frac{1}{6}-\frac{2}{3}} \times 7^{\frac{1}{3}-\frac{1}{3}} = 3^0 \times 7^0 = 1 \times 1 = 1$

Try to solve

4 Prove that:

a $\dfrac{(343)^{2x-\frac{1}{3}} \times (4)^{3x+1}}{(196)^{3x} \times 4} = \dfrac{1}{7}$

b $\dfrac{125 \times \sqrt[8]{4^3} \times 10^{-\frac{1}{4}}}{4^{\frac{5}{8}} \times \sqrt[4]{6^{-3}} \times 15^{\frac{3}{4}}} = 25$

Solving exponential equations in R

Example

4 Find in R the solution set of each of the following equations:

a $x^{\frac{7}{2}} = 128$

b $(2x+3)^{\frac{4}{3}} = 81$

c $x^{\frac{4}{3}} - 10\,x^{\frac{2}{3}} + 9 = 0$

d $\sqrt[3]{x^5} - \sqrt[6]{x^5} = 6$

Notice

If $x^{\frac{m}{n}} = a$

Then $x = a^{\frac{n}{m}}$

m is an odd number

If $x^{\frac{m}{n}} = a$

Then $x = \pm a^{\frac{n}{m}}$

n is an even number

And m, n hasn't common factor

Solution

a $x^{\frac{7}{2}} = 128$

$x = (128)^{\frac{2}{7}}$ $\qquad$ $x = (2^7)^{\frac{2}{7}}$

$x = 2^2 = 4$ $\qquad$ $\therefore$ solution set $= \{4\}$

b $(2x+3)^{\frac{4}{3}} = 81$

$((2x+3)^{\frac{4}{3}})^3 = (3^4)^3$ raise both sides to the power 3

$(2x+3)^4 = 3^{12}$

$2x + 3 = \pm (3^{12})^{\frac{1}{4}}$ $\qquad$ $2x + 3 = \pm 3^3$

Either $2x + 3 = 27$ $\qquad$ $2x = 24$ $\qquad$ $\therefore x = 12$

Or $2x + 3 = -27$ $\qquad$ $2x = -30$ $\qquad$ $\therefore x = -15$

solution set $= \{12, -15\}$

c $x^{\frac{4}{3}} - 13x^{\frac{2}{3}} + 36 = 0$

$(x^{\frac{2}{3}} - 9)(x^{\frac{2}{3}} - 4) = 0$

either $x^{\frac{2}{3}} - 9 = 0$ **or** $x^{\frac{2}{3}} - 4 = 0$

$x^{\frac{2}{3}} = 3^2$ $\quad$ $x^{\frac{2}{3}} = 2^2$

$x = \pm (3^2)^{\frac{3}{2}}$ $\quad$ $x = \pm (2^2)^{\frac{3}{2}}$

$x = \pm 27$ $\quad$ $x = \pm 8$

solution set = {27, -27, 8 , -8}

d $\sqrt[3]{x^5} - 31\sqrt[6]{x^5} - 32 = 0$

$x^{\frac{5}{3}} - 31x^{\frac{5}{6}} - 32 = 0$

$(x^{\frac{5}{6}} - 32)(x^{\frac{5}{6}} + 1) = 0$

either $x^{\frac{5}{6}} - 32 = 0$ **or** $x^{\frac{5}{6}} + 1 = 0$

$x^{\frac{5}{6}} = 2^5$ $\quad$ $x^{\frac{5}{6}} = -1$ **refused**

$x = (2^5)^{\frac{5}{6}}$

$x = 64$

solution set = {64}

Try to solve

5 Find in R the solution set of each of

a $x^{\frac{4}{3}} = 81$ **b** $(x+1)^{\frac{-5}{2}} = 32^{\frac{-1}{2}}$ **c** $\sqrt[5]{x^4} - 3\sqrt[5]{x^2} = 4$

Exercises 2 - 1

1 Simplify $\dfrac{\sqrt{8} \times 4^{-1} \times 2^{-\frac{3}{2}}}{6^{-2} \times 3^2}$

2 Show when the relation $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$ is true for all real values of a, b?.

3 Complete each if the following:

a $(8)^{\frac{2}{3}}$ in the simplest form ....

b $(6\frac{1}{4})^{\frac{3}{2}}$ in the simplest form ....

c $(\frac{16}{625})^{-\frac{3}{4}}$ in the simplest form ....

d $\sqrt[3]{(3\frac{3}{8})^{-1}}$ in the simplest form ....

e $(5^2 - 3^2)^{\frac{1}{2}}$ in the simplest form ....

4 Choose the correct answer from those given:

a If $5^x = 2$ then $25^x =$ .... (10 , 625 , 4 , 2)

b $(2^7 \div 2^5)^{-\frac{1}{2}} =$ .... $(2 , -2 , \frac{1}{2} , -\frac{1}{2})$

c If $x^{\frac{3}{2}} = 64$ then $x =$ .... (512 , 16 , 4 , 2)

d Which of the given is not equal to $(\sqrt[5]{x^4})$ $((\sqrt[5]{x})^4 , \sqrt[4]{x^5} , x^{\frac{4}{5}} , (x^{\frac{1}{5}})^4)$

e If $4x^5 = 128$ then $x =$ .... (4 , ±2 , 2 , -2)

f The real roots of the equation $(x - 2)^4 = 16$ are .... ({0} , {4} , {8} , {0, 4})

g If $3^a = 4^b$ then $9^{\frac{a}{b}} + 16^{\frac{b}{a}} =$ .... (7 , 12 , 20 , 25)

5 **Find the error:**

a) $-9 = (-9)^{\frac{2}{2}} = \sqrt{(-9)^2} = \sqrt{81} = 9$

b) If $x^4 = 81$ then $x = \sqrt[4]{81}$ $\therefore\ x = 3$

6 **Geometry:** If the length of the radius (r) of the sphere is given by the relation $r = (\frac{3V}{4\pi})^{\frac{1}{3}}$. where (V) is the volume of the sphere find the increase in the radius length when the volume changes from $\frac{32}{3}\pi$ to 36π cubic units.

7 Find the solution set of each of following equations:

a) $x^{\frac{5}{2}} = \frac{1}{32}$

b) $x^4 = 81$

c) $\sqrt[3]{(x-1)^5} = 32$

d) $(x^2 - 5x + 9)^{\frac{5}{2}} = 243$

e) $x^{\frac{4}{5}} - 5x^{\frac{2}{5}} + 4 = 0$

f) $x + 15 = 8\sqrt{x}$

g) $\sqrt[3]{x^2} - 25\sqrt[3]{x} - 54 = 0$

h) $(2x - 1)^4 = (x + 3)^4$

8 If $x^{\frac{3}{2}} = 3y^{\frac{2}{3}} = 27$ find the value of $x + y$

9 **Creative thinking:** Choose the correct answer:

a) If $x < 0$ then: $\sqrt{x^2} - \sqrt[3]{x^3} - \sqrt{x^2 - 2x + 1} + 1 =$ .... (x , -x , zero , - 1)

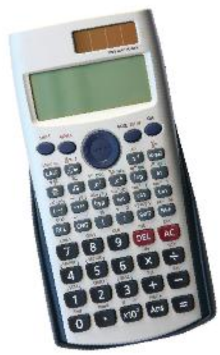
b) If $a = \sqrt[4]{\frac{\sqrt[3]{2}}{\sqrt{7}}}$ then which of the following is rational .... (a^{12} , a^{16} , a^{18} , a^{24})

Activity

Use the calculator to evaluate the following (Approximating the answer to two decimals)

a) $\sqrt[3]{\frac{7^4 \times 3^{-5}}{2^{-7}}}$

b) $(23)^{-\frac{3}{2}} + (0.01)^{-\frac{5}{3}}$



Unit Two

2 - 2 Exponential Function and its Applications

We will learn

- The Exponential function
- The graphical representation of the exponential function
- Properties of the exponential function

Key - term

- Expontential Function
- Exponential Growth
- Exponential Decay

Matrials

- Scientific cal.
- Computer program for graph .

Remember that

The algebrail function:
the independent variable (x) is the base and the exponent t is a real number..

The exponential function:
The independent variable (x) is the exponent and the base is a positive real number doesn't equal to one.

Activity

Bacteria cells multiply by direct division to two cells during a limited period of time then the two cells divide into four cells, then the four cells divide into eight cells and cell division continues that way through the same period of time and in the same circumstances.

The following table shows the time that bacteria cells divide per hour and the number of the producing cells.

Time in hour	0	1	2	3	4	5	6
No. of cells	1	2	4	....	....	....	64

1) Complete the table.
2) Express the number of cells in the exponential form with base 2 in each division.
3) Find the expected number of cells after 8 hours.
4) Express in the exponential form the number of cells after x hours.

Learn

Exponential Function

Definition

The function f such that $f(x) = a^x$, $a > 0$, $a \neq 1$, $x \in R$ **is called exponential function.**

Example:

1

$f(x) = 2^x$ **its base** (2) and **its power** (x).

$f(x) = 5^{x+1}$ **its base** (5) and **its power** ($x+1$).

$f(x) = (\frac{1}{3})^{2x}$ **its base** $(\frac{1}{3})$ and **its power** ($2x$)

Try to solve

1. Determine which of the following is an exponential function.

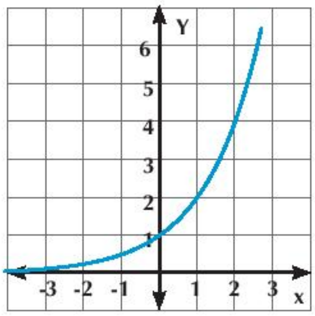
a) $f(x) = x^2$ b) $f(x) = (2)^x$

c) $f(x) = \frac{3}{x+1}$ d) $f(x) = x^3 - 1$

e) $f(x) = (\frac{3}{4})^{x-1}$ f) $f(x) = (-2)^x$

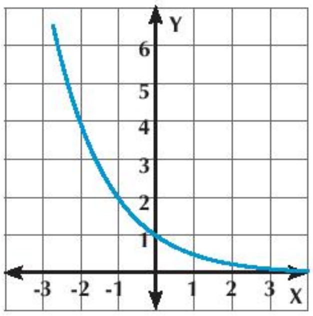
Graphical Representation of Exponential Function

Draw the graph of each of the two function $f(x) = 2^x$, $g(x) = (\frac{1}{2})^x$ where $x \in [-3, 3]$



$f(x) = 2^x$

x	$f_1(x)$	$f_2(x)$
-3	$\frac{1}{8}$	8
-2	$\frac{1}{4}$	4
-1	$\frac{1}{2}$	2
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$



$g(x) = (\frac{1}{2})^x$

Properties of the exponential function $f(x) = a^x$, $a > 0$, $a \neq 1$

1) The domain of $f(x) = a^x$ is R and its range is $]0, \infty[$

2) If $a > 1$ then the function is increasing on its domain and named by exponential growth

if $0 < a < 1$ then the function is decreasing on its, domain and it is named by exponential decay.

3) The curve of $f(x) = a^x$ passes through the point $(0, 1)$ for all $a > 0, a \neq 1$

4) $f(x) = a^x$ is One - to - One function

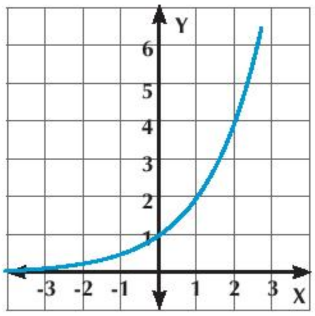
5) The curve of the function $f(x) = a^x$ is image of the curve $f(x) = (\frac{1}{a})^x$ by reflection in y-axis

6) $a^x \longrightarrow \infty$ when $x \longrightarrow \infty$ if $a > 1$

$a^x \longrightarrow 0$ when $x \longrightarrow \infty$ if $0 < a < 1$

Try to solve

(2) In the opposite figure f is defined on R , Such that $f(x) = (3)^x$. Draw on the same figure the curve of the function g which is defined on R , such that $g(x) = (\frac{1}{3})^x$, then find the domain and the range of each function. also determine which function is increasing and which is decreasing and state the reason.



(3) **Critical Thinking:** If $f(x) = a^x$ where $0 < a < 1$ arrange the following in a ascending order $f(7)$, $f(-2)$, $f(\sqrt{5})$, $f(0)$.

Example

(1) If $f(x) = 3^x$ **then complete the following:**

(a) $f(2) =$ .... (b) $f(x + 2) =$ .... $\times f(x)$ (c) $f(x) \times f(-x) =$ ....

Solution

a $f(2) = 3^2 = 9$ **b** $f(x + 2) = 3^{x+2} = 3^x \times 3^2 = 9\,f(x)$

c $f(x) \times f(-x) = 3^x \times 3^{-x} = 3^{x-x} = 3^0 = 1$

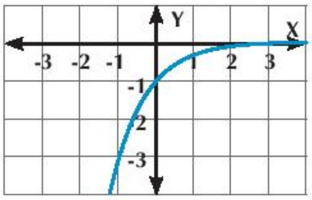
Try to solve

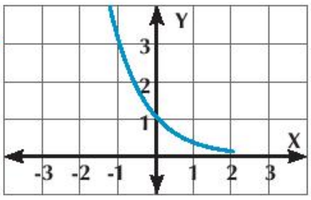
4 Write the rule of each function under its suitable graph:

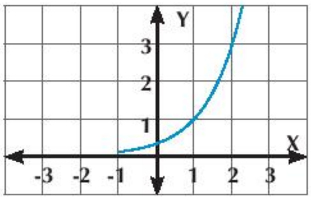
a $y = 3^x$ **b** $y = 3^{-x}$ **c** $y = -3^x$

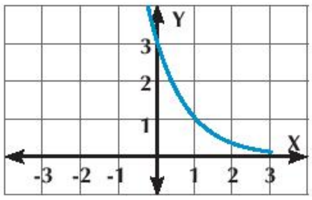
d $y = -3^{-x}$ **e** $y = (3^x)-1$ **f** $y = 3^{x-1}$

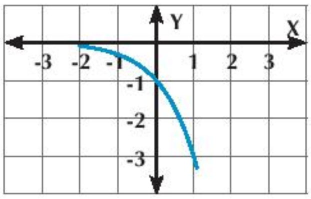
g $y = 3^{1-x}$ **h** $y = 1 - 3^x$

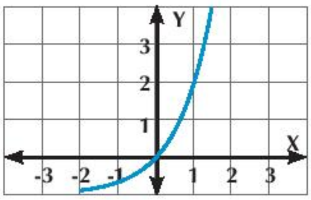

Graph (1)

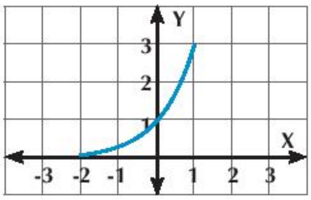

Graph (2)

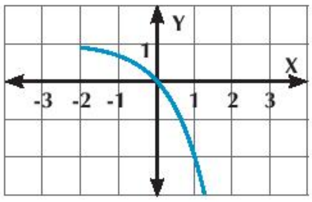

Graph (3)


Graph (4)


Graph (5)


Graph (6)


Graph (7)


Graph (8)

Enrich

Applications tends to equations in the form $a^x = b$

Growth and Decay

In our daily life there are a lot of phenomena expressing growth and decay by time such as the study of population, bacteria, viruses, radiation substances, electricity and temperature. In algebra, there are two functions, representing the growth and decay which are exponential growth function and exponential decay function.

First :Exponential growth

We can use the function f, such that $f(t) = a\,(1 + r)^t$ to represent the exponential growth with a constant percentage during constant intervals of time, where (t) is the time, (a) is the intial value, (r) is the growth percentage per interval of time (Discuss your teacher to conclude the previous relation).

Example

2 **The compound interest:** If principal P is deposited in one of the banks at interest rate r (percentage) and compounded n times per year for a period of t years, then the accumulated value A is given by:

$A = P\,(1 + \frac{r}{n})^{nt}$

Enrich

Example: A man deposited a capital of 5000 L.E in one of the banks with annual compound interest 8%. Find the sum of the capital after 10 years in each of the following.

a) The interest compounded annually
b) The interest compounded quarter annually
c) The interest compounded monthly.

Solution

Use the relation $A = P(1 + \frac{r}{n})^{nt}$

(a) The interest is compounded annually $\therefore$ n = 1

$C = 5000(1 + 0.08)^{10} = 10794.62$ L.E

(b) The interest is compounded quarter annually $\therefore$ n = 4

$C = 5000(1 + \frac{0.08}{4})^{10 \times 4} = 11040.2$ L.E

(c) The interest is compounded monthly $\therefore$ n = 12

$C = 5000(1 + \frac{0.08}{12})^{10 \times 12} = 11098.2$ L.E

Try to solve

5) Number of breeding Bees in a bee cell increases at rate 25% weekly. If the number of bees at the beginning is 60 bees. Write the exponential function that describes the number of bees after t weeks. Then estimate this number after 6 weeks.

Second : Exponential decay

We can use the function f: $f(t) = a(1 - r)^t$ to represent the exponential decay with a constant percentage during constant intervals of time, where t is the time, a is the intial value, r is the decay percentage per interval of time

Example

3) **Connedted with trade:** Kareem bought a car at 120000 L.E, and its price decreases at the rate of 12% per a year.

1st: Write the exponential function which represents the price of the car after t years.

2nd: Estimate to the nearest pound the price of the car after 6 years.

Solution

$a = 120000$, $r = \frac{12}{100} = 0.12$, $t = 6$ years

1st: the exponential decay function is: $f(t) = a(1 - r)^t$ **by substituting then**

$f(t) = 120000(1 - 0.12)^t$ **then:** $f(t) = 120000(0.88)^t$

2nd: put t = 6 in the exponential growth function

$f(5) = 120000(0.88)^6 = 55728.49041$

The expected price of the car after 6 years is 55728 l.E

Try to solve

6) **Connected with medicine:** A patient gets 40 milligram of a medicine. The body gets rid of 10% of this medicine every hour.

a Write the exponential function which represents the quantity of medicine left in the body after t hours.

b Estimate this quantity of medicine left in the body after 4 hours.

Exercises 2 - 2

1 Complete each of the following:

a The function f such that $f(x) = a^{x}$ is an exponential function if a ...., x ....

b The exponential function g where $g(x) = 3^{x-1}$ its base is ....

c The function K where $K(x) = (\frac{-1}{2})^{x+1}$ is not exponential because ....

d The coordinates of the point of intersection of the curve of the function $f(x) = a^{x}$ with the straight line x = 0 is the point (........ , ...)

e The equation of the line of symmetry of the graph of the two functions f and g where, $g(x) = 3^{x}$, $g(x) = (\frac{1}{3})^{x}$ is ....

2 Choose the correct answer from those given :

a The exponential function of base a is increasing if

(a) $a > 0$ **(b)** $a > 1$ **(c)** $0 < a < 1$ **(d)** $a = 1$

b The exponential function of base a is decreasing if :

(a) $a > 0$ **(b)** $a < 0$ **(c)** $0 < a < 1$ **(d)** $-1 < a < 0$

c The exponential function $f(x) = a^{x}$, $a > 1$ its curve approaches:

(a) the x-axis (positive direction) **(b)** the x-axis (negative direction)

(c) the y-axis (positive direction) **(d)** the y-axis (negative direction)

d In the exponential function $f(x) = a^{x}$, $a > 1$ then $f(x) > 1$ when:

(a) $x \in R$ **(b)** $x \in R^{+}$ **(c)** $x \in R^{-}$ **(d)** $x \in Z$

e In the exponential function $g(x) = a^{x}$, $(0 < a < 1)$ then $0 < a^{x} < 1$ when $x \in$

(a) $]0 , \infty[$ **(b)** $]-\infty , 0]$ **(c)** $]1 , \infty[$ **(d)** $]-\infty , 1]$

3 Show which of the following is an exponential function then determine the base and the power of each:-

a $f(x) = 2x^{3}$ **b** $f(x) = \frac{2}{3}(5)^{x}$ **c** $f(x) = \frac{1}{x - 1}$

d $f(x) = 3x^{2} - 1$ **e** $f(x) = (\frac{2}{3})^{x-1}$ **f** $f(x) = (-7)^{x}$

4 Represent graphically each of the following functions then find the domain and the range of each. Also determine which is increasing and which is decreasing:

a $f(x) = 3^{x}$ **b** $f(x) = (\frac{1}{2})^{x}$ **c** $f(x) = -3(2)^{x}$ **d** $f(x) = 2^{x+1} + 1$

e $f(x) = (\frac{1}{2})^{x+2} - 2$ **f** $f(x) = 2(\frac{2}{3})^{x-1} + 1$ **g** $f(x) = -(\frac{1}{2})^{2x} + \frac{3}{4}$

Exponential Equations

2 - 3

Discover

From that table, show when 2^x equals $2x$:

x	-2	-1	0	1	2	3	4
$2x$	-4	-2	0	....	....	....	....
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	....	....	....	....

We will learn

- Solving the exponential equations algebraically.
- Solving the exponential equations graphically.

Learn

Exponential Equation

If the power of the equation contains the unknown "x" then it is called exponential equation i.e. ($3^x = 27$) thus:

1st: if $a^m = a^n$, $a \notin \{0, 1, -1\}$ **then** $m = n$.

Key - term

- Exponential Equation
- Graphical Solution

Example

1) Find in R the solution set of each of the following equations:

a) $3^{x+1} = \frac{1}{27}$ b) $(2\sqrt{2})^{x-3} = 8^{x-2}$

Solution

a) $3^{x+1} = \frac{1}{27}$ $\therefore 3^{x+1} = 3^{-3}$

$\therefore x + 1 = -3$, $x = -4$ $\therefore$ solution set = **{-4}**

b) $(2\sqrt{2})^{x-3} = 8^{x-2}$ $\therefore (2 \times 2^{\frac{1}{2}})^{x-3} = (2^3)^{x-2}$

$\therefore (2^{\frac{3}{2}})^{x-3} = (2^3)^{x-2}$ $\therefore 2^{\frac{3}{2}(x-3)} = 2^{3(x-2)}$

$\therefore \frac{3}{2}x - \frac{9}{2} = 3x - 6$ **multiply by 2**

$\therefore 3x - 9 = 6x - 12$ $\therefore 3x - 6x = 9 - 12$

$-3x = -3$ $x = 1$ $\therefore$ solution set = **{1}**

Matrials

- Computer program for graph.
- Scientific Calculator

Try to solve

1) Find in R the solution set of each of the following equations :

a) $2^{3-x} = \frac{1}{16}$ b) $\frac{(3^2 + 5^2)^x}{68^x} = \frac{1}{32}$

2nd: If $a^m = b^m$, $a, b \notin \{0, 1, -1\}$, then

either: $m = 0$

or: $a = b$ if **m** is an odd , $a = \pm b$ if **m** is an even.

Example

(2) Find in R the solution set of each of the following equations:

(a) $2^{x-3} = 5^{x-3}$ (b) $7^{x+1} = 3^{2x+2}$

Solution

(a) $2^{x-3} = 5^{x-3}$ $\therefore\ x - 3 = 0$ $x = 3$

$\therefore$ solution set = **{3}**

(b) $7^{x+1} = 3^{2x+2}$ $\therefore\ 7^{x+1} = 3^{2(x+1)}$ $\therefore\ 7^{x+1} = 9^{x+1}$

$\therefore\ x + 1 = 0$ $x = -1$ $\therefore$ solution set = **{-1}**

Try to solve

(2) Find in R the solution set of each of the following equations:

(a) $3^{x-5} = 2^{x-5}$ (b) $4^{x+2} = 3^{2x+4}$

Critical thinking: Find all possible solution of the equation $x^{x-2} = 4^{x-2}$

Example

(3) If $f(x) = 3^x$

(a) Prove that $f(x+2) \times f(x-2) = f(2x)$ (b) If $f(x+1) - f(x-1) = 72$ find x

Solution

(a) **L.H.S** $= f(x+2) \times f(x-2)$ $= 3^{x+2} \times 3^{x-2}$

$= 3^{x+2+x-2}$ $= 3^{2x}$ $= f(2x) =$ **R.H.S.**

(b) $\because f(x+1) - f(x-1) = 72$ $\therefore\ 3^{x+1} - 3^{x-1} = 72$

$\therefore\ 3^{x+1} - 3^{x-1} = 72$

$\therefore\ 3^{x-1}(3^2 - 1) = 72$

$\therefore\ 3^{x-1} = 9 = 3^2$ $\therefore\ x - 1 = 2$ $\therefore\ x = 3$

Try to solve

(3) If $f_1(x) = 8^x$, $f_2(x) = 4^x$

(a) prove that $\dfrac{f_1(2x+1) + f_2(3x+2)}{f_1(2x-1) + f_2(3x-2)} = 128$

(b) Solve the equation $f_1(2x) + f_2(3x-1) = 80$

Example

(4) If $f(x) = 2^x$

find x which satisfies the equation: $f(x) + f(5 - x) = 12$:

Solution

By substituting in the equation $f(x) + f(5 - x) = 12$

$2^x + 2^{5-x} = 2$

$2^x \times 2^x + 2^{5-x} \times 2^x = 12 \times 2^x$ multiply both sides by 2^x

$2^{x+x} + 2^{5-x+x} - 12 \times 2^x = 0$

$2^{2x} - 12 \times 2^x + 32 = 0$ by factorizing trinomial

$(2^x - 4)(2^x - 8) = 0$

either : $2^x = 2^2$ then $x = 2$

or: $2^x = 2^3$ then $x = 3$

Try to solve

(4) In the previous example prove that: $\dfrac{f(x+1)}{f(x-1)} + \dfrac{f(x-1)}{f(x+1)} = \dfrac{18}{25}$

Solving Exponential Equastions Graphically

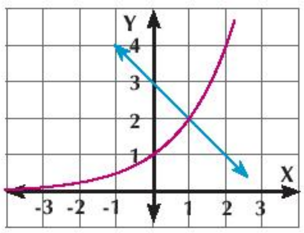
Activity

(5) Using a graph program draw in one figure the two curves of the two functions $f_1(x) = 2^x$, $f_2(x) = 3 - x$, then find from the graph the solution set of the equation $2^x = 3 - x$

Solution

Using the GeoGebra program draw the two curves of the two functions and from the graph we find that the point of intersection is (1, 2)

So the solution set of the equation $2^x = 3 - x$ is {1}.



Try to solve

(5) Using a graph program draw the graph of each of the two functions:

$f_1(x) = 2^x$, $f_2(x) = x + 2$ in one figure and then find from the graph the solution set of the equation $2^x = x + 2$

Exercises 2 - 3

1. Choose the correct answer:

   a. If $2^{x+1} = 8$, then $x =$ ....

   (a) 1 (b) 2 (c) 4 (d) 3

   b. If $5^{x-1} = 4^{x-1}$, then $x =$ ....

   (a) 5 (b) 1 (c) -1 (d) 0

   c. $(\frac{1}{2})^{a^2 - a - 2} = 1$ where $a > 0$, then $a =$ ....

   (a) 1 (b) -3 (c) 2 (d) 3

2. Find the solution set of each of the following equations:

   a. $2^{x+1} = 4$ b. $3^{x-1} = \frac{1}{9}$ c. $7^{x-2} = 1$

   d. $5^{x+3} = 4^{x+3}$ e. $(3\sqrt{3})^{|x|} = 27$ f. $3^{x+3} - 3^{x+2} = 162$

   g. $5^{2x} + 25 = 26 \times 5^x$ h. $2^x + 2^{5-x} = 12$ i. $(\frac{1}{2})^{x+1} + (\frac{1}{2})^{x+3} + (\frac{1}{2})^{x+5} = 84$

3. Find the S.S of the two equations:

   $3^x \times 5^y = 75$, $3^y \times 5^x = 45$

4. If $f_1(x) = 3^x$, $f_2(x) = 9^x$ find the value of x which satisfy $f_1(2x - 1) + f_2(x + 1) = 756$

5. If $f(x) = 7^{x+1}$ find x which satisfy $f(2x - 1) + f(x - 2) = 50$

6. find graphically the solution set of the equation:

   a. $3^{x-2} = 3 - x$ b. $2^x = 2x$

7. **Creative thinking:** If $x^3 = y^2$ and $x^{n+1} = y^{n-1}$ find the value of n ?

8. **Numbers:** If the sum of $2 + 4 + 8 + 16 + ... + 2^n$ is given by the relation $s_n = 2(2^n - 1)$

   a. Find the sum of the first ten numbers in this pattern

   b. Find the number of terms of this pattern starting from the first term to give the sum 131070

9. Solve each of the following equations

   a. $3^{x^2 - 42} = (\frac{1}{3})^x$ b. $7^{2-x} + 7^{-x} = 50$

10. **Creative thinking:**

    Find the solution set of the equation:

    $9^{x+1} - 3^{x+3} - 3^x + 3 = 0$.

The Inverse Function

2 - 4

We will learn

- The inverse function
- The graphical representation of the inverse function
- Finding the inverse function algebraically and graphically

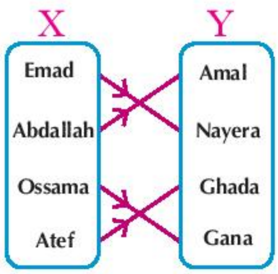
Key - term

- Function
- Inverse Function
- One - to One Function
- Domain
- Range
- Reflection

Matrials

- Scientific Calculator
- Graph program
- Computer

Think and discuss



The opposite figure represents relation (father) between a set of fathers (x)= { Emad, Abdallah, Ossama, Atef} and a set of daughters (y)= {Amal , Nayera, Ghada, Gana}. using the figure :

1) Write the relation representing (father) from X to Y. does the relation represent a function ? and if so is it one to one Function?

2) Write the relation representing (daughter) from Y to X. is it a function?

Learn

The Inverse Function

If the function f is (one-to-one) function from X to y then f^{-1} is an inverse function of f from Y to X if :

for $(x, \mathbf{y}) \in f$ **then** $(\mathbf{y}, x) \in f^{-1}$

Example

1 If the function f is as follows: $f = \{ (1, 2), (2, 4), (3, 6), (4, 8)\}$. Find the inverse function of f and represent both in one figure.

Solution

The function f is one to one

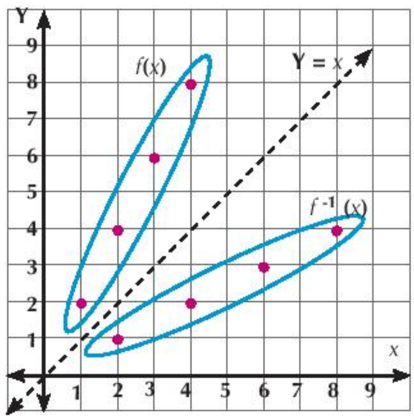
So, it has an inverse

$\because f(x) = \{(1, 2), (2, 4), (3, 6), (4, 8)\}$

$\therefore f^{-1}(x) = \{(2, 1), (4, 2), (6, 3), (8, 4)\}$

We notice that the function f and the inverse function f^{-1} are symmetric about the straight line $y = x$

Thus $f^{-1}(x)$ is the image of $f(x)$ by reflection in the straight line $y = x$

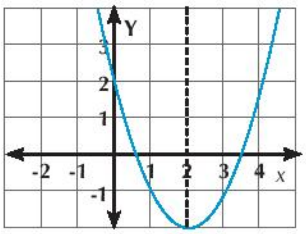


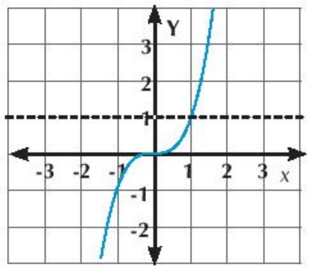
Try to solve

(1) Find the inverse function of the function represented by the following table:

x	-3	-2	-1	0	1
$f(x)$	7	3	1	0	$\frac{1}{2}$

The vertical line test	The horizontal line test
If a vertical line cuts the curve at one point then the curve represents a function.	If any horizontal line cuts the curve at a point then the curve represents a one-to-one function

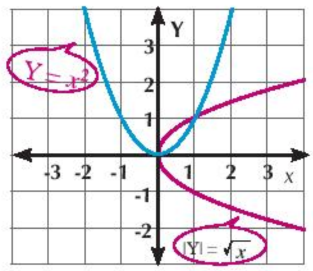




Notice that:

If the function is not one-to-one (Doesn't satisfy the horizontal line test..) then its inverse doesn't represent a function.

i.e $y = x^2$ (is not one to one) then its inverse $|y| = \sqrt{x}$ is not a function.



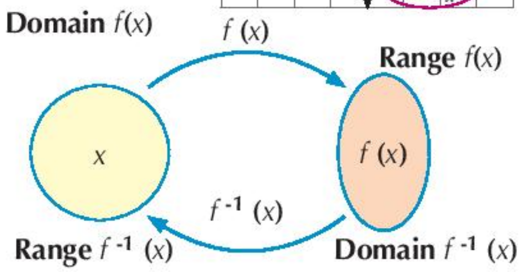
Properties of the inverse function:

1) we said that $f(x)$, $g(x)$ each one is inverse function to the other if

$(f \circ g)(x) = x$ **and** $(g \circ f)(x) = x$

2) the domain of $f(x)$ = the range of inverse function $f^{-1}(x)$

the range of $f(x)$ = the domain of inverse function $f^{-1}(x)$



Critical thinking:

What is the domain of the function f such that $f(x) = x^2$ in which the function f has an inverse function and find that inverse function.

Notice

to find the inverse function, first we exchange variables, then we find y in terms of x.

Example

(2) Find the inverse function of the function f such that $f(x) = 2x + 1$ and represent f(x) and its inverse graphically in one figure.

Solution

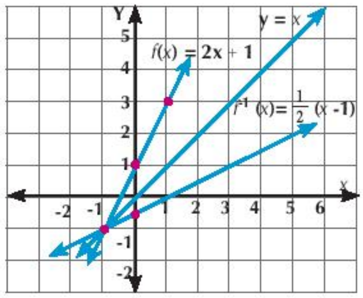
$y = 2x + 1$

$\therefore\ x = 2y + 1$ **exchange variables**

$\therefore\ 2y = x - 1$

$\therefore\ y = \frac{1}{2}(x - 1)$ $f^{-1}(x) = \frac{1}{2}(x - 1)$

x	1	0	-1
$f(x)$	3	1	-1
$f^{-1}(x)$	0	$\frac{-1}{2}$	-1



Notice that The curves of the function f and its inverse function f^{-1} are symmetric about the straight line $y = x$

Try to solve

2) Find the inverse function of the function $y = x^3$ and represent both in one figure.

Example

3) If $f(x) = 3 + \sqrt{x - 1}$ find

a) The domain and the range of the function f.

b) $f^{-1}(x)$ and find its domain and its range

c) Using a graph program draw the graph of each of $f(x)$ and its inverse function $f^{-1}(x)$

Solution

a) $f(x)$ is defined for all values $x - 1 \geq 0$ i.e. $x \geq 1$

$\therefore$ The domain of $f(x) = [1, \infty[$

$\because\ \sqrt{x - 1} \geq 0$ **for all values of x which belong to the domain**

$\therefore\ 3 + \sqrt{x - 1} \geq 3 \Rightarrow f(x) \geq 3$

$\therefore$ The range of $f(x) = [3, \infty[$

b) $\because\ y = 3 + \sqrt{x - 1}$

by exchange the variables x, y

$x = 3 + \sqrt{y - 1}$ $x - 3 = \sqrt{y - 1}$ **by squared both sides**

$(x - 3)^2 = y - 1$

$\therefore\ y = (x - 3)^2 + 1$ $\therefore\ f^{-1}(x) = (x - 3)^2 + 1$

The domain of $f^{-1}(x) = R$ And the range of it $= [1, \infty[$

Try to solve

3) if $f: R \longrightarrow R^+$ $\quad f(x) = \frac{1}{x^2+1}$

a) Find $f^{-1}(x)$ and determine its domain and its range

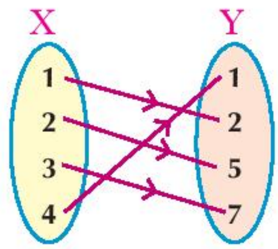
b) Using a graph program draw the graph of each of $f(x)$, $f^{-1}(x)$

Exercises 2 - 4

1) Complete:

a) If the function $f = \{(1, 4), (2, -3), (3, 1), (4, 0)\}$ then $f^{-1} =$ ....

b) The opposite figure represents a function f: x $\longrightarrow$ Y then $f^{-1}(2) =$ ....



c) The image of the point (2, 1) by reflection in the straight line $y = x$ is ....

d) If f is a one to one function and $f(2) = 6$ then $f^{-1}(6) =$ ....

e) If $f: x \longrightarrow 4x$ then $f^{-1}: x \longrightarrow$ ....

2) Put (✔) for the correct statement and (✘) for the incorrect statement :

a) The domain of the function is the domain of its inverse function. (..................)

b) The increasing function on its domain always has an inverse function.. (..................)

c) The even function always has an inverse function. (..................)

d) The odd function always has an inverse function. (..................)

3) Find the inverse function (if possible) of each of the following:

a) $f(x) = \frac{1}{2}x + 4$

b) $f(x) = 4x$

c) $f(x) = 5 + \frac{4}{x}$

d) $f(x) = \frac{3}{x}$

e) $f(x) = 8x^3 - 1$

f) $f(x) = \sqrt[3]{4 - x}$

g) $f(x) = 2 + \sqrt{3 - x}$

h) $f(x) = x^2$ where $x > 0$

i) $f(x) = (x - 1)^2 + 2$ where $x > 1$

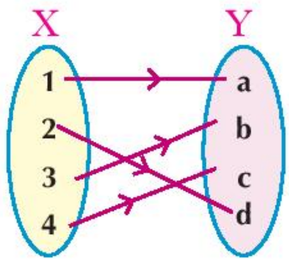
j) $f(x) = x^2 + 8x + 7$ where $x > -4$

k) $f(x) = \sqrt{9 - x^2}$ where $-3 \leqslant x \leqslant 0$

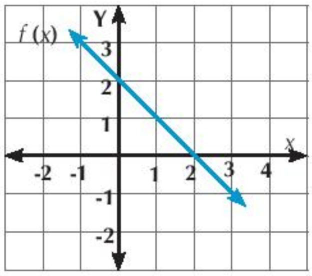
l) $f(x) = \sqrt{9 - x^2}$ where $0 \leqslant x \leqslant 3$

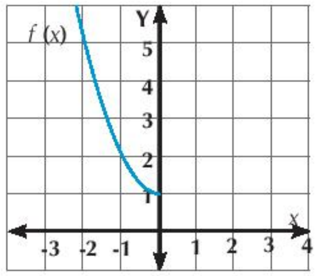
4 **a** If $f(x) = 5x$. find $f^{-1}(x)$ and represent it graphically.

b The opposite figure represents the function f from X to Y find $f^{-1}(b) + 2f^{-1}(c)$.



5 In each of the following figures draw in the same figure the curve of the inverse function $f^{-1}(x)$





6 **Discover the Error:**

Wael and Rana tried to find the inverse function of the function $f(x) = \frac{x - 5}{x}$

Wael's solution

$\because f(x) = \frac{x - 5}{x}$

$\because f^{-1}(x) = \frac{1}{f(x)}$

$\therefore f^{-1}(x) = 1 \div \frac{x - 5}{x}$

$= 1 \times \frac{x}{x - 5}$

$f^{-1}(x) = \frac{x}{x - 5}$

Rana's solution

$\because y = \frac{x - 5}{x}$

$\therefore x = \frac{y - 5}{y}$ **exchange variables**

$\therefore yx = y - 5$ **cross multiplication**

$\therefore yx - y = -5$

$\therefore y(x - 1) = -5$

$\therefore f^{-1}(x) = \frac{-5}{x - 1}$

Which solution is correct? Why?

7 **Open question:** Is it possible for a function f to be itself the inverse function f^{-1}? If it's possible give examples.

8 Determine the domain at which the function f has an inverse for each?

a $f(x) = x^2$ **b** $f(x) = x^3$ **c** $f(x) = \frac{1}{2}x$

Unit Two

2 - 5 Logarithmic Function and Its Graph

We will learn

- Definition of the logarithmic function .
- The graphical representation of the logarithmic function
- Transformation from the exponential from to the logarithmic form and conversely.
- Solving simple exponential equation

Key - term

- Logarithm
- Inverse Function
- Domain
- Common Logarithm

Matrials

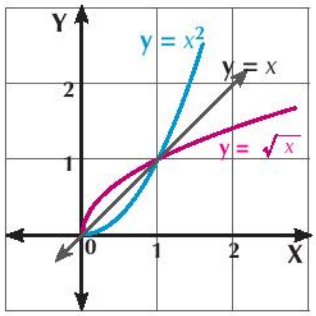
- Scientific Cal.
- Computer

Note

$\text{Log}_a\, x = y$ is a logarithmic form, its corresponding exponential form is $a^y = x$

Graphical representation of the inverse function of the exponential function

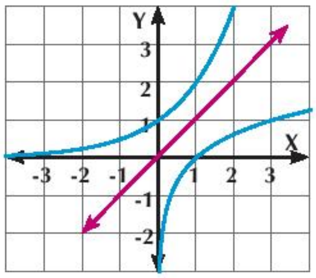
Discover



You know that the function $y = \sqrt{x}$ is the inverse function of $y = x^2$ for $x > 0$ (its image by reflection in the line $y = x$)

Can you represent the inverse function of the exponential function f such that $f(x) = 2^x$ graphically by representing the values of x, y for the ordered pairs of the function.

$y = 2^x$		$x = 2^y$	
x	y	x	y
-3	$\frac{1}{8}$	$\frac{1}{8}$	-3
-2	$\frac{1}{4}$	$\frac{1}{4}$	-2
-1	$\frac{1}{2}$	$\frac{1}{2}$	-1
0	1	1	0
1	2	2	1
2	4	4	2
3	8	8	3



We find that the inverse of $y = 2^x$ is $x = 2^y$ and the variable y in the equation $x = 2^y$ is called logarithm x. **and it's written as** $y = \text{Log}_a\, x$ **and it's read as logarithm x to the base a**

Learn

Logarithmic Function

If $a \in R^+ - \{1\}$ then the function $f: R^+ \longrightarrow R$ where $f(x) = \log_a x$ is the inverse function of the exponential function $y = a^x$

$f(x) = \text{Log}_a\, x$ is called the logarithmic function

- Domain of logarithmic function = R^+
- Range of logarithmic function = R
- The form $y = \text{Log}_a\, x$ is equivalent to $a^y = x$

Converting to the logarithmic form:

(a) $2^4 = 16$ equivalent $\log_2 16 = 4$ (b) $5^2 = 25$ equivalent $\log_5 25 = 2$

(c) $(\frac{1}{2})^4 = \frac{1}{16}$ equivalent $\log_{\frac{1}{2}} \frac{1}{16} = 4$ (d) $10^{-2} = 0.01$ equivalent $\log_{10} 0.01 = -2$

Try to solve

(1) put each of the following in the logarithmic form

(a) $7^2 = 49$ (b) $(\sqrt{2})^{10} = 32$ (c) $(\frac{3}{5})^4 = \frac{81}{625}$ (d) $5^{-3} = \frac{1}{125}$

The common logarithms of base 10.

If the base of the logarithm is 10 it is named by common logarithm and written without base such as $\log_{10} 7$ is written Log 7, $\log_{10} 127$ is written log 127

Change to the exponential form:

(a) $\log_3 81 = 4$ equivalent $3^4 = 81$ (b) $\log_2 128 = 7$ equivalent $2^7 = 128$

(c) $\log \frac{1}{100} = -2$ equivalent $10^{-2} = \frac{1}{100}$ (d) $\log_{81} 27 = \frac{3}{4}$ equivalent $81^{\frac{3}{4}} = 27$

Try to solve

(2) Put in exponential form:

(a) $\log_5 125 = 3$ (b) $\log_3 \frac{1}{243} = -5$ (c) $\log_4 1 = 0$ (d) $\log 1000 = 3$

Evaluating the value of the logarithmic form of given base:

Example

(1) Find the value of:

(a) $\log 0.001$ (b) $\log_3 \sqrt[4]{27}$

Solution

(a) **Putting $y = \log 0.001$**

change to exponential form:

$10^y = 0.001$

$10^y = (\frac{1}{10})^3$ from the properties of exponents

$10^y = (10)^{-3}$ from properties in exponents

$y = -3$ thus $\log 0.001 = -3$

(b) **Putting $y = \log_3 \sqrt[4]{27}$**

Change to exponential form

$3^y = 3^{\frac{3}{4}}$ from the properties of exponents

$y = \frac{3}{4}$

$\therefore \log_3 \sqrt[4]{27} = \frac{3}{4}$

Try to solve

(3) Find the value of: (a) $\log 0.00001$ (b) $\log_{\frac{1}{2}} 128$

Example

2 Find in R the solution set of each of the following equations:

a $\log_3 (2x - 5) = 1$ b $\log_x (x + 2) = 2$

Solution

a The equation is valid when $2x - 5 > 0$ i.e. $x > \frac{5}{2}$ (the equation validity domain)

converting the equation into exponential form

$\therefore 3^1 = 2x - 5$ $\therefore 2x = 8$

$\therefore x = 4 \in$ the equation validity domain $\therefore$ solution set = $\{4\}$

b The equation is valid when $x \begin{cases} x + 2 > 0 \\ x > 0 \\ x \neq 1 \end{cases}$ **i.e.** $\begin{cases} x > -2 \\ x > 0 \\ x \neq 1 \end{cases}$

then $]0,\infty[- \{1\}$ (the equation validity domain)

converting the equation into the exponential form

$\therefore x^2 = x + 2$ $\therefore x^2 - x - 2 = 0$

$\therefore (x - 2)(x + 1) = 0$ $\therefore x = 2$ or $x = -1$

$\because x = -1 \notin$ the equation validity domain $\therefore$ solution set = $\{2\}$

Try to solve

4 Find in R the solution set of each of the following equations:

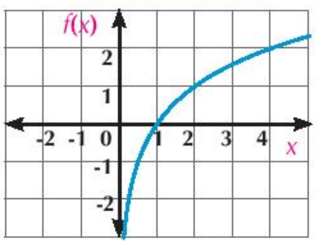
a $\log_{81} x = \frac{3}{4}$ b $\log_x 5x = 2$

Learn

Graphical Representation of the Logarithmic Function

The function f such that $f(x) = \log_a x$, $a \neq 1$ is represented graphically as in the following figures:

when a > 1



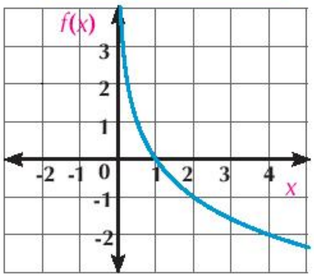
The domain: R^+

The Range: R

Intersection With x-axis: (1, 0)

Increasing on: R^+

when $0 < a < 1$



The domain: R^+

The Range: R

Intersection With x-axis: (1, 0)

Decreasing on: R^+

Critical thinking: Can you deduce the relation between the exponential functional and logarithmic function? Show this.

Example

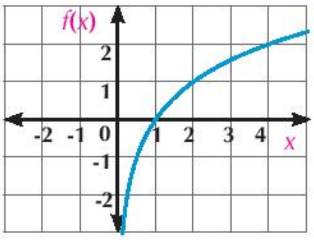
3) Represent the following functions graphically:

a) $f(x) = \text{Log}_2 x$ b) $f(x) = \text{Log}_{\frac{1}{2}} x$

Solution

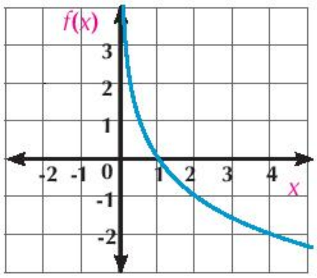
a) **Notice that the base** $2 > 1$

x	2	1	4
$f(x)$	1	0	2



b) **Notice that:** the base: $0 < \frac{1}{2} < 1$

x	2	1	4
$f(x)$	-1	0	-2



Using the calculator:

The scientific calculator can be used to find the values of the logarithms as follows:

1) to find $\text{Log}_2 4$ we press the key with the following sequence:

log 2 REPLAY 4 = 2

2) to find Log 38 we press the keys with the following sequence:

log 3 8 = 1.579783597

Drill :

Using the calculator to find

a) $\text{Log}_3 12$ b) $\text{Log}_3 24$ b) $\text{Log}_{\sqrt{3}} \frac{1}{27}$ d) Log 128

Exercises 2 - 5

1 Express each of the following in the equivalent logarithmic form:

a $3^{-2} = \frac{1}{9}$ **b** $(\frac{2}{5})^4 = \frac{16}{625}$ **c** $5^0 = 1$ **d** $(\sqrt{2})^4 = 4$

2 Express each of the following in the equivalent exponential form:

a $\text{Log } 100 = 2$ **b** $\text{Log}_2 4\sqrt{2} = \frac{5}{2}$ **c** $\text{Log}_7 1 = 0$ **d** $\text{Log}_{\sqrt{11}} 121 = 4$

3 Find the domain of each of the following functions :

a $f(x) = \text{Log}_3 (2x + 1)$ **b** $f(x) = 2\text{Log}x$ **c** $f(x) = \text{Log}_{(5-x)} (x-3)$

4 Without using calculator find the value of:

a $\text{Log}_2 16$ **b** $\text{Log}_{\sqrt{5}} 5$ **c** $\text{Log}_8 1$ **d** $\text{Log}_3 3\sqrt{3}$

5 Find in R the solution set of each of the following equations:

a $\text{Log}_3 27 = x + 2$ **b** $\text{Log}_x (2x + 3) = 2$ **c** $\text{Log } (2x + 1) = 0$

d $\text{Log}_2 (\text{Log}_3 x) = 1$ **e** $\text{Log}_4 [13 + \text{Log}_2 (x - 1)] = 2$ **f** $\text{Log}_2 (4^x - 2) = x$

6 Represent graphically each of the following functions:

a $f(x) = \text{Log}_3 x$

x	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$
$f(x)$					

b $f(x) = \text{Log}_{\frac{1}{2}} (x+1)$

x	$-\frac{7}{8}$	$-\frac{1}{2}$	0	1	3	7
$f(x)$						

7 Draw in one diagram the curves of each of the two functions g, f where $g(x) = \text{Log}_2 x$, $f(x) = 6 - x$, then use the graph to find the solution set of the equation $\text{Log}_2 x = 6 - x$.

Choose the correct answer:

8 If $\text{Log}_3 x = 2$ then $x =$ ....

a 9 **b** 8 **c** 3 **d** 5

9 If $\text{Log}_a 16 = 4$ then a ∈ ....

a {16} **b** {2} **c** {2, -2} **d** {1}

10 $\text{Log}_5 125 =$ ....

a $\sqrt{3}$ **b** 3 **c** 5 **d** 125

(11) The domain of the function f such that $f(x) = \text{Log}_{(1-x)} 3$ is ....

(a) $]-\infty, 0[\cup]0, 1[$ (b) $]-\infty, 1[$ (c) $]1, \infty[$ (d) $]-1, 1[$

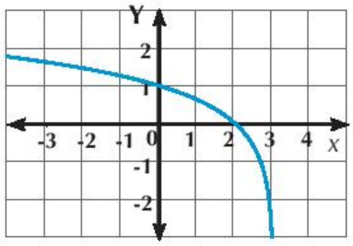
(12) Log100 = ...

(a) 1 (b) 2 (c) 3 (d) -1

(13) If the curve of the function f where $f(x) = \text{Log}_a x$ passes through the point (8, 3): then $f(4)$ = ...

(a) 1 (b) 2 (c) $\frac{1}{4}$ (d) -2

(14) the opposite figure represents the function ....



(a) $y = 3^{x-1}$ (b) $y = 3^{x+1}$

(c) $y = \text{Log}_3(2-x)$ (d) $y = \text{Log}_3(3-x)$

(15) Find the value of each of the following then check the result by using calculator:

(a) $\text{Log}_3 81$ (b) $\text{Log}_2 \frac{1}{8}$ (c) $\text{Log}_{\sqrt{7}} 343$ (d) $\text{Log}\ 0.001$

(16) Find the value of each of the following then check the result by using calculator:

(a) $\text{Log}_{81} x = \frac{3}{4}$ (b) $\text{Log}_3(2x - 5) = 0$ (c) $\text{Log}_x(x + 6) = 2$

(d) $\text{Log}_5 \text{Log}_2 \text{Log}_3 x = 0$ (e) $\text{Log}_3(\frac{x^2}{2x-3}) = 1$ (f) $\text{Log}_5 |2x + 1| = 1$

(17) **Education:** if the relation between retention of materials of a student in the first secondary form and the number of months (t) starting from the end of study of the class is: $f(t) = 70 - 4\ \text{Log}_2 (t + 1)$

find the score of the student:

first: at the end of the study of the class (t = 0)

second: after 7 months from the end of the study of the class.

(18) **Application:** In a study to measure the students retain what has been studied in a certain subject they re - examed from time to time in the same subject. If the student score follows the relation $f(t) = 85 - 25\ \text{Log}(t + 1)$, where t is the period after studying in monthes, f(t) is the student score in percentage. Find:

(a) The score of the student in the first exam for this subject.

(b) The score of the student after 3 months from studying this subject.

(c) The score of the student after one year from studying this subject.

Unit Two

2 - 6 Some Properties of Logarithms

We will learn

- Using properties of logarithms
- Solving logarithmic equations.
- Using scientific cal. in solving exponential equations.
- Life applications on logarithms.

Key - term

- Logarithmic Equations
- Richter Scale

Matrials

- Scientific Ca.
- Computer and a graph program

Discover

Use the calculator to evaluate each of :

1) $(\log_2 4 + \log_2 8)$, $\log_2 32$ **2)** $(\log 40 + \log \frac{5}{2})$, $\log 100$

3) $(\log_2 27 - \log_3 9)$, $\log_3 3$ What do you deduce?

Learn

Some Properties of Logarithms

If: $a \in R^+ - \{1\}$ **1)** $\log_a a = 1$ **2)** $\log_a 1 = 0$

Try to prove each of **1** and **2** using the definition of logarithm.

3) Multiplication property in logarithms:

$\log_a xy = \log_a x + \log_a y$ where $x, y \in R^+$

To prove this relation:

put $b = \log_a x$, $c = \log_a y$

and from the definition of logarithm:

$x = a^b$, $y = a^c$

then $xy = a^b \times a^c$ **i.e.** $xy = a^{b+c}$

converting the last form into logarithmic: $\log_a xy = b + c$

By Substituting the values of b and c, **we get** $\log_a xy = \log_a x + \log_a y$

Example

(1) Find the value of $\log_2 10$ in the simplest form. If the value of $\log_2 5 \simeq 2.3219$, verify your answer using the calculator.

Solution

$\log_2 10 = \log_2 (2 \times 5)$

$= \log_2 2 + \log_2 5$ **using the multiplication property**

$= 1 + 2.3219 \simeq 3.3219$ **using property (1) and by substituting $\log_2 5 \simeq 2.3219$**

Check by using calculator:

log 2 REPLAY 1 0 = 3.321928095

Try to solve

1) Find the value of $\log_3 15$ in the simplest form, if $\log_3 5 \simeq 1.465$. verify the result using calculator.

4) The division property in logarithms

$\log_a \frac{x}{y} = \log_a x - \log_a y$ where $x, y \in R^+$ **(Try to prove this relation)**

Example

2) Find the value of the expression: Log30 - Log3 .

Solution

$\log 30 - \log 3 = \log \frac{30}{3} = \log 10 = 1$

Try to solve

2) By using the division property in logarithms prove that: Log2 = 1 - Log5

5) The power property:

$\log_a x^n = n \log_a x$ **where** $n \in R$, $x > 0$, $a \in R^+$, $a \neq 1$

Example

3) Find in the simplest form the value of $\log_5 \sqrt[4]{125}$

Solution

$\log_5 \sqrt[4]{125} = \log_5 (5)^{\frac{3}{4}} = \frac{3}{4} \log_5 5 = \frac{3}{4} \times 1 = \frac{3}{4}$

Notice

$(\sqrt[4]{125} = \sqrt[4]{5^3} = 5^{\frac{3}{4}})$

Try to solve

(3) Simplify

$\log_3 \sqrt[7]{243}$, $\log_7 \sqrt[4]{343}$

Notice that: $\log_a(\frac{1}{x}) = -\log_a x$ where $x \in R^+$

6) Base changing property:

If $x \in R^+$ and y , $a \in R^+ - \{1\}$, prove that: $\log_y x = \frac{\log_a x}{\log_a y}$

proof **(not required)**

let: $z = \log_y x$

$y^z = x$ converting into the exponential form

$z \log_a y = \log_a x$ by taking $\log_a$ to both sides

then $z = \frac{\log_a x}{\log_a y}$ **i.e:** $\log_y x = \frac{\log_a x}{\log_a y}$

Try to solve

(4) use property 6 to find the value of: (a) $\log_4 8$ (b) $\log_9 243$

7- The multiplicative inverse property: $\log_a b = \frac{1}{\log_b a}$

Critical thinking: If a, b $\in R^+ - \{1\}$ prove that $\log_a b = \frac{1}{\log_b a}$ hence find the value of: $\log_3 7 \times \log_7 3$ in simplest form.

Simplifying logarithmic expressions

Example

(4) Simplify:

(a) $2 \log 25 + \log(\frac{1}{3} + \frac{1}{5}) + 2 \log 3 - \log 30$ (b) $\log_5 49 \times \log_8 5 \times \log_9 8 \times \log_7 9$

Solution

a The expression $= \text{Log } 25^2 + \text{Log}\frac{8}{15} + \text{Log}3^2 - \text{Log}30$ **property 5**

$= \text{Log } (25^2 \times \frac{8}{15} \times 3^2 \times \frac{1}{30})$ **properties 3, 4**

$= \text{Log}100 = 2$

b The expression $= \frac{\text{Log}49}{\text{Log}5} \times \frac{\text{Log}5}{\text{Log}8} \times \frac{\text{Log}8}{\text{Log}9} \times \frac{\text{Log}9}{\text{Log}7}$ **properties 6**

$= \frac{\text{Log}49}{\text{Log}7} = \frac{2\text{Log}7}{\text{Log}7} = 2$

Try to solve

5) Simplify: $\text{Log}0.009 - \text{Log}\frac{27}{16} + \text{Log } 15\frac{5}{8} - \text{Log }\frac{1}{12}$

6) Prove that: $\frac{\text{Log}729 - \text{Log}64}{\text{Log}9 - \text{Log}4} = 3$

7) If $x^2 + y^2 = 8x\,y$, prove that: $2\text{ Log}(x + y) = 1 + \text{Log x} + \text{Log y}$

Solving Logarithmic Equations

Example

5) Find the solution set of each of the following equations In R:

a $\text{Log}_3(x - 1) + \text{Log}_3(x + 1) = \text{Log}_3 8$ **b** $\text{Log}_3 x + \text{Log}_x 3 = 2$

Solution

a The equation is valid $x \in \{x{:}x - 1 > 0\} \cap \{x{:}x + 1 > 0\}$

then $x > 1$ (equation validity domain)

$\text{Log}_3(x - 1) + \text{Log}_3(x + 1) = \text{Log}_3 8$

$\therefore \text{Log}_3(x - 1)(x + 1) = \text{Log}_3 8$ **property 3**

$\therefore x^2 - 1 = 8$ $\therefore x^2 = 9$ then $x = \pm 3$

$x = -3 \notin$ equation validity domain $\therefore$ Solution set $= \{3\}$

b The equation is valid at $x > 0$, $x \neq 1$

$\therefore \text{Log}_3 x + \frac{1}{\text{Log}_3 x} = 2$ **property 7**

$\therefore (\text{Log}_3 x)^2 + 1 = 2\text{Log}_3 x$ **multiply by $\text{Log}_3 x$**

$\therefore (\text{Log}_3 x)^2 - 2\text{Log}_3 x + 1 = 0$ $\therefore (\text{Log}_3 x - 1)^2 = 0$

$\therefore \text{Log}_3 x = 1$ $\therefore x = 3 \in$ the equation validity domain

$\therefore$ Solution set $= \{3\}$

Try to solve

8. Find the solution set of each of the following equations In R:

a) $\text{Log}_3 x + \text{Log}_3 (x + 2) = 1$ b) $\text{Log}(8 - x) + 2\text{Log}\sqrt{x - 6} = 0$ c) $\text{Log}x - \text{Log}_x 100 = 1$

Solving Exponential Equations Using Logarithms

Example using calculator to solve exponential equations

6. Find the value of x in each of the following (Round the result to the nearest hundredth).

a) $2^{x+1} = 5$ b) $5^{x-2} = 3 \times 4^{x+1}$

Solution

a) $2^{x+1} = 5$ **by taking log to both sides**

$\therefore \text{Log}2^{x+1} = \text{Log}5$ $\therefore (x + 1)\text{ Log}2 = \text{Log}5$

$\therefore x + 1 = \dfrac{\text{Log}5}{\text{Log}2}$ **i.e. :** $x = \dfrac{\text{Log}5}{\text{LOg}2} - 1$ $\therefore x \simeq 1.32$

Using the calculator:

log 5 (÷ log 2) - 1 = 1.321928095

b) $5^{x-2} = 3 \times 4^{x+1}$ **by taking log to both sides**

$\therefore \text{Log}5^{x-2} = \text{Log}(3 \times 4^{x+1})$ $\therefore \text{Log}5^{x-2} = \text{Log}3 + \text{Log}4^{x+1}$

$\therefore (x - 2)\text{ Log}5 = \text{Log}3 + (x + 1)\text{ Log}4$ $\therefore x\text{ Log}5 - 2\text{Log}5 = \text{Log}3 + x\text{ Log}4 + \text{Log}4$

$\therefore x\text{ Log}5 - x\text{ Log}4 = \text{Log}3 + \text{Log}4 + 2\text{Log}5$ $\therefore x(\text{Log}5 - \text{Log}4) = \text{Log}3 + \text{Log}4 + 2\text{Log}5$

$$\therefore x = \frac{\text{Log}3 + \text{Log}4 + 2\text{Log}5}{\text{Log}5 - \text{Log}4} \simeq 25.56$$

Using the calculator:

log 3 (+ log 4 (+ 2 log 5 (= ÷) log) 5 (- log 4 = 25.56104553

Try to solve

9. Find the value of x in each of the following approximating the result to the nearest $\frac{1}{10}$:

a) $3^{7-2x} = 13.4$ b) $7^{x-2} = 4^{x+3}$

Example Applications on logarithmic laws

(7) **Geology:** If the magnitude of the intensity M(I) of an earthquake on Richter scale is given by $M(I) = \text{Log}\left(\frac{I}{I_\circ}\right)$, where I is the earthquake intensity, $I_\circ$ represents the smallest earth movement that can be recorded, called the reference intensity.

(a) Find on Richter scale the magnitude of the earthquake of intensity 10^6 times the reference intensity .

(b) In 1989 An earthquake measuring 7.1 on Richter scale occurred. Determine its intensity.

Solution

(a) $\because M = \text{Log}\left(\frac{I}{I_\circ}\right)$, $I_\circ = 10^6\, I_\circ$

$$\therefore M = \text{Log}\left(\frac{10^6 I_\circ}{I_\circ}\right) = \text{Log}10^6 = 6\,\text{Log}10 = 6$$

i.e. the magnitude of the earthquake on the Richter scale is 6.

(b) $\because M = 10$

$\therefore \quad 7.1 = \text{Log}\left(\frac{I}{I_\circ}\right) \qquad \therefore \quad \frac{I}{I_\circ} = 10^{7.1}$

$\therefore I = 10^{7.1}\, I_\circ$

i.e. the earthquake intensity is 12590000 times the reference intensity.

Try to solve

(10) If the population of a city starting from 2010 is given by $N = 10^5\,(1.3)^{t-2010}$, where N is the number population , t the year

(a) Find the population of this city in 2015.

(b) In which year the population of this city is 1.4 million people.

Exercises 2 - 6

1 Without using calculator find

a) $\log 1000$ b) $\log_2 32$ c) $\log_{\frac{1}{4}} 16$ d) $\log_7 49$

e) $\log 0.001$ f) $\log_8 2$ g) $\log_8 1$ h) $\log_y \sqrt{y}$

2 Simplify to the simplest form

a) $\log 2 + \log 5$ b) $\log_5 15 - \log_5 3$ c) $\dfrac{\log 25}{\log 5}$

d) $\log_2 5 \times \log_5 2$ e) $\log 54 - 3\log 3 - \log 2$ f) $1 + \log 3 - \log 2 - \log 15$

g) $\log_{abc} a + \log_{abc} b + \log_{abc} c$ h) $\dfrac{1}{\log_2 12} + \dfrac{1}{\log_8 12} + \dfrac{1}{\log_9 12}$

3 If $x, y \in R_+$, $a, b \in R_+ - \{1\}$ Put (✔) in front of the correct statement and (✘) in front of the incorrect statement :

a) $\log_a(x + y) = \log_a x + \log_a y$ ( )

b) $\log_a(x + y) = \log_a x \times \log_a y$ ( )

c) $\log_a(x y) = \log_a x + \log_a y$ ( )

d) $\log 2x^5 = 5\log 2x$ ( )

e) $\log_a(\frac{x}{y}) = \log_a x + \log_a y^{-1}$ ( )

f) $\dfrac{\log_a x}{\log_a y} = \dfrac{\log_b x}{\log_b y}$ ( )

g) If $x > 0$ then $\log_a x^4 = 4\log_a x$ ( )

4 If $\log 2 = x$, $\log 3 = y$ find in terms of x, y each of: $\log 6$, $\log_{18} 12$

5 Find the value of x in each of the following approximate the result to the nearest hundredth.

a) $7^{3x-2} = 5$ b) $7^{x+1} = 3^{x-2}$ c) $\dfrac{5}{10^{2x}} = 7$ d) $x^{\log x} = 100x$

6 Find in R the solution set of each of the following equations:

a) $\log_4 x = 1 - \log_4 (x - 3)$ b) $\log_3 (x + 6) = 2\log_3 x$ c) $\log(x + 8) - \log(x - 1) = 1$

d) $(\log x)^2 - \log x^2 = 3$ e) $(\log x)^3 = \log x^9$ f) $3^{\log x} = 2^{\log 3}$

g) $\log_3 x = \log_x 3$ h) $\log_2 x + \log_x 2 = 2$ i) $\log_2 (2^x - 4) + x - 5 = 0$

7 Use the calculator to calculate:

a) $\log 3.15$ b) $\log_2 25$ c) $2\log 5 - 3\log 7$ d) $\dfrac{3^{150} \times 5^{200}}{7^{250}}$

8 **Discover the Error:** Amira and Esraa solved the problem : simplify: $\text{Log}\,x^3 + \text{Log}\,y^4 - \text{Log}\,x\,y^2$

Esraa's solution

the expression $= \log \frac{x^3 \times y^4}{x\,y^2} = \log x^2 y^2$

$= \log (x\,y)^2 \quad = 2 \log x\,y$

$= 2 (\log x + \log y)$

Amira's solution

the expression $= 3 \log x + 4 \log y - 2 \log x\,y$

$= 3 \log x + 4 \log y - 2 (\log x + \log y)$

$= 3 \log x + 4 \log y - 2 \log x - 2 \log y$

$= \log x + 2 \log y$

which answer is correct ? why?

9 **Creative thinking:** without using the calculator calculate:

$\text{Log}\,(\tan 1^\circ) + \text{Log}\,(\tan 2^\circ) + \text{Log}\,(\tan 3^\circ) + \ldots\ldots + \text{Log}\,(\tan 89^\circ)$

General Exercises

For more exercises, please visit the website of Ministry of Education.

Unit three

Limits and continuity

Unit preface

First ideas of calculus appeared in the works of mathematical greek "Archimedes" who has developed a number of laws in geometry such as volume and surface area of the sphere using some ways which considered as a beginning to these methods that used in integration, and in ١٦th and ١٧th century A.D lots of mathematical scientists were busy studying problems which related to calculus until each of Newton and leibniz has discovered the basic theory of differentiation and integral calculus. Calculus is the branch of mathematics that concerned with limits, derivative, integration and infinite series It is the science that used to study the change in the function and analyze it . We find that the calculus related to lots of applications in geometry and various sciences that often needed to study the behavior and the change of the function and to solve the problems that can not be solved easily by algebra

Unit objectives

By the end of this unit , the student should be able to :

- Recognize an introduction of limits.
- Recognize some unspecified quantities like $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, ...
- Determine a method to calculate the limit : direct substitution - factorization - long division - multiply by the conjugate.
- Finding a limit of a function using the law $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = n\, a^{n-1}$
- Deduce the limit of a function using the law $\lim\limits_{x \to a} \frac{x^n - a^n}{x^m - a^m} = n\, a^{n-m}$
- Finding the limit of a function at infinity.
- Finding the limits of trigonometric functions.
- Use the graphic calculator to verify the result of a limit of a function as an activity.
- Recognize the right hand limit and the left hand limit .
- Recognize the definition of the continuity.
- Recognize the continuity of a function at a point- continuity of a function at an interval - some types of continuous function - redefine some discontinues function to be continuous.
- Recognize applications on the concepts of limits, continuity (exercise and activities).

Key terms

- Unspecified quantity
- Undefined
- Right limit
- Left limit
- Limit of a function
- Direct substitution
- Polynomial function
- Limit of a function at infinity
- Trigonometric function
- Limit of a trigonometric function
- Continuity of a function

Materials

Scientific calculator - computer - graphic programs

Lessons of the unit

Lesson (3 - 1): Introduction to limits of functions.

Lesson (3 - 2): Finding the limit of a function algebraically.

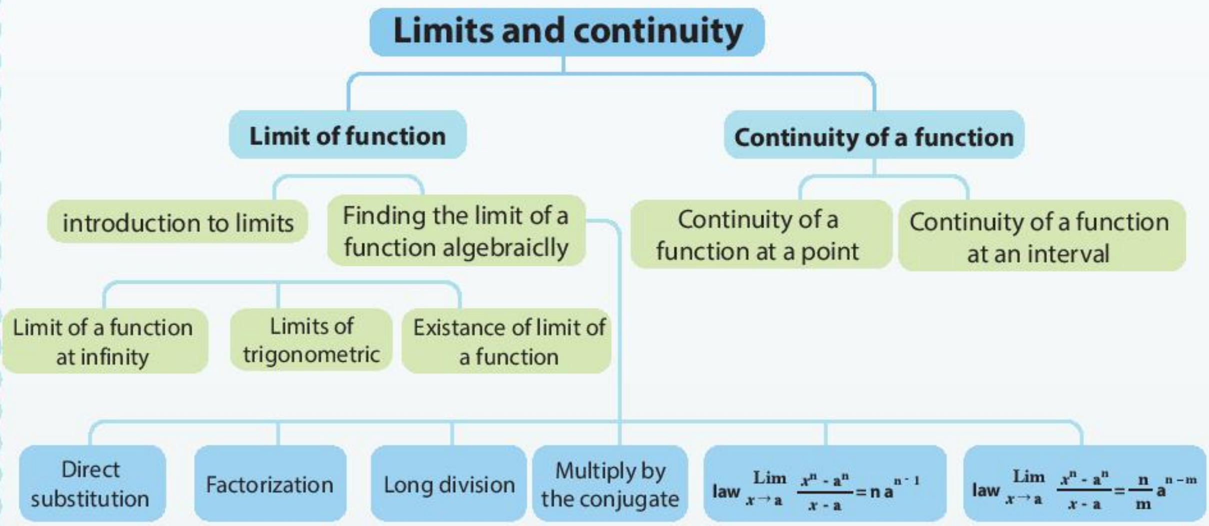
Lesson (3 - 3): Limit of a function at infinity.

Lesson (3 - 4): Limits of trigonometric functions.

Lesson (3 - 5): Existance of limit of a function at a point .

Lesson (3 - 6): Continuity.

Chart of the unit



Unit three

3 - 1 Introduction to Limits of Functions

You will learn

- Unspecified quantities
- Limit of a function at a point

Key terms

- Unspecified quantities
- Undefine
- Extended real numbers
- Right limit
- Left limit
- Value of a function
- Limit of a function

Materials

- Scientific calculator
- Graphic programs

The concept of the limit of a function at a point is one of the basic concepts in Calculus. In this unit we will recognize the concept of the limit of the function graphically and algebraically. But before that, let's identify the types of quantities in the set of real numbers.

Remember

$+\infty$ is a symbol that indicates unbounded quantity larger than any real number that can be imagined.

Think and discuss

Find the result of each of the following (if possible):

1. 3×5
2. $28 \div 4$
3. $4 - 9$
4. $7 \div 0$
5. $0 \div 0$
6. $\infty + 3$
7. $\infty \div \infty$
8. $\infty - \infty$

Unspecified quantities

Learn

In (think and discuss) we see that some results of operations are completely determined like number 1, 2, 3 while the other operations have no specified result.

Notice that $7 \div 0$ undefined (the division by zero meaningless) also the operation $0 \div 0$ has no specified result because there exist infinite number of numbers which multiplied by zero to give zero .

So the quantity $\frac{0}{0}$ is called unspecified quantity, similarly the quantitics $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$ **(why) ?**

Add to your informations

The operations on $\mathbb{R}$ and two symbols ∞, $-\infty$ is performed as follows for all $a \in \mathbb{R}$, then :

1- $\infty + a = \infty$

2- $-\infty + a = -\infty$

3- $\infty \times a = \begin{cases} \infty & \text{when } a > 0 \\ -\infty & \text{when } a < 0 \end{cases}$

4- $-\infty \times a = \begin{cases} \infty & \text{when } a < 0 \\ -\infty & \text{when } a > 0 \end{cases}$

Example

1. Find (if possible) the result of each of the following operations:

a) $4 + \infty$ b) $3 - \infty$ c) $0 \div 3$ d) $-5 \div 0$
e) $\infty + \infty$ f) $0 \div 0$ g) $5 \times \infty$ h) $-6 \times -\infty$

Solution

a) ∞ b) $-\infty$ c) 0 d) undefined
e) ∞ f) unspecified g) ∞ h) ∞

Try to solve

1. Find (if possible) the result of each of the following operations:

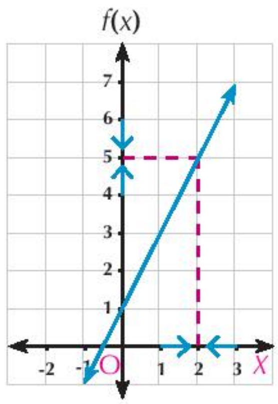
a) $0 \div (-2)$ b) $7 \div 0$ c) $9 \div \infty$ d) $\infty \times 0$
e) $(-7) \times \infty$ f) $(-\infty) + 12$ g) $\infty + \infty$ h) $\infty \div \infty$

Limit of a function at a point:

Study the values of the function f where $f(x) = 2x + 1$ when x close to 2 from the next data:

$x > 2$	$f(x)$
2.1	5.2
2.01	5.02
2.001	5.002
2.0001	5.0002
....	....
↓	↓
$x \to 2^+$	$f(2^+) \to 5$

$x < 2$	$f(x)$
1.9	4.8
1.99	4.98
1.999	4.998
1.9999	4.9998
....	....
↓	↓
$x \to 2^-$	$f(2^-) \to 5$



We see that:

when x approaches to **2** from the right and from the left then, $f(x)$ approaches to **5**. We express that mathematically as $\lim_{x \to 2} (2x + 1) = 5$ and the graphical representation of f illustrating that.

Definition 1

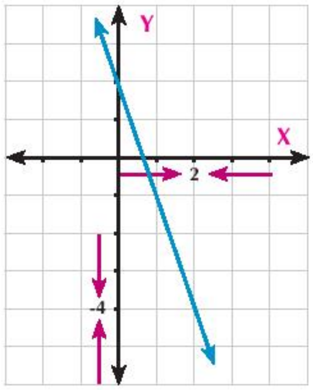
If the value of the function $f(x)$ approaches to the real number l when x close to the real number a, then $\lim_{x \to a} f(x) = l$

And it is read limit $f(x)$ when x approaches to the real number a equals l

Example **Estimating the limit (The limit is equal to the value of the function)**

(2) Estimate $\lim_{x\to 2} (2 - 3x)$ graphically and numerically.

Solution



Algebraic: The linear function : $y = 2 - 3x$ represented graphically as the opposite graph:

From the graph we see:

When $x \to 2$ then $f(x) \to -4$

i.e $\lim_{x\to 2} (2 - 3x) = -4$

Numerical: Form the table of the values $f(x)$ by choosing x values near to the number 2 from the right side and the left side as follows:

x	2.1	2.01	2.001	⟶	2	⟵	1.999	1.99	1.9
$f(x)$	-4.3	-4.03	-4.003	⟶	-4	⟵	- 3.997	- 3.97	-3.7

➢ From the table when x approaches to the number 2 from right and from the left, The values of f(x) approaches to the number –4

Try to solve

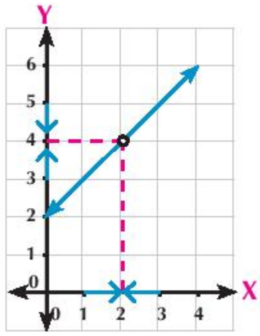
(2) Estimate the following limits graphically and numerically:

(A) $\lim_{x\to 2} (1 - 3x)$ (B) $\lim_{x\to 0} (x^2 - 2)$

Example **Estimating the limit (The limit isn't equal to the value of the function)**

(3) Estimate graphically and algebrically $\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$.

Solution



Graphically: The opposite figure represents : $f(x) = \frac{x^2 - 4}{x - 2}$ where $x \neq 2$.

From the graph we notice that when $x \to 2$ then $f(x) \to 4$

then: $\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = 4$

Algebrcally: form a table to the values $f(x)$, by choosing values of x closer to 2.

x	2.1	2.01	2.001	⟶	2	⟵	1.999	1.99	1.9
$f(x)$	4.1	4.01	4.001	⟶	4	⟵	3.999	3.99	**3.9**

➢ The table shows that $f(x) \to 4$ when $x \to 2$ from the right and from the left.

From the previous example we notice that :

1- The hall on the graph represents unspecified quantity $(\frac{0}{0})$, at $x = 2$

2- It is not necessary that the function is defined at x = 2 to have a limit as x ⟶ 2

Try to solve

(3) Estimate graphically and algebrically the limit of each :

(A) $\lim\limits_{x \to 2} \frac{x^2 - 1}{x + 1}$ (B) $\lim\limits_{x \to 3} \frac{x^2 - 2x - 3}{x - 3}$

Using of technology to find the limit of a function at a point (graphic calculator)

Activity

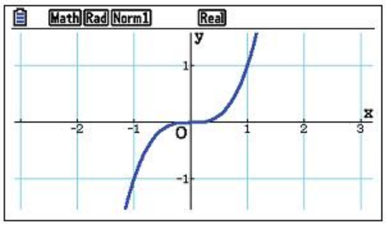
Use the graphic calculator to draw the curve of the function f, then estimate the limit of the function at the specific point.

1) $f(x) = x^3$ when $x \longrightarrow$ zero

2) $f(x) = (\frac{x^3 - 1}{x - 1}) - 2$ when $x \longrightarrow 1$

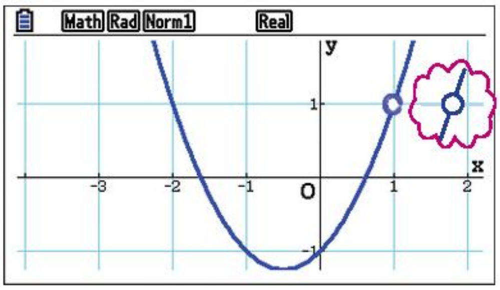
3) $f(x) = \frac{\sin\ x}{x}$ when $x \longrightarrow$ zero

We can use the graphic calculator or a graphic program like (*Geogebra*) in a computer or in a tablet to draw the function curve as follows:



1) Using the graphic calculator, represent the curve of the function f where $f(x) = x^3$

From the graph $\lim\limits_{x \to 0} f(x) =$ zero

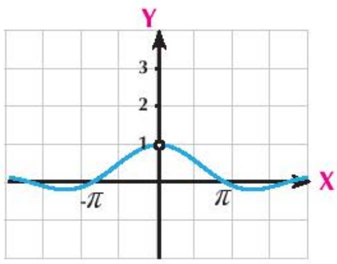


2) Using the graphic calculator, represent the curve of the function f where

$f(x) = (\frac{x^3 - 1}{x - 1}) - 2$

From the graph $\lim\limits_{x \to 1} f(x) = 1$

Note the hall at the point (1 , 1))



3) Using the graphic calculator, represent the curve of the function f where:

$f(x) = \frac{\sin x}{x}$

From the graph $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

We conclude from the previous activity:

The existence of $\lim\limits_{x \to a} f(x)$ doesn't necessary mean the function be defined at $x = a$

Creative thinking: If the function f is defined at $x = a$ does it mean that it has a limit at a. Explain your answer.

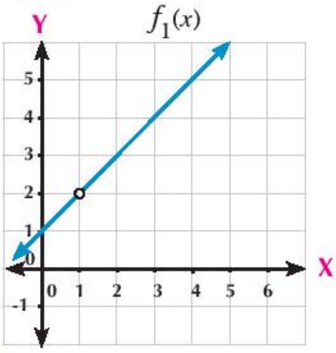
Exercise on the activity: Using the graphic calculator or with a graphic program in a computer or a tablet, estimate all of the following:

a $\lim\limits_{x \to 0} (2 - x^2)$ **b** $\lim\limits_{x \to -2} \frac{x^3 + 8}{x + 2}$ **c** $\lim\limits_{x \to 0} \frac{\sin 3x}{x}$

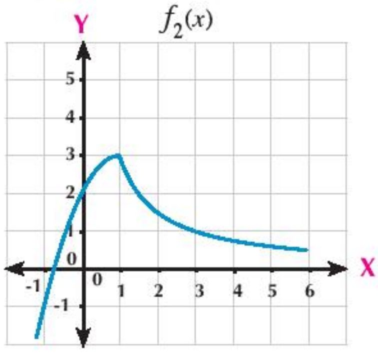
Exercises 3 - 1

1 Estimate the limit of each of the following functions as $x \longrightarrow 1$

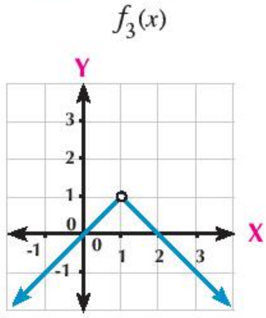
a



b

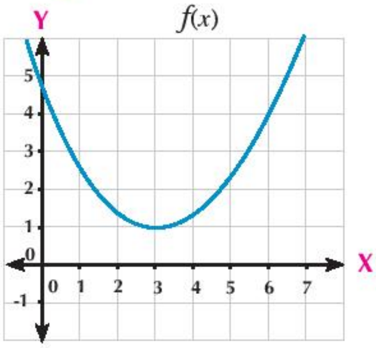


c



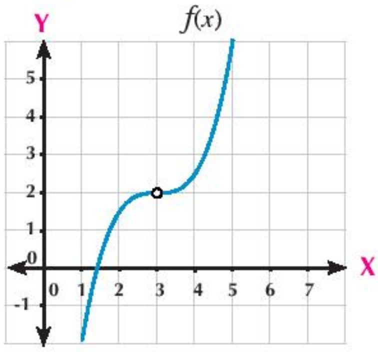
2 Estimate the limit of each of the following functions at the indicated point:

a



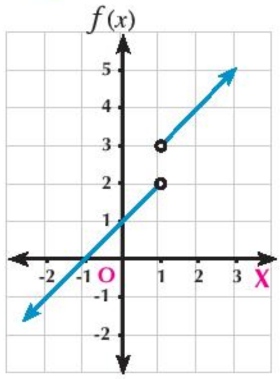
$\lim\limits_{x \to 3} f(x) = \ldots\ldots\ldots$

b



$\lim\limits_{x \to 3} f(x) = \ldots\ldots\ldots$

c

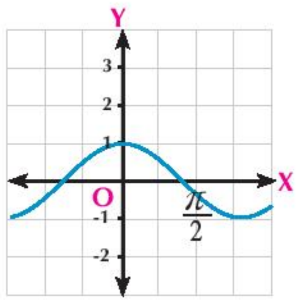


$\lim\limits_{x \to 1} f(x) = \ldots\ldots\ldots$

3 From the opposite graph, find:

a $\lim\limits_{x \to 0} f(x)$

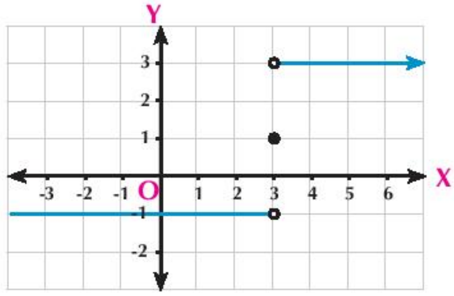
b $f(0)$



4. From the opposite graph , find

a. $\lim_{x \to 3} f(x)$

b. $f(3)$



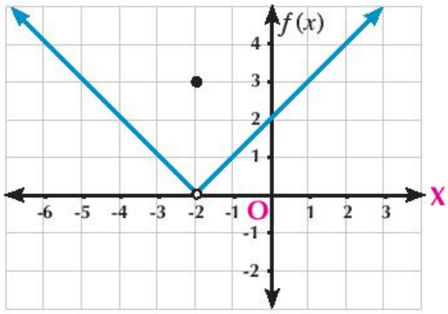
5. From the opposite graph , find:

a. $\lim_{x \to -2} f(x)$

b. $f(-2)$

c. $\lim_{x \to 0} f(x)$

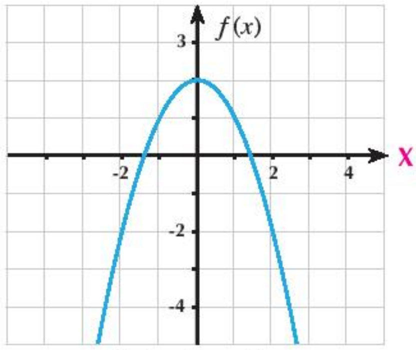
d. $f(0)$



6. From the opposite graph , find:

a. $\lim_{x \to 0} (2 - x^2)$

b. $f(0)$

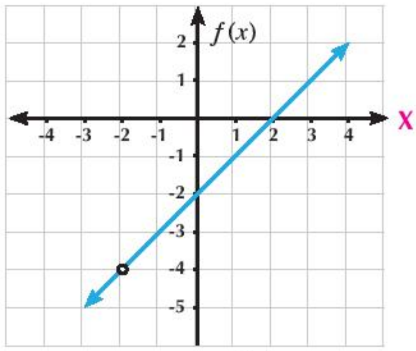


$f(x) = 2 - x^2$

7. From the opposite graph , find:

a. $\lim_{x \to -2} \frac{x^2 - 4}{x + 2}$

b. $f(-2)$



$f(x) = \frac{x^2 - 4}{x + 2}$

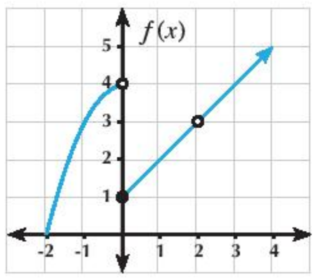
8. From the opposite graph , find:

a. $f(0)$

b. $\lim_{x \to 0} f(x)$

c. $f(2)$

d. $\lim_{x \to 2} f(x)$



(9) Complete the following table and deduce $\lim_{x \to 2} f(x)$ where $f(x) = 5x + 4$

x	1.9	1.99	1.999	⟵	2	⟶	2.001	2.01	2.1
$f(x)$					?				

(10) Complete the following table and deduce $\lim_{x \to -1} (3x + 1)$

x	- 0.9	- 0.99	- 0.999	⟵	- 1	⟶	- 1.001	- 1.01	- 1.1
$f(x)$					?				

(11) Complete the following table and deduce $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$

x	- 0.9	- 0.99	- 0.999	⟵	- 1	⟶	- 1.001	- 1.01	- 1.1
$f(x)$					?				

(12) Complete the following table and deduce $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$

x	1.9	1.99	1.999	⟵	2	⟶	2.001	2.01	2.1
$f(x)$					?				

(13) Use the graphic calculator or a graphic program to estimate the limit of all of the following then check your answers using guiding values.

(a) $\text{Lim}_{x \to 2} (3x - 4)$

(b) $\text{Lim}_{x \to 1} (x^2 - 4)$

(c) $\text{Lim}_{x \to -1} \frac{x^3 + 1}{x + 1}$

(d) $\text{Lim}_{x \to -2} \frac{x^3 + 8}{x^2 - 2}$

(e) $\text{Lim}_{x \to 0} (x + \text{Sin } x)$

(f) $\text{Lim}_{x \to 0} \frac{\sin x - x}{x}$

(g) $\text{Lim}_{x \to 0} \frac{1}{x^2}$

(h) $\text{Lim}_{x \to 0} \frac{1}{|x|}$

Unit (3)

Finding the Limit of a Function Algebraically

3 - 2

You have learned how to find the limit of a function graphically or numerically by studying the values of the function near $x = a$. Here are some theorems and corollaries that help in finding the limit of a function without making a graph or studying the values of the function.

Activity

Use one of the graph programs to graph each of the following function on the same figure:

$$f_1(x) = \frac{x^2 - x - 2}{x - 2}, \quad f_2(x) = x + 1$$

What do you notice?

Find: $\lim\limits_{x \to 2} f_1(x)$, $\lim\limits_{x \to 2} f_2(x)$

What do you conclude?

Learn

Limit of a polynomial function:

Theorem 1

➤ **If** $f(x)$ is a polynomial , $a \in \mathbb{R}$

Then: $\lim\limits_{x \to a} f(x) = f(a)$

Example **Direct substitution**

1. Find each of the following limits:

a) $\lim\limits_{x \to 2} (x^2 - 3x + 5)$

b) $\lim\limits_{x \to 3} (-4)$

Solution

a) $\lim\limits_{x \to 2} (x^2 - 3x + 5)$

$= 4 - 6 + 5 = 3$ **(direct substitution)**

b) $\lim\limits_{x \to 3} (-4) = -4$ $f(x) = -4$ **is a constant function for all** $x \in \mathbb{R}$

Remember

The function f is called polynomial function if it is at the form

$f(x) = c_0 + c_1 x + c_2 x^2 \dots + c_n x^n$

where: $n \in N$,

$c_n \neq$ **zero,**

$c_0, c_1, \dots c_n \in R$

You will learn

- The limit of the polynomial function.
- Some of limits theorems
- Find the limit fa function using the long division .
- The theorem $\lim\limits_{x \to a} = \frac{x^n - a^n}{x - a} = n a^{n-1}$

Key terms

- Limit of a function
- Polynomial function
- Direct substitution
- Factorization
- Synthetic division
- Conjugate

Materials

- Scientific calculator
- Graph programs computer

Try to solve

(1) Find each of the following limits:

(a) $\lim\limits_{x\to 3} (2x - 5)$ (b) $\lim\limits_{x\to -2} (3x^2 + x - 4)$ (c) $\lim\limits_{x\to -2} (7)$

Theorem 2

➤ **If** $\lim\limits_{x\to a} f(x) = l$, $\lim\limits_{x\to a} g(x) = m$ **then:**

1- $\lim\limits_{x\to a} k\,f(x) = k.l$ **where** $k \in \mathbb{R}$

2- $\lim\limits_{x\to a} [f(x) \pm g(x)] = l \pm m$

3- $\lim\limits_{x\to a} f(x)\,.\,g(x) = l.m$

4- $\lim\limits_{x\to a} \frac{f(x)}{x(x)} = \frac{l}{m}$ **where** $m \neq 0$

5- $\lim\limits_{x\to a} (f(x))^n = l^n$ **where** $l^n \in \mathbb{R}$

Example **Using theorem**

(2) Find the following limits:

(a) $\lim\limits_{x\to -1} \frac{3x + 7}{x^2 + 2x - 5}$ (b) $\lim\limits_{x\to -2} (\sqrt{4x^2 - 3})$

Solution

(a) $\lim\limits_{x\to -1} \frac{3x + 7}{x^2 + 2x - 5} = \frac{\lim\limits_{x\to -1} (3x + 7)}{\lim\limits_{x\to -1} (x^2 + 2x - 5)} = \frac{3 \times -1 + 7}{(-1)^2 + 2(-1) - 5} = \frac{4}{-6} = \frac{-2}{3}$

(b) $\lim\limits_{x\to -2} \sqrt{4x^2 - 3} = \sqrt{\lim\limits_{x\to -2} (4x^2 - 3)} = \sqrt{4(-2)^2 - 3} = \sqrt{16 - 3} = \sqrt{13}$

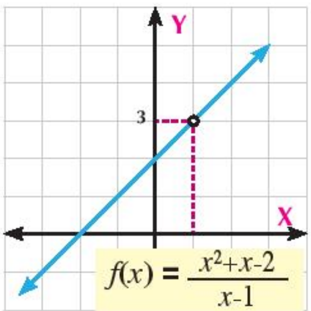
Try to solve

(2) Calculate the following limits:

(a) $\lim\limits_{x\to 2} \frac{x^2 - 3}{2x + 1}$ (b) $\lim\limits_{x\to -2} \sqrt{2x^2 + 1}$

Finding the limit of a function at cases of unspecified:

To find $\lim\limits_{x\to 1} f(x)$ where $f(x) = \frac{x^2 + x - 2}{x - 1}$ using direct substitution we get one of the cases of the unspecified $\frac{\text{zero}}{\text{zero}}$, The opposite graph shows the graphical representation of the function f and we see that $\lim\limits_{x\to 1} f(x) = 3$ So we search for another equivalent function to the function f let it g where $g(x) = x + 2$ which obtained by cancelling non-zero common factors in the numerator and denominator.



Theorem 3

➢ **If** $f(x) = g(x)$ for all $x \in \mathbf{R} - \{ a \}$

and if $\lim\limits_{x\to a} g(x) = l$, **then** $\lim\limits_{x\to a} f(x) = l$

Example

3) Use factorization to find the following limits:

a) $\lim\limits_{x\to 1} \dfrac{x^3 - 1}{x - 1}$ b) $\lim\limits_{x\to 1} \dfrac{x^3 - 2x^2 + 1}{x^2 + x - 2}$

Solution Using Factorization

a) We notice that $f(x) = \dfrac{x^3 - 1}{x - 1}$ unspecified at $x = 1$

Factorize and canceling the non zero common factors then $f(x)$ can be written at the form.

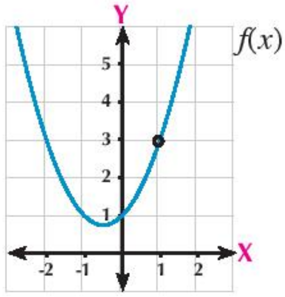
$$f(x) = \frac{(x-1)(x^2 + x + 1)}{(x-1)} = x^2 + x + 1 = g(x)$$

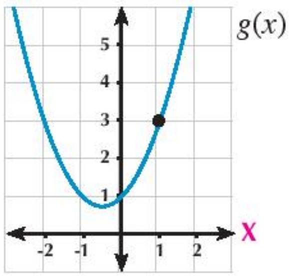
then $f(x) = g(x)$ for all $x \neq 1$

$\because \lim\limits_{x\to 1} g(x) = 3$

and according to **theorem 3 ,** $\lim\limits_{x\to 1} f(x) = 3$

$\therefore \lim\limits_{x\to 1} \dfrac{x^3 - 1}{x - 1} = 3$





Hint

In the long division :

1) Terms of both divisor and divident should be arranged in ascending or descending order.
2) Divide the first term of the divident by the first term of the divisor.
3) Multiply the quotient by the divisor and subtract the result from the divident to get the remainder .
4) Repeat the steps 2, 3 until you finish the division.

Long division method

b) We see that the numerator function $f(x) = 0$ when $x = 1$,

also the denominator function $g(x) = 0$ when $x = 1$

$\therefore (x - 1)$ is a common factor of the numerator and denominator.

$\because$ The numerator is difficult to be factorized into factors containing $(x - 1)$, So we use the long division to find the other factor of the expression $x^3 - 2x^2 + 1$ as follow:

$x - 1$	$x^3 - 2x^2 \quad + 1$
$x^2 - x - 1$	$x^3 - x^2$
	$0 - x^2 \quad + 1$
	$- x^2 + x$
	$0 - x + 1$
	$- x + 1$
	zero

We can use a simple method to perform the division process called synthetic division:

In this method we use the cofficients of the polynomials:

step 1: write coefficients of dividend arranged descendinly and let the divisor = zero to find value of x :

Value of x → 1	1	-2	0	1	← **coefficients**

step 2: leave 1st coefficent and multiply the first coeffient by the value of x and write the result down the 2nd coeffient then add.

1	1	-2	0	1
	↓	1 (1×1)		
	1	-1		

step 3: repeat the multiplication and addition operations, you will find the cofficant of the quoients 1 , - 1 , -1

1	1	-2	0	1
	↓	1 (1×1)	-1 (1×1)	-1 (-1×1)
	1	-1	-1	0

i.e. the quoient $x^2 - x - 1$

$\therefore x^3 - 2x^2 + 1 = (x - 1)(x^2 - x - 1)$

$$\therefore \lim_{x\to 1} \frac{(x-1)(x^2-x-1)}{(x-1)(x+2)} = \lim_{x\to 1} \frac{x^2-x-1}{x+2} = -\frac{1}{3}$$

Try to solve

(3) Find:

(a) $\lim_{x\to -2} \frac{x^3+8}{x+2}$

(b) $\lim_{x\to 4} \frac{2x-8}{x^2-x-12}$

(c) $\lim_{x\to 2} \frac{x^3-x^2-5x+6}{x-2}$

(d) $\lim_{x\to -3} \frac{x^3-10x-3}{x^2+2x-3}$

Example Using the conjugate

(4) Find the following limits:

(a) $\lim_{x\to 4} \frac{\sqrt{x-3}-1}{x-4}$

(b) $\lim_{x\to 5} \frac{x^2-5x}{\sqrt{x+4}-3}$

Solution

(a) **notice that:** $f(x) = \frac{\sqrt{x-3}-1}{x-4}$ unspecified at $x = 4$

So we should eleminate the factor $(x - 4)$ from both of numerator and denominator.

$$\lim_{x\to 4} \frac{\sqrt{x-3}-1}{x-4} \times \frac{\sqrt{x-3}+1}{\sqrt{x-3}+1} = \lim_{x\to 4} \frac{x-3-1}{(x-4)(\sqrt{x-3}+1)}$$

$$= \lim_{x\to 4} \frac{(x-4)}{(x-4)(\sqrt{x-3}+1)}$$

$$= \lim_{x\to 4} \frac{1}{\sqrt{x-3}+1} = \frac{1}{2}$$

b $\lim_{x \to 5} \frac{x^2 - 5x}{\sqrt{x+4} - 3} = \lim_{x \to 5} \frac{x^2 - 5x}{\sqrt{x+4} - 3} \times \frac{\sqrt{x+4} + 3}{\sqrt{x+4} + 3}$

$= \lim_{x \to 5} \frac{x(x-5)(\sqrt{x+4} + 3)}{x + 4 - 9} = \lim_{x \to 5} \frac{x(x-5)(\sqrt{x+4} + 3)}{(x - 5)}$

$= \lim_{x \to 5} x(\sqrt{x+4} + 3) = 5(3 + 3) = 30$

Try to solve

4) Find the following limits:

a $\lim_{x \to 5} \frac{\sqrt{x-1} - 2}{x - 5}$

b $\lim_{x \to -1} \frac{x + 1}{\sqrt{x+5} - 2}$

Theorem 4

If the function f at the form $f(x) = \frac{x^n - a^n}{x - a}$ then $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n\, a^{n-1}$

Activity

Ask for your teacher's help to search on the Internet for methods to proof theorem (4).

Example **Finding the limit of a function at a point using theorem (4)**

5) Find $\lim_{x \to 3} \frac{x^4 - 81}{x - 3}$

Solution

$\lim_{x \to 1} \frac{x^4 - 3^4}{x - 3} = 4(3)^3 = 108$

Corollaires

Corollaries on theorem (4):

1- $\lim_{x \to 0} \frac{(x + a)^n - a^n}{x} = n\, a^{n-1}$

2- $\lim_{x \to a} \frac{x^n - a^n}{x^m - a^m} = \frac{n}{m} a^{n-m}$

Example

6) Find:

a $\lim_{x \to 0} \frac{(x + 5)^4 - 625}{x}$

b $\lim_{x \to 2} \frac{x^5 - 32}{x^2 - 4}$

c $\lim_{x \to 2} \frac{(x - 4)^5 + 32}{x - 2}$

d $\lim_{x \to 16} \frac{\sqrt[4]{x^5} - 32}{\sqrt{x^3} - 64}$

Solution

a $\lim\limits_{x\to 0} \dfrac{(x+5)^4 - 5^4}{x} = 4 \times 5^3 = 500$

b $\lim\limits_{x\to 2} \dfrac{x^5 - 2^5}{x^2 - 2^2} = \dfrac{5}{2} \times 2^3 = 20$

c $\lim\limits_{x\to 2} \dfrac{(x-4)^5 + 32}{x-2} = \lim\limits_{x\to 2} \dfrac{(x-4)^5 - (-2)^5}{(x-4) - (-2)}$

$= 5\,(-2)^4 = 80$

d $\lim\limits_{x\to 16} \dfrac{\sqrt[4]{x^5} - 32}{\sqrt{x^3} - 64} = \lim\limits_{x\to 16} \dfrac{x^{\frac{5}{4}} - (16)^{\frac{5}{4}}}{x^{\frac{3}{2}} - (16)^{\frac{3}{2}}}$

$= \dfrac{\frac{5}{4}}{\frac{3}{2}} \times (16^{\frac{5}{4} - \frac{3}{2}}) = \dfrac{5}{6} \times 16^{-\frac{1}{4}} = \dfrac{5}{12}$

Notice

$16^{\frac{5}{4}} = (2^4)^{\frac{5}{4}} = 2^{4 \times \frac{5}{4}}$
$= 2^5 = 32$

thus :

$16^{\frac{3}{2}} = 64$

Try to solve

5) Find:

a $\lim\limits_{x\to -5} \dfrac{x^4 - 625}{x+5}$ **b** $\lim\limits_{x\to 16} \dfrac{\sqrt[4]{x^7} - 128}{x - 16}$ **c** $\lim\limits_{x\to 7} \dfrac{\sqrt[5]{x+25} - 2}{x - 7}$

Creative thinking:

If $\lim\limits_{x\to 2} \dfrac{x^n - 64}{x - 2} = \ell$ **What is the value of:** n , ℓ

Exercises 3 - 2

Complete:

1) $\lim\limits_{x\to 2} (3x - 1) =$ ....

2) $\lim\limits_{x\to 1} \dfrac{x-3}{x+1} =$ ....

3) $\lim\limits_{x\to 0} \dfrac{x^2 - x}{x} =$ ....

4) $\lim\limits_{x\to 2} \dfrac{x^2 - 4}{x - 2} =$ ....

5) $\lim\limits_{x\to a} \dfrac{x^5 - a^5}{x - a} =$ ....

6) $\lim\limits_{x\to 2} \dfrac{\sqrt[3]{x} - \sqrt[3]{2}}{x - 2} =$ ....

7) $\lim\limits_{x\to 1} \dfrac{\frac{1}{x^3} - 1}{x^4 - 1} =$ ....

8) $\lim\limits_{x\to 2} \dfrac{x^{-1} - 2^{-1}}{x^{-3} - 2^{-3}} =$ ....

Choose the correct answer from those given:

9) $\lim\limits_{x\to -1} \dfrac{x^3 - 1}{x + 1}$ equals:

a -3 **b** -2 **c** 3 **d** has no limit

10) $\lim_{x \to \frac{\pi}{2}} \frac{\sin x}{x}$ equals:

a) 1 b) $\frac{\pi}{2}$ c) $\frac{2}{\pi}$ d) has no limit

11) $\lim_{x \to 16} \frac{\sqrt{x} - 1}{x - 16}$ equals:

a) zero b) $\frac{1}{2}$ c) 1 d) has no limit

12) $\lim_{x \to \frac{\pi}{4}} \frac{\tan x}{x}$ equals:

a) 0 b) 1 c) $\frac{4}{\pi}$ d) has no limit

13) $\lim_{x \to 3} \frac{x^5 - 243}{x^3 - 27}$ equals:

a) 0 b) $\frac{5}{3}$ c) 15 d) 9

14) $\lim_{x \to 2} \frac{x^2 - 4a}{x - 2}$ exists then a equals:

a) -1 b) 1 c) 2 d) 4

15) $\lim_{x \to 2} \frac{5}{(x-2)^3}$ equals:

a) $-\frac{5}{2}$ b) zero c) 5 d) has no limit

Find each of the following limits (if exist):

16) $\lim_{x \to 3} (x^2 - 3x + 2)$

17) $\lim_{x \to -2} \frac{x^2 + 1}{x - 3}$

18) $\lim_{x \to \frac{\pi}{2}} (2x - \sin x)$

19) $\lim_{x \to \pi} \frac{\cos 2x}{x}$

20) $\lim_{x \to -1} \frac{x + 1}{x^3 + 1}$

21) $\lim_{x \to 9} \frac{9 - x}{x^2 - 81}$

22) $\lim_{x \to 2} \frac{x^2 - 7x + 10}{x^2 - 2x}$

23) $\lim_{x \to -1} \frac{x^2 + 6x + 5}{x^2 - 3x - 4}$

24) $\lim_{x \to 9} \frac{x + \sqrt{x} - 12}{x - 9}$

25) $\lim_{x \to 0} \frac{\frac{1}{2 + x} - \frac{1}{2}}{x}$

26) $\lim_{x \to -2} \frac{x + 2}{x^4 - 16}$

27) $\lim_{x \to 1} (\frac{2}{x} + \frac{x^2 - x}{x - 1})$

28) $\lim_{x \to 2} \frac{x^3 + 3x^2 - 12x + 4}{x^3 - 4x}$

29) $\lim_{x \to 4} \frac{x^3 - 4x^2 - x + 4}{2x^2 - 7x - 4}$

30) $\lim_{x \to 3} \frac{x^2 - 9}{x^3 - 2x^2 + 2x - 15}$

31) $\lim_{x \to 2} \frac{x^5 - 32}{x - 2}$

32) $\lim_{x \to 2} \frac{x^{-8} - (16)^{-2}}{x - 2}$

33) $\lim_{x \to 1} \frac{\sqrt[7]{x} - 1}{x - 1}$

34) $\lim_{x \to 2} \frac{x^{-9} - \frac{1}{512}}{x - 2}$

35) $\lim_{x \to \sqrt{5}} \frac{x^7 - 125\sqrt{5}}{x^4 - \sqrt{25}}$

36) $\lim_{x \to 81} \frac{\sqrt[4]{x} - 3}{x - 81}$

37) $\lim_{x \to 2} \frac{x^7 - 128}{x^5 - 32}$

38) $\lim_{x \to 6} \frac{(x-5)^7 - 1}{x - 6}$

39) $\lim_{x \to 7} \frac{\sqrt[5]{x+25} - 2}{x - 7}$

40) $\lim_{x \to 2} \frac{(x-3)^6 - 1}{x - 2}$

41) $\lim_{h \to 0} \frac{(x-2h)^{17} - x^{17}}{51\,h}$

42) $\lim_{x \to -3} \frac{x^4 - 81}{x^5 + 243}$

43) $\lim_{h \to 0} \frac{(3+h)^4 - 81}{6\,h}$

44) $\lim_{x \to -1} \frac{\sqrt{x^2+8} - 3}{x + 1}$

45) $\lim_{x \to -3} \frac{\sqrt{x+7} - 2}{x + 3}$

46) $\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{\sqrt{x+6} - 3}$

47) $\lim_{x \to -1} \frac{x^2 - x - 2}{x + 1}$

48) $\lim_{x \to 0} \frac{(2x-1)^2 - 1}{5x}$

49) $\lim_{x \to 1} \left(\frac{1}{x-1} - \frac{3}{x^3 - 1}\right)$

50) $\lim_{x \to -2} \frac{x^2 + 4x + 4}{x^3 + x^2 - 8x - 12}$

51) $\lim_{x \to 4} \frac{\sqrt{2x+1} - 3}{x - 4}$

Activity

52) **Volume** A piece of cardboard in the form of a square of side length 24 cm is used to make a box without a lid by cutting four squares each of side length x cm around the four corners :

1) Draw the figure of the cardboard.

2) Prove that the volume of the box is given by $V = x(24 - 2x)^2$

3) Find the volume of the box when $x = 4$ by studying the values of the function when $x \to 4$ using the following table :

x	3	3.5	3.9	→	4	←	4.1	4.5	5
$f(x)$	....	....	....	→	....	←	....	....	....

4) Use one of the graph programs to graph the function and verify the maximum value of the volume at $x = 4$

Creative thinking :

53) If $\lim_{x \to 2} \frac{f(x) - 5}{x - 2} = 1$ **find :** $\lim_{x \to 2} f(x)$

54) If $\lim_{x \to 0} \frac{f(x)}{x^2} = 5$ **find :**

A) $\lim_{x \to 0} f(x)$

B) $\lim_{x \to 0} \frac{f(x)}{x}$

55) **Trade:** A company found that if it spends x L.E for advertising one of its production, its profit is given by the function $f(x) = 0.2\,x^2 + 40\,x + 150$. Find the profit of the company when its expenditure approaches to 100L.E.

Limit of the Function at Infinity

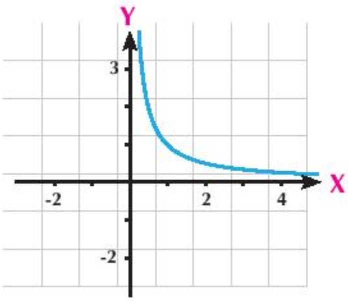
In many life applications, we need to know the behavior of a function when $x \longrightarrow \infty$ the next activity illustrate that.

Activity

Use one of the graph programs to represent the function f where :

$f(x) = \frac{1}{x}$, $x > 0$

What do you notice from the graph when $x \longrightarrow \infty$?



From the graph we notice that:

- When x increase and approaches to ∞ then $f(x)$approaches to certain number.

Complete the table to find the number that $f(x)$ approaches to $f(x)$

x	10	100	1000	10000	100000	$x \to \infty$
$f(x)$	0.1	0.01				$x \to ?$

Learn

Limit of a Function at Infinity

From the previous activity we find that when x approaches to Infinity, the values of $f(x)$ approaches to zero.

Theorem 1

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Corollary

$$\lim_{x \to \infty} \frac{a}{x^n} = 0 \quad \text{where} \quad n \in \mathbf{R}^+, \text{ a is constant}$$

Some basic rules :

- $\lim_{x \to \infty} c = c$ where c constant.
- If n is a positive integer then $\lim_{x \to \infty} x^n = \infty$

Notice that: theorem (2) which studied before which relating to limit of the sum, difference product and quotient of two functions when $x \to a$ is correct when $x \to \infty$

You will learn

- Limit of a function at infinity
- Finding the limit of a function at infinity algebraically.
- Finding the limit of a function at infinity graphically

Key terms

- Limit of a function at infinity

Materials

- Secientific calculator
- Graph programs.

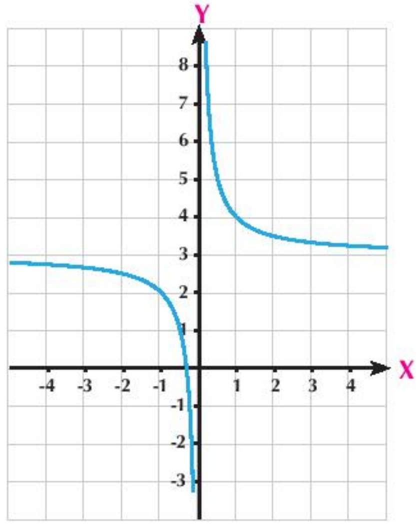
Example

1 Find:

a $\lim_{x \to \infty} (\frac{1}{x} + 3)$

b $\lim_{x \to \infty} (4 - \frac{3}{x^2})$

➢ Then check your results using one of the graph programs.

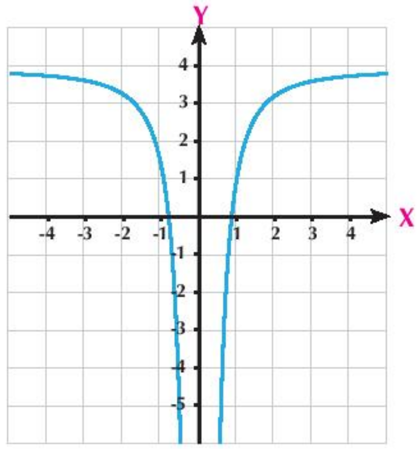


Solution

a $\lim_{x \to \infty} (\frac{1}{x} + 3) = \lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} 3$

$= 0 + 3 = 3$

$\therefore \lim_{x \to \infty} (\frac{1}{x} + 3) = 3$



b $\lim_{x \to \infty} (4 - \frac{3}{x^2}) = \lim_{x \to \infty} 4 - \lim_{x \to \infty} \frac{3}{x^2}$

$= 4 - 3 \lim_{x \to \infty} \frac{1}{x^2} = 4 - 3 \times 0 = 4$

$\therefore \lim_{x \to \infty} (4 - \frac{3}{x^2}) = 4$

Try to solve

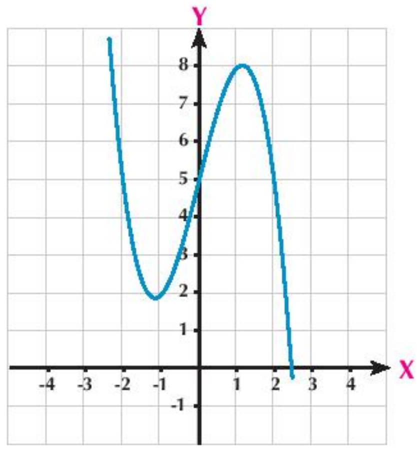
1 Find:

a $\lim_{x \to \infty} (\frac{5}{x} + 2)$

b $\lim_{x \to \infty} (\frac{2}{x^2} + 5)$

Example

2 Find: $\lim_{x \to \infty} (4x - x^3 + 5)$



Solution

$\lim_{x \to \infty} x^3 (\frac{4}{x^2} - 1 + \frac{5}{x^3})$

$= \lim_{x \to \infty} x^3 \times \lim_{x \to \infty} (\frac{4}{x^2} - 1 + \frac{5}{x^3})$

$= \infty \times -1 = -\infty$

Try to solve

(2) Find each of the following limits:

(a) $\lim\limits_{x \to \infty} (x^3 + 7x^2 + 2)$ (b) $\lim\limits_{x \to \infty} (4 - 3x - x^3)$

Example

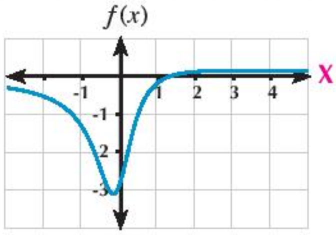
(3) Find each of the following limits:

(a) $\lim\limits_{x \to \infty} \frac{2x - 3}{3x^2 + 1}$ (b) $\lim\limits_{x \to \infty} \frac{2x^2 - 3}{3x^2 + 1}$ (c) $\lim\limits_{x \to \infty} \frac{2x^3 - 3}{3x^2 + 1}$

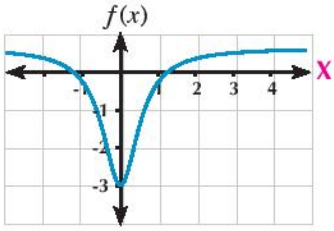
Solution

Divide each of the denominator and numerator by x^2 (the greatest power of x in the denominator).

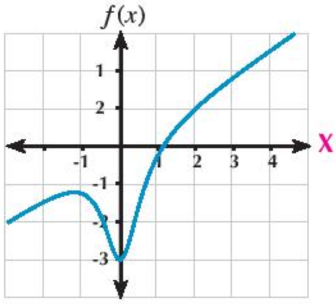
(a) $$\lim_{x \to \infty} \frac{2x - 3}{3x^2 + 1} = \frac{\lim\limits_{x \to \infty} (\frac{2}{x} - \frac{3}{x^2})}{\lim\limits_{x \to \infty} (3 + \frac{1}{x^2})} = \frac{0 - 0}{3 + 0} = 0$$



(b) $$\lim_{x \to \infty} \frac{2x^2 - 3}{3x^2 + 1} = \frac{\lim\limits_{x \to \infty} (2 - \frac{3}{x^2})}{\lim\limits_{x \to \infty} (3 + \frac{1}{x^2})} = \frac{2 - 0}{3 + 0} = \frac{2}{3}$$



(c) $$\lim_{x \to \infty} \frac{2x^3 - 3}{3x^2 + 1} = \frac{\lim\limits_{x \to \infty} (2x - \frac{3}{x^2})}{\lim\limits_{x \to \infty} (3 + \frac{1}{x^2})} = \frac{\infty - 0}{3 + 0} = \infty$$



From the previous example, we can conclude that when finding $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)}$ where $f(x)$, $g(x)$ are polynomials then :

- The limit equals a non-zero real number if the numerator and the denominator have the same degree.
- The limit equals zero if the numerator's degree is smaller than the denominator's degree.
- The limit gives $\pm \infty$ if the numerator's degree is greater than the denominator's degree.

Try to solve

(3) Find:

(a) $\lim\limits_{x \to \infty} \frac{5x^2 - 3x + 1}{2x}$ (b) $\lim\limits_{x \to \infty} \frac{4x^3 - 5x}{8x^4 + 3x^2 - 2}$ (c) $\lim\limits_{x \to \infty} \frac{-6x^2 + 1}{3x^2 + x - 2}$

Example

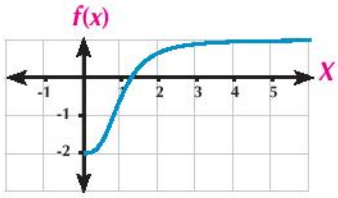
4 Find the following limits :

a $\lim_{x \to \infty} \frac{x^3 - 2}{|x|^3 + 1}$

b $\lim_{x \to \infty} (x - \sqrt{x^2 + 4})$

Solution

a $\lim_{x \to \infty} \frac{x^3 - 2}{|x|^3 + 1}$



$f(x) = \frac{x^3 - 2}{|x|^3 + 1}$

$\because x \longrightarrow \infty$

$\therefore x > 0$ **then** $|x| = x$

$\therefore \lim_{x \to \infty} \frac{x^3 - 2}{x^3 + 1}$

Dividing both numerator and denominator by x^3

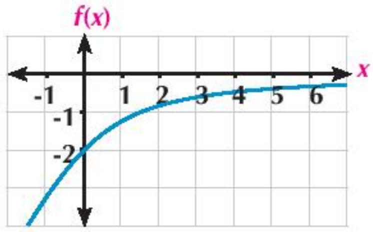
$$= \frac{\lim_{x \to \infty} (1 - \frac{2}{x^3})}{\lim_{x \to \infty} (1 + \frac{1}{x^3})} = \frac{1 - 0}{1 + 0} = 1$$

b $\lim_{x \to \infty} (x - \sqrt{x^2 + 4})$

$$= \lim_{x \to \infty} \frac{(x - \sqrt{x^2 + 4})}{1} \times \frac{(x + \sqrt{x^2 + 4})}{(x + \sqrt{x^2 + 4})}$$

$$= \lim_{x \to \infty} \frac{x^2 - x^2 - 4}{x + \sqrt{x^2 + 4}}$$

$$= \lim_{x \to \infty} \frac{-4}{x + \sqrt{x^2 + 4}}$$



$f(x) = x - \sqrt{x^2 + 4}$

$\because x \longrightarrow \infty$

$\therefore x > 0 \longrightarrow \sqrt{x^2} = |x| = x$

Dividing both numerator and denominator by $x = \sqrt{x^2}$

$$\therefore \lim_{x \to \infty} \frac{-4}{x + \sqrt{x^2 + 4}} = \frac{\lim_{x \to \infty} -\frac{4}{x}}{\lim_{x \to \infty} (1 + \sqrt{1 + \frac{4}{x^2}})} = \frac{0}{1 + 1} = 0$$

Try to solve

4 Find the following limits :

a $\lim_{x \to \infty} \frac{x - 3}{\sqrt{4x^2 + 25}}$

b $\lim_{x \to \infty} (\sqrt{3x^2 + 5x} - \sqrt{3}\, x)$

Exercises 3 - 3

Complete the following :

1. $\lim_{x\to\infty} (1 + \frac{3}{x}) =$ ....

2. $\lim_{x\to\infty} (\frac{3}{x^2} - 2) =$ ....

3. $\lim_{x\to\infty} (-7) =$ ....

4. $\lim_{x\to\infty} (x^2 - 3) =$ ....

5. $\lim_{x\to\infty} \frac{2x+1}{x}$ ....

6. $\lim_{x\to\infty} \frac{x^3 - 5}{x^2 + 1} =$ ....

7. $\lim_{x\to\infty} \frac{x^5 + 3}{x^3 - 5} =$ ....

8. $\lim_{x\to\infty} \frac{3x}{\sqrt{x^2 - 1}} =$ ....

9. $\lim_{x\to\infty} (3 - \frac{7}{x} + \frac{4}{x^2}) =$ ....

10. $\lim_{x\to\infty} (\sqrt{x^2 + 1} - x) =$ ....

Choose the correct answer from those given:

11. $\lim_{x\to\infty} \frac{6x}{2x + 3}$ equals
 a) 0 b) 2 c) 3 d) $+\infty$

12. $\lim_{x\to\infty} \sqrt{\frac{4}{x} + 1}$ equals
 a) 0 b) 1 c) 2 d) $+\infty$

13. $\lim_{x\to\infty} \frac{x + 3}{2 - x^2}$ equals
 a) 0 b) $\frac{1}{2}$ c) $\frac{3}{2}$ d) $+\infty$

14. $\lim_{x\to\infty} \frac{x^2 + 1}{2x - 1}$ equals
 a) 0 b) $\frac{1}{2}$ c) 1 d) $+\infty$

15. $\lim_{x\to\infty} \sqrt{\frac{1 + x}{4x - 1}}$ equals
 a) -1 b) $\frac{1}{4}$ c) $\frac{1}{2}$ d) 1

Find

16. $\lim_{x\to\infty} \frac{3}{x^2}$

17. $\lim_{x\to\infty} (x^3 + 5x^2 + 1)$

18. $\lim_{x\to\infty} \frac{2 - 7x}{2 + 3x}$

19. $\lim_{x\to\infty} \frac{x^2}{x + 3}$

20. $\lim_{x\to\infty} \frac{4x^2}{x^2 + 3}$

21. $\lim_{x\to\infty} \frac{5 - 6x - 3x^2}{2x^2 + x + 4}$

(22) $\lim\limits_{x \to \infty} \dfrac{2x - 1}{x^2 + 4x + 1}$ (23) $\lim\limits_{x \to \infty} \dfrac{x^3 - 2}{3x^2 + 4x - 1}$ (24) $\lim\limits_{x \to \infty} \dfrac{2x^2 - 1}{4x^3 - 5x - 1}$

(25) $\lim\limits_{x \to \infty} \dfrac{2x^2 - 6}{(x - 1)^2}$ (26) $\lim\limits_{x \to \infty} \left(7 + \dfrac{2x^2}{(x + 3)^2}\right)$ (27) $\lim\limits_{x \to \infty} \left(\dfrac{1}{3x^2} - \dfrac{5x}{2 + x}\right)$

(28) $\lim\limits_{x \to \infty} \left(\dfrac{x}{2x+1} + \dfrac{3x^2}{(x-3)^2}\right)$ (29) $\lim\limits_{x \to \infty} \dfrac{-x}{\sqrt{4 + x^2}}$ (30) $\lim\limits_{x \to \infty} \dfrac{x^3 - 4x + 5}{(2x - 1)^3}$

(31) $\lim\limits_{x \to \infty} \left(\sqrt{4x^2 - 2x + 1} - 2x\right)$ (32) $\lim\limits_{x \to \infty} \left(\sqrt{5x^2 + 4x + 7} - \sqrt{5x^2 + x + 3}\right)$

(33) $\lim\limits_{x \to \infty} x\left(\sqrt{4x^2 + 1} - 2x\right)$ (34) $\lim\limits_{x \to \infty} \dfrac{x^2 + x - 1}{8x^2 - 3}$ (35) $\lim\limits_{x \to \infty} \dfrac{4 - 3x^3}{\sqrt{x^6 + 9}}$

(36) $\lim\limits_{x \to \infty} \dfrac{(x + 2)^3(3 - 2x^2)}{3x(x^2 + 7)^2}$

(37) If $\lim\limits_{x \to \infty} \left(\sqrt{ax^2 + 3bx + 5} - 2x\right) = 3$ find the value of each of a, b.

(38) $\lim\limits_{x \to \infty} \dfrac{x^{-1} + 3x^{-2} + 5}{2x^{-2} - x^{-3} + 1}$ (39) $\lim\limits_{x \to \infty} \dfrac{2x^{-1} - 3x^{-2} + x^{-3}}{4x^{-3} + x^{-1}}$

(40) **Creative thinking** : One of the companies produces greeting cards with initial cost of 5000 L.E. and extra cost of a half pound for every single card. If the total cost is given by $C = \frac{1}{2}x + 5000$ where x is the number of produced cards.

Find :

(1) The cost of the card when the production is:

(a) 10000 card (b) 100000 card

(2) The cost of the card when the company produces an infinite number of cards.

Limits of Trigonometric Functions

Unit (3)

3 - 4

Activity

If f is a function where $f(x) = \frac{\sin x}{x}$, and the required is to study the values of the function f when $x \to 0$ (x is the measure of the angle in radians).

Make a table to study the behavior of the function $f(x) = \frac{\sin x}{x}$ when x approaches to zero using radians

x	1	0.01	0.001	→	0	←	-0.001	-0.01	-1
$\frac{\sin x}{x}$	0.8415	0.9983	....	→	....	←	....	0.9983	0.8415

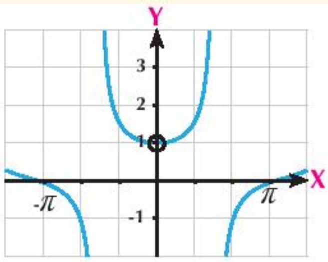
From the previous table deduce $\lim\limits_{x \to 0} \frac{\sin x}{x}$

Learn

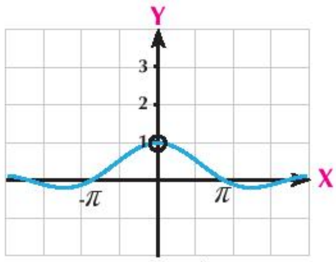
Theorem 1

If x is the measure of an angle in radian then :

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{\tan x}{x} = 1$$



$f(x) = \frac{\sin x}{x}$



$f(x) = \frac{\tan x}{x}$

Oral discussion:

If x is the measure of the angle in degree measure, can we find $\lim\limits_{x \to 0} \frac{\sin x}{x}$ Explain your answer.

Corollary 1:

$$\lim_{x \to 0} \frac{\sin a x}{x} = a \, , \quad \lim_{x \to 0} \frac{\tan ax}{x} = a$$

You will learn

- The limit of some trigonometric Functions

Key terms

- Trigonometric Function
- Limit of a Trigonometric Function

Materials

- Scientific calculator.
- Graph programs

Add to your information

This theorem has more than one proof you can check that ussing the link:
http://math.stackexchange.com/question/75130

Example

1 a $\lim_{x \to 0} \frac{\sin 3x}{x} = 3$ b $\lim_{x \to 0} \frac{\tan 2x}{7x} = \frac{1}{7} \lim_{x \to 0} \frac{\tan 2x}{x} = \frac{2}{7}$

c $\lim_{x \to 0} \frac{\sin 5x \cos 2x}{x} = \lim_{x \to 0} \frac{\sin 5x}{x} \times \lim_{x \to 0} \cos 2x = 5 \times 1 = 5$

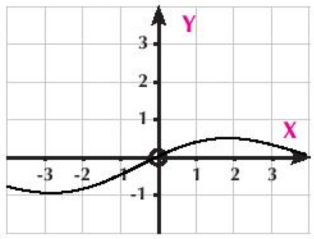
Try to solve

1 Find:

a $\lim_{x \to 0} \frac{\sin 2x}{3x}$ b $\lim_{x \to 0} \frac{\tan 4x}{5x}$ c $\lim_{x \to 1} \frac{\sin \pi x}{1 - x}$

Corollary 2

a $\lim_{x \to 0} \frac{1 - \cos x}{x} = \text{zero}$



➢ **Ask for your teacher's help to prove corollary (2)**

Example

2 Find:

a $\lim_{x \to 0} \frac{1 - \cos x}{\tan x}$ b $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

Solution

a $\lim_{x \to 0} \frac{1 - \cos x}{\tan x} = \lim_{x \to 0} \frac{1 - \cos x}{x} \times \frac{x}{\tan x}$

$= \lim_{x \to 0} \frac{1 - \cos x}{x} \times \frac{x}{\tan x} = 0 \times 1 = 0$

b $\lim_{x \to 0} \frac{1 - \cos x}{x^2} \times \frac{1 + \cos x}{1 + \cos x}$

$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)} = \lim_{x \to 0} \frac{\sin^2 x}{x^2(1 + \cos x)}$

$= \lim_{x \to 0} \frac{\sin^2 x}{x^2} \times \lim_{x \to 0} \frac{1}{1 + \cos x} = (1)^2 \times \frac{1}{1 + 1} = \frac{1}{2}$

Remember

$\sin^2 x + \cos^2 x = 1$

Try to solve

2 Find the following limits:

a $\lim_{x \to 0} 6x^2 \csc 2x \cot x$ b $\lim_{x \to 0} \frac{1 - \cos x}{\sin 2x}$

Example

3) Find the following limits :

a) $\lim\limits_{x\to 0} \dfrac{x^2 - x + \sin x}{2x}$

b) $\lim\limits_{x\to 0} \dfrac{x + x\cos x}{\sin x \cos x}$

c) $\lim\limits_{x\to 0} \dfrac{\sin(1 - \cos x)}{1 - \cos x}$

d) $\lim\limits_{x\to 0} \dfrac{\sin^2 3x}{5x^2}$

Solution

a) $= \frac{1}{2} \lim\limits_{x\to 0} \left(\frac{x^2}{x} - \frac{x}{x} + \frac{\sin x}{x}\right)$

$= \frac{1}{2} \lim\limits_{x\to 0} \left(x - 1 + \frac{\sin x}{x}\right)$

$= \frac{1}{2} (0 - 1 + 1) = 0$

b) $\lim\limits_{x\to 0} \dfrac{x + x\cos x}{\sin x \cos x}$

dividing both of numerator and denominator

by $x = \lim\limits_{x\to 0} \dfrac{1 + \cos x}{\frac{\sin x}{x} \times \cos x}$

$= \dfrac{\lim\limits_{x\to 0} (1 + \cos x)}{\lim\limits_{x\to 0} \frac{\sin x}{x} \times \cos x}$

$= \dfrac{1 + \cos 0}{1 \times \cos 0} = \dfrac{1 + 1}{1} = 2$

c) $\lim\limits_{x\to 0} \dfrac{\sin(1 - \cos x)}{1 - \cos x}$

let $1 - \cos x = y$

when $x \longrightarrow 0$ then $y \longrightarrow 0$

$\lim\limits_{y\to 0} \dfrac{\sin y}{y} = 1$

$\therefore \lim\limits_{x\to 0} \dfrac{\sin(1 - \cos x)}{1 - \cos x} = 1$

d) $\lim\limits_{x\to 0} \dfrac{\sin^2 3x}{5x^2}$

$= \frac{1}{5} \left(\lim\limits_{x\to 0} \frac{\sin 3x}{x}\right)^2$

$= \frac{1}{5} \times (3)^2$

$= \frac{9}{5}$

Try to solve

3) Find the following limits :

a) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{5x^3 - 4x}$

b) $\lim\limits_{x\to 0} \dfrac{\sin(x-1)}{x^2 + x - 2}$

c) $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{3x^2}$

d) $\lim\limits_{x\to 1} \dfrac{x \sin 2x + \sin^2 2x}{\tan^2 3x + x^2}$

Exercises 3 - 4

Complete:

1. $\lim\limits_{x\to 0} \cos 3x =$ ....

2. $\lim\limits_{x\to \frac{\pi}{2}} \sin 2x =$ ....

3. $\lim\limits_{x\to 0} \tan x =$ ....

4. $\lim\limits_{x\to 0} \frac{\sin 7x}{x} =$ ....

5. $\lim\limits_{x\to 0} \tan \frac{3x}{4x} =$ ....

6. $\lim\limits_{x\to 5} \frac{\sin (x-5)}{3(x-5)} =$ ....

7. $\lim\limits_{x\to 0} \frac{\sin \pi x}{2x} =$ ....

8. $\lim\limits_{x\to 0} \frac{\sin^2 x}{x^2} =$ ....

9. $\lim\limits_{x\to 0} \frac{3+2x}{\cos 4x} =$ ....

10. $\lim\limits_{x\to 0} \frac{1-\cos x}{3x} =$ ....

11. $\lim\limits_{x\to 0} \frac{2x}{\tan 3x} =$ ....

12. $\lim\limits_{x\to 0} 3x \csc 2x =$ ....

13. $\lim\limits_{x\to 0} \frac{\sin^2 2x}{3x^2} =$ ....

14. $\lim\limits_{x\to 0} \frac{\sin^2 x \tan 3x}{4x^2} =$ ....

Choose the correct answer from those given:

15. $\lim\limits_{x\to 0} \frac{\sin 3x}{x} =$ ....
   - a) 0
   - b) $\frac{1}{3}$
   - c) 1
   - d) 3

16. $\lim\limits_{x\to 0} \frac{\tan 4x}{5x} =$ ....
   - a) 0
   - b) $\frac{4}{5}$
   - c) 1
   - d) $\frac{5}{4}$

17. $\lim\limits_{x\to 0} \frac{\sin 2x + 3\tan x}{5x} =$ ....
   - a) 1
   - b) $\frac{5}{6}$
   - c) $\frac{6}{5}$
   - d) 2

18. $\lim\limits_{x\to 0} \frac{\tan^2 2x}{x \sin 3x} =$ ....
   - a) $\frac{4}{9}$
   - b) $\frac{1}{2}$
   - c) $\frac{2}{3}$
   - d) $\frac{4}{3}$

19. $\lim\limits_{x\to 0} \frac{\sin \frac{1}{2}x}{\sin \frac{3}{4}x} =$ ....
   - a) $\frac{1}{6}$
   - b) $\frac{3}{8}$
   - c) $\frac{1}{2}$
   - d) $\frac{2}{3}$

20. $\lim\limits_{x\to 0} \frac{\sin x}{x} =$ .... where x in degree measure
   - a) 1
   - b) $\frac{\pi}{180}$
   - c) $\frac{180}{\pi}$
   - d) π

Find the following limits

21 $\lim_{x\to 0} \frac{\sin x}{5x}$

22 $\lim_{x\to 0} \frac{\sin x}{\tan x}$

23 $\lim_{x\to 0} \frac{\tan 2x}{x}$

24 $\lim_{x\to 0} \frac{3(1-\cos x)}{x}$

25 $\lim_{x\to 0} \frac{\sin x\,(1-\cos x)}{x^2}$

26 $\lim_{x\to 0} \frac{\cos x \tan x}{x}$

27 $\lim_{x\to 0} \frac{\sin^2 x}{x}$

28 $\lim_{x\to 0} \frac{\tan x}{x^2}$

29 $\lim_{x\to 0} \frac{(1-\cos^2 x)}{x^2}$

30 $\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$

31 $\lim_{x\to 0} \frac{x - x\cos x}{\sin^2 3x}$

32 $\lim_{x\to 0} \frac{1-\tan x}{\sin x - \cos x}$

33 $\lim_{x\to \frac{1}{2}} \frac{x\cos(-2x+1)}{x^2+x}$

34 $\lim_{x\to 0} \frac{x^2 - 3\sin x}{x}$

35 $\lim_{x\to -0} (1+\cos x) \times \frac{1-\cos x}{x^2}$

36 $\lim_{x\to 0} \frac{\tan 3x^2 + \sin^2 5x}{x^2}$

37 $\lim_{x\to 0} \frac{\sin 5x^3 + \sin^3 5x}{2x^3}$

38 $\lim_{x\to 0} \left(\frac{2x^2 + \sin 3x}{2x^2 + \tan 6x}\right)^4$

39 $\lim_{x\to 0} \frac{2x^3 + x\sin 5x}{x^2 - \tan 3x^2}$

40 $\lim_{x\to 0} \frac{1-\cos x + \sin x}{1-\cos x - \sin x}$

41 $\lim_{x\to 0} \frac{\tan 2x + 5\sin 3x}{2\sin 3x - \tan 5x}$

42 $\lim_{x\to 0} \frac{x\tan 2x}{x^2 + \sin^2 3x}$

43 $\lim_{x\to 0} \frac{1-\cos 3x}{\cos^2 2x - 1}$

44 $\lim_{x\to 0} \frac{2-\cos 3x - \cos 4x}{x}$

45 $\lim_{x\to 0} \frac{\tan 3x^2 + \sin^2 5x}{x^2}$

46 $\lim_{x\to 0} \frac{1-\cos 3x}{\cos^2 5x - 1}$

47 $\lim_{x\to 0} \frac{\sin(\sin x)}{5\sin x}$

48 $\lim_{x\to 0} \frac{x}{\cos(\frac{\pi}{2} - x)}$

49 $\lim_{x\to 0} x(\csc 2x - \cot 3x)$

Unit three

3 - 5 Existence of Limit of a Function at a Point

You will learn

- The right limit
- the left limit
- Discuss existence of Limit of a function at apoint.

Key terms

- Right limit
- Left limit

Materials

- Scientific calc.
- Graph program

Think and discuss

Figure (1) :

Represents the graph of the function f, such that:

$$f(x) = \begin{cases} x - 1 & \text{If } x > 2 \\ 3 - x & \text{If } x < 2 \end{cases}$$

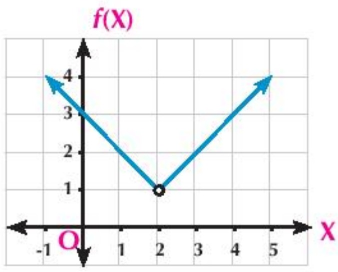


Fig (1)

- **Discuss** the existence of $\lim_{x \to 2^+} f(x)$
- **Discuss** the existence of $\lim_{x \to 2^-} f(x)$

is $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$?

Figure (2) :

Represents the graph of the function g, Such that :

$$g(x) = \begin{cases} 2 & \text{If } x > 0 \\ -2 & \text{If } x < 0 \end{cases}$$

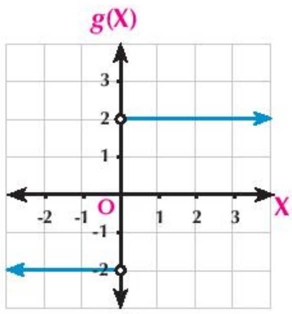


Fig (2)

- **Discuss** the existence of $\lim_{x \to 0^+} g(x)$
- **Discuss** the existence of $\lim_{x \to 0^-} g(x)$

is $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^-} g(x)$?

Learn limit of a function

The right limit and the left limit

The limit of the function f when x tends to a equals l iff the right limit and the left limit when x tends to a are equal and each of them equals l where $l \in \mathbf{R}$:

$\lim_{x \to a} f(x) = l$ iff: $f(a^+) = f(a^-) = l$

where

$$f(a^+) = \lim_{x \to a^+} f(x)$$

$$f(a^-) = \lim_{x \to a^-} f(x)$$

Illustrating examples

a **From the figure (1) we notice that:**

$f(1^-) = 3$ $\qquad f(1^+) = -1$

$\therefore f(1^-) \neq f(1^+)$

$\therefore$ The function has no limit when $x \longrightarrow 1$

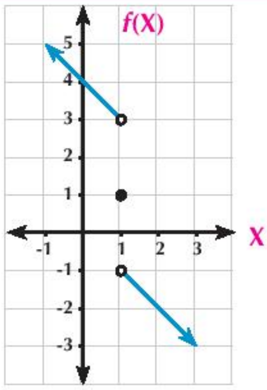


Fig (1)

b **In the figure (2) we notice that:**

$f(-1^-) = 3$ $\qquad f(-1^+) = 3$

$\therefore f(-1^-) = f(-1^+) = 3$

$\therefore \lim\limits_{x \to -1} f(x) = 3$

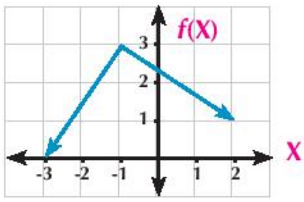
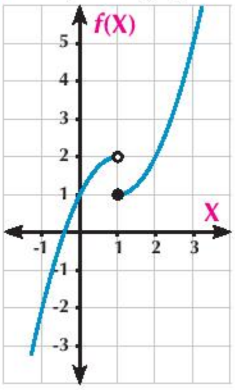
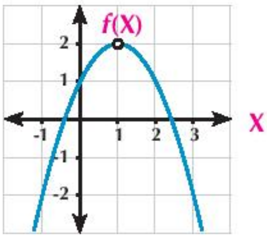


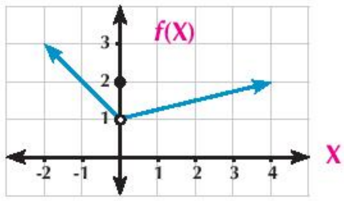
Fig (2)

Try to solve

1. Study the graphs of the given functions then find:







a $f(1^-)$	**a** $f(1^-)$	**a** $f(0^-)$
b $f(1^+)$	**b** $f(1^+)$	**b** $f(0^+)$
c $\lim\limits_{x \to 1} f(x)$	**c** $\lim\limits_{x \to 1} f(x)$	**c** $\lim\limits_{x \to 0} f(x)$

Example

1. Find the limit of the function f when $x \longrightarrow 0$ where $f(x) = \begin{cases} x\,|x| - 1 & \text{If } x < 0 \\ \dfrac{|x|}{x} - 2 & \text{If } x > 0 \end{cases}$

Solution

Redefine the function, then

$$f(x) = \begin{cases} x(-x) - 1 & \text{If } \quad x < 0 \\ \dfrac{x}{x} - 2 & \text{If } \quad x > 0 \end{cases} \qquad = \begin{cases} -x^2 - 1 & \text{for all } \quad x < 0 \\ -1 & \text{for all } \quad x > 0 \end{cases}$$

$\because f(0^-) = \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-x^2 - 1) = -1$

$f(0^+) = \lim\limits_{x \to 0^+} -1 = -1$ $\qquad \because f(0^-) = f(0^+) = -1$ $\qquad \therefore \lim\limits_{x \to 0} f(x) = -1$

Try to solve

(2) If $f(x) = \begin{cases} |x-3| & x \neq 3 \\ 2 & x = 3 \end{cases}$

find $\lim\limits_{x \to 3} f(x)$ (if possible)

Example

(2) Discuss the existence of the limit of the function f when $x \to 0$ such that :

$$f(x) = \begin{cases} \dfrac{x^2 + 2x}{x} & \text{If } x < 0 \\[2ex] \dfrac{x + \tan 2x}{3x - \sin 2x} & \text{If } x > 0 \end{cases}$$

Solution

$$f(0^-) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x^2 + 2x}{x} = \lim_{x \to 0^-} \frac{x(x+2)}{x}$$

$$= \lim_{x \to 0^-} (x + 2) = 2$$

$$f(0^+) = \lim_{x \to 0^+} \frac{x + \tan 2x}{3x - \sin 2x} = \lim_{x \to 0^-} \frac{\frac{x}{x} + \frac{\tan 2x}{x}}{\frac{3x}{x} - \frac{\sin 2x}{x}} = \lim_{x \to 0^-} \frac{1 + \frac{\tan 2x}{x}}{3 - \frac{\sin 2x}{x}}$$

$$= \frac{1+2}{3-2} = 3$$

$\because f(0^-) \neq f(0^+) = 2$ $\qquad \therefore \lim\limits_{x \to 0} f(x)$ does not exist.

Try to solve

(3) Discuss the existence of the limit of the function f when $x \longrightarrow \pi$ where

$$f(x) = \begin{cases} \dfrac{\sin x}{x - \pi} & \text{If } x < \pi \\[2ex] \cos x & \text{If } x > \pi \end{cases}$$

Example

(3) Discuss the existence of the limit of the function f when $x \longrightarrow 1$ where: $f(x) = \sqrt{x - 1}$

Solution

$\because$ $f(x)$ is defined for all $x - 1 \geq 0$

$\therefore$ The domain of $f(x)$ is $[1 + \infty[$

$\because$ $f(1^+) = \lim\limits_{x \to 1^+} f(x)$ $\qquad \therefore f(1^+) = \lim\limits_{x \to 1^+} \sqrt{x - 1} = 0$

$f(1^-)$ is undefined because f (x) is not defined at the left of 1

$\therefore f(x)$ has no limit to $x \longrightarrow 1$

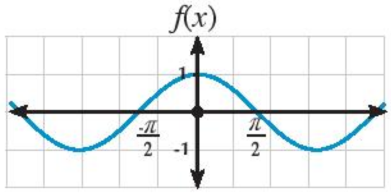
Try to solve

(4) Discuss the existence of the limit of the function f when $x \longrightarrow 3$ where $f(x) = \sqrt{3 - x}$

Exercises 3 - 5

Complete the following :

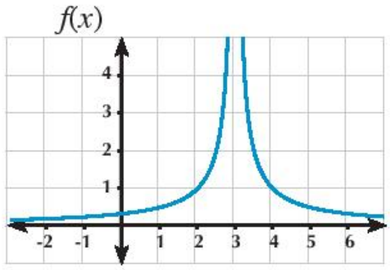
1. In the opposite graph :



A $\lim_{x \to 0^-} f(x) =$ ....

B $\lim_{x \to 0^+} f(x) =$ ....

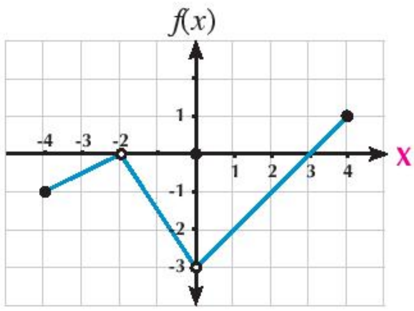
2. In the opposite graph :



A $\lim_{x \to 3^-} f(x) =$ ....

B $\lim_{x \to 3^+} f(x) =$ ....

3. In the opposite graph :



A $\lim_{x \to -2} f(x) =$ ....

B $\lim_{x \to 0} f(x) =$ ....

C $\lim_{x \to 1} f(x) =$ ....

D $\lim_{x \to -4} f(x) =$ ....

E $\lim_{x \to 4} f(x) =$ ....

4. The function f is defined on $\mathbb{R}$ such that $f(x) = \begin{cases} 2 & \text{If } x > 0 \\ 2 - x & \text{If } x < 0 \end{cases}$

A $\lim_{x \to 0^-} f(x) =$ .... B $\lim_{x \to 0^+} f(x) =$ ....

5. The function f is defined on $\mathbb{R}$ such that $f(x) = \begin{cases} 3 & \text{If } x > 0 \\ -3x & \text{If } x \leqslant 0 \end{cases}$

A $\lim_{x \to 0^+} f(x) =$ .... B $\lim_{x \to 0^-} f(x) =$ ....

6. The function f is defined on $\mathbb{R}$ such that $f(x) = \begin{cases} x & \text{If } x \leqslant 0 \\ \frac{1}{x} & \text{If } x < 0 \end{cases}$

A $\lim_{x \to 0^+} f(x) =$ .... B $\lim_{x \to 0^-} f(x) =$ ....

Discuss the existence of limit of each of the following functions

7) $\lim_{x\to 2} f(x)$ where $f(x) = \begin{cases} 2x & \text{when } x < 2 \\ x^2 & \text{when } x > 2 \end{cases}$

8) $\lim_{x\to 3} f(x)$ where $f(x) = \begin{cases} x^2+1 & \text{when } x < -3 \\ 3x+1 & \text{when } x > -3 \end{cases}$

9) $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} x^2+2 & \text{when } x < 0 \\ 3x+1 & \text{when } x > 0 \end{cases}$

10) $\lim_{x\to -1} f(x)$ where $f(x) = \begin{cases} 2x+1 & \text{when } x < -1 \\ -1 & \text{when } x > -1 \end{cases}$

11) If the function f where

$f(x) = \begin{cases} \dfrac{(x-1)^2}{|x-1|} & \text{when } x < 1 \\ 6x - 3\text{m} & \text{when } x > 1 \end{cases}$ has a limit at $x = 1$. find the value of m

12) Discuss the existence of $\lim f(x)$ when $x \longrightarrow \pi$ where

$f(x) = \begin{cases} \dfrac{2\sin x}{\pi - x} & \text{when } x < \pi \\ 1 + \cos x & \text{when } x > \pi \end{cases}$

13) If $\lim_{x\to 2} f(x) = 7$ where $f(x) = \begin{cases} x^2+3\text{m} & \text{when } x < 2 \\ 5x+\text{k} & \text{when } x > 2 \end{cases}$ find the values of m , k

14) If the function f where , $f(x) = \begin{cases} x^2+\text{k} & \text{If } x < -1 \\ x+4 & \text{If } x > -1 \end{cases}$

has a limit at $x = -1$. find the value of k

15) Discuss the existence of $\lim f(x)$ when $x \longrightarrow 0$ where ,

$f(x) = \begin{cases} \dfrac{5x+\tan 2x}{6x+\sin x} & \text{when } x > 0 \\ \cos x & \text{when } x < 0 \end{cases}$

16) Discuss the existence of $\lim f(x)$ where $f(x) = \begin{cases} \dfrac{3x}{\tan x} & \text{when } -\dfrac{\pi}{3} < x < 0 \\ 3\cos x & \text{when } 0 < x < \dfrac{\pi}{3} \end{cases}$

A) at $x \longrightarrow \dfrac{-\pi}{3}$ B) at $x \longrightarrow \dfrac{\pi}{3}$ C) at $x \longrightarrow 0$

17) Discuss the existence of $\lim_{x\to 2} f(x)$ where $f(x) = \dfrac{1}{x-2}$

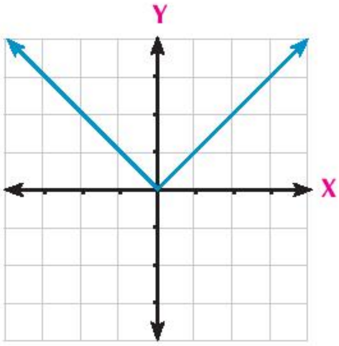
Unit (3)

Continuity

3 - 6

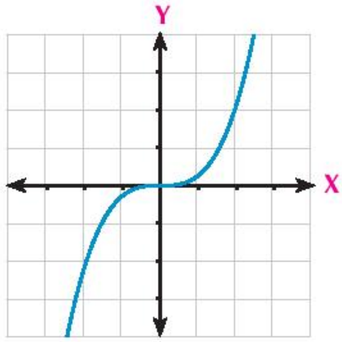
Think and discuss

Look at the following graphs and write your notes?



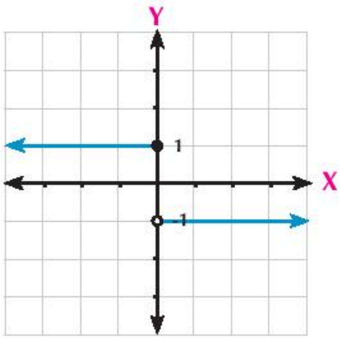
$f_1(x) = |x|$

Fig (1)



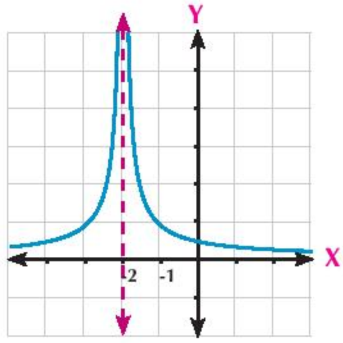
$f_2(x) = 2x^3$

Fig (2)



$$f_3(x) = \begin{cases} 1 & \text{If } x < 1 \\ -1 & \text{If } x > 1 \end{cases}$$

Fig (3)



$$f_4(x) = \frac{1}{|x + 2|}$$

Fig (4)

In **fig**. **(1)** , **(2)**, the curves are continuous and unbroken at any point.

In **fig (3)**, the curve of the function is discontinuous at $x =$ ....

In **fig (4)** the curve of the function is discontinuous at $x =$ ....

from the previous we can conclude that the function f is continuous at $x = a$ if the curve of the function is unbroken at this point and the function is discontinuous at $x = a$ if its curve is broken at this point

You will learn

- Continuity of a function at a point.
- Continuity of a function on an interval

Key terms

- Continuity of a function
- Continuity of a function at a point
- Continuity of a function on an interval

Materials

- Scientific calculator.
- Graph programs

Continuity of a function at a point:

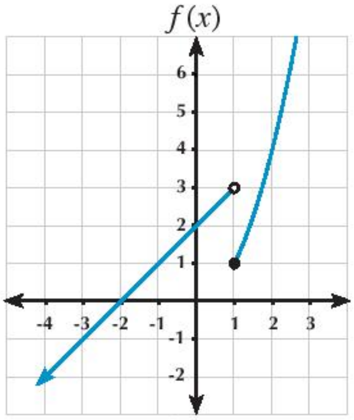


Fig (1)

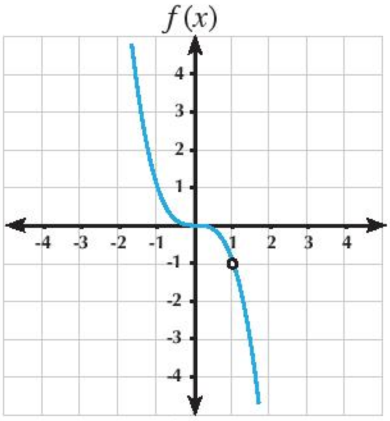


Fig (2)

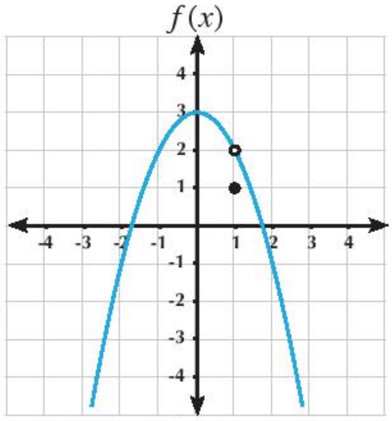


Fig (3)

Look at the previous graphs then find $\lim_{x\to 1} f(x)$, $f(1)$ if exist.

In fig (1) : $\lim_{x\to 1^+} f(x) = 1$, $\lim_{x\to 1^-} f(x) = 3$ **then** $\lim_{x\to 1} f(x)$ doesn't exist, $f(1) = 1$

In fig (2) : $\lim_{x\to 1^+} f(x) = -1$, $\lim_{x\to 1^-} (x) = -1$ **then** $\lim_{x\to 1} f(x) = -1$, $f(1)$ is undefined.

In fig (3) : $\lim_{x\to 1^+} f(x) = 2$, $\lim_{x\to 1^-} (x) = 2$ **then** $\lim_{x\to 1} f(x) = 2$, $f(1) = 1$

then $\lim_{x\to 1} f(x) \neq f(1)$

The function f in each of the previous figures is discontinuous at $x = 1$

Try to put a definition of continuity of a function at a point.

Definition

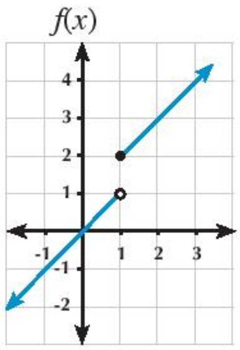
The function f is said to be continuous at $x = a$ if the following conditions satisfied:

- $\lim_{x\to a} f(x)$ exist
- $f(a)$ defined
- $\lim_{x\to a} f(x) = f(a)$

Example **Discussing the continuity of a function**

1 Discuss the continuity of the function f where $f(x) = \begin{cases} x & \text{if } x \leqslant 1 \\ x+1 & \text{if } x > 1 \end{cases}$

a at $x = 0$ **b** at $x = 1$



Solution

a Discuss the continuity at $x = 0$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x = \text{zero}$,

$f(0) = 0$ i.e $\lim_{x\to 0} f(x) = f(0)$

So, the function is continuous at $x = 0$

b Discuss the continuity at $x = 1$

Notice that the rule of the function at the right of the point $x = 1$ differ from it's rule at the left of this point so we discuss the existence of right limit and left limit at $x = 1$

$f(1^+) = \lim\limits_{x \to 1} (x + 1) = 2$, $f(1^-) = \lim\limits_{x \to 1} x = 1$

i.e: $f(1^+) \neq f(1^-)$ and this Condition is enough to discontinuity of the function f at $x = 1$ and the graph illustrates the discontinuity of the function f at $x = 1$

Try to solve

1) Discuss the continuity of the function f where $f(x) = \begin{cases} 4x - 1 & \text{if } x \leqslant 1 \\ x^2 + 2 & \text{if } x > 1 \end{cases}$ at $x = 1$

Example **Check the continuity of the function at a point**

2) Discuss the continuity of each of the following functions at the indicated points infront of each:

a $f(x) = \dfrac{x + 3}{x - 2}$ at $x = 2$, $x = 3$

b $f(x) = 5 - |x - 3|$ at $x = 3$

Solution

a **First:** Discuss the continuity of the function at $x = 2$

$\because$ Since the function domain = R - {2} $\therefore f(x)$ is undefined at $x = 2$

$\therefore f(x)$ is discontinuous at $x = 2$

Second: Discuss the continuity of the function at $x = 3$

$\because f(3) = \dfrac{3 + 3}{3 - 2} = 6$

$\because \lim\limits_{x \to 3} \dfrac{x + 3}{x - 2} = \dfrac{3 + 3}{3 - 2} = 6$

$\therefore \lim\limits_{x \to 3} f(x) = f(3)$

$\therefore f(x)$ continuous at $x = 3$

b By redifining the function $f(x) = \begin{cases} 8 - x & \text{at } x > 3 \\ x + 2 & \text{at } x < 3 \end{cases}$

$\because f(3) = 5$, $\because f(3^+) = \lim\limits_{x \to 3^+} (8 - x) = 5$, $f(3^-) = \lim\limits_{x \to 3^-} (x + 2) = 5$

$\therefore \lim\limits_{x \to 3} f(x) = 5$ **i.e:** $f(3) = \lim\limits_{x \to 3} f(x)$

$\therefore$ f(x) is continuous at $x = 3$

Try to solve

2) Discuss the continuity of the following functions at indicated points:

a $f(x) = \dfrac{x^2 - 4}{x - 2}$ at $x = 1$, $x = 2$

b $f(x) = 3 - |x - 2|$ at $x = 2$

Redefinition of a function to be continuous at a point (if possible)

Example

(3) Redefine (if possible) each of the following function to be continuous at $x = 1$

(a) $f(x) = \frac{x^2 + 2x - 3}{x - 1}$ (b) $f(x) = \begin{cases} x^3 + 2x & , x > 1 \\ 5x - 1 & , x < 1 \end{cases}$

Solution

(a) In order to the function f to be continuous at $x = 1$ then $\lim_{x \to 1} f(x) = f(1)$

$\therefore \lim_{x \to 3} \frac{(x-1)(x+3)}{x-1} = \lim_{x \to 1} (x+3)$

i.e: $\lim_{x \to 1} f(x) = 4$

Therefore, we can redefine the function f to be continuous as:

$$f(x) = \begin{cases} \frac{x^2 + 2x - 3}{x - 1} & \text{at } x \neq 1 \\ 4 & \text{at } x = 1 \end{cases}$$

(b) In order to the function f to be continuous at $x = 1$, then $f(1) = \lim_{x \to 1} f(x)$

$\therefore f(1^+) = \lim_{x \to 1} (x^3 + 2x) = 1 + 2 \times 1 = 3$, $f(1^-) = \lim_{x \to 1^-} (5x - 1) = 5 - 1 = 4$

$\because f(1^-) \neq f(1^+)$ therefore, the function has no limit at $\to 1$

and we can't redefine the function to be continuous at $x = 1$

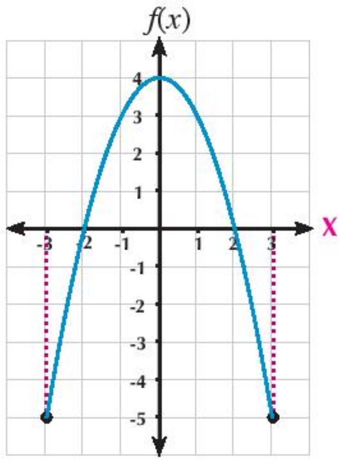
Try to solve

(3) Redefine (if possible) the following function to be continuous at $x = 3$ where $f(x) = \frac{x^2 - 5x + 6}{x - 3}$

(4) Show that $f(x) = \frac{x^2 + 2x - 15}{x - 3}$ is discontinuous at $x = 3$

Then find the value of h that makes $f(x) = \begin{cases} \frac{x^2 + 2x - 15}{x - 3} & \text{at } x \neq 3 \\ h + 1 & \text{at } x = 3 \end{cases}$ continuous at $x = 3$

Continuity of a function on an Interval



The opposite graph represents the curve of the function f where $f(x) = 4 - x^2$ on the interval [-3, 3]. the function f is said to be continuous on [-3, 3] , if it is continuous at all the points of the interval [– 3 ; 3].

i.e: $\lim_{x \to a} f(x) = f(a)$ for all $a \in]-3, 3[$

$\lim_{x \to -3^+} f(x) = f(-3)$ $\lim_{x \to 3^-} f(x) = f(3)$

From the previous we can conclude the following definition:

Definition 2

If $f(x)$ is defined on [a, b] , and

1- $f(x)$ is continuous on]a, b[

2- $\lim_{x \to a^+} f(x) = f(a)$, $\lim_{x \to b^-} f(x) = f(b)$

then f is continuous on [a, b]

Depending on the above definition and the limits of the functions, we can declare some of the continuous functions

1- Polynomial function: continuous on R or on its domain.

2- Rational function: continuous on R except set of zeros of the denominator.

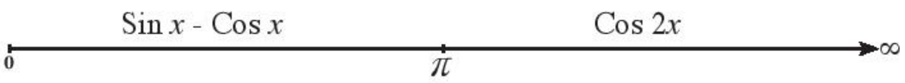
3- Sine and cosine function: continuous on R.

4- Tangent function: $f(x) = \tan x$ continuous on $R - \{x : x = \frac{\pi}{2} + n\pi\}$, $n \in Z$

Example

4 Discuss the continuity of the function f on $[0, \infty[$ where $f(x) = \begin{cases} \sin x - \cos x & \text{when } 0 \leqslant x \leqslant \pi \\ \cos 2x & \text{when } x > \pi \end{cases}$

Solution



$f(x)$ is defined on the interval $[0, \infty[$

To discuss the continuity of the function we will discuss the continuity on its subintervals, also discuss the continuity at the point at which the function changes its rule, also at the right of zero.

1) $f(x) = \sin x - \cos x$ is continuous on $]0, \pi[$

also $f(x) = \cos 2x$ is continuous on $]\pi, \infty[$

2) $f(0) = \sin 0 - \cos 0 = -1$, $\lim_{x \to 0^+} (\sin x - \cos x) = -1$

i.e: $f(0) = \lim_{x \to 0^+} f(x)$ the function is continuous from the right at $x = 0$

3) We discuss the continuity $x = \pi$

$f(\pi) = \sin\pi - \cos\pi = 1$

$f(\pi^-) = \lim_{x \to \pi^-} (\sin x - \cos x) = 1$, $f(\pi^+) = \lim_{x \to \pi^+} \cos 2x = 1$

$f(\pi^-) = f(\pi^+)$, $f(\pi) = 1$ $\therefore f(\pi^-) = f(\pi^+) = f(\pi)$

$\therefore f$ is continuous at $x = \pi$ **from (1) , (2) , (3)** the function is continuous on $[0 , \infty[$

Try to solve

(5) Discuss the continuity of each of the function on its domain

$$f(x) = \begin{cases} 1 + \sin x & 0 \leqslant x < \frac{\pi}{2} \\ 2 + (x - \frac{\pi}{2})^2 & x > \frac{\pi}{2} \end{cases}$$

Example

(5) Discuss the continuity of the following functions :

(a) $f(x) = x^2 - 3x + 2$ (b) $f(x) = \frac{x^2 - 4}{x + 4}$

(c) $f(x) = \frac{\sin x + \cos x}{x^2 - 1}$ (d) $f(x) = \frac{\tan x}{x^2 + 1}$

Solution

(a) $f(x) = x^2 - 3x + 2$ is a polynomial of second degree, then it is continuous on $\mathbb{R}$

(b) $f(x) = \frac{x^2 - 4}{x + 4}$ is fraction function whose domain R - {-4}

$\therefore$ Set of zeroes of denominator = {-4}

$\therefore$ the function is continuous on $\mathbb{R}$ - { -4}

(c) $f(x) = \frac{\sin x + \cos x}{x^2 - 1}$

$\because \sin x$, $\cos x$ are continuous on $\mathbb{R}$

$\because (x^2 - 1)$ is continuous on $\mathbb{R}$

$\therefore$ The zeroes of the denominator are{1, -1) the function f is continuous on $\mathbb{R}$ - {-1, 1}

(d) $f(x) = \frac{\tan x}{x^2 + 4}$

the numerator function: $\tan x$ is continuous on $\mathbb{R} - \{x: x = \frac{\pi}{2} + n\pi, n \in z\}$

the denominator function $x^2 + 1 > 0$ for all x, then there is no zeroes of the denominator.

the function f is continuous on $\mathbb{R} - \{x: x = \frac{\pi}{2} + n\pi, n \in z\}$

Try to solve

(6) Discuss the continuity of each of the following functions:

a) $f(x) = 7$

b) $f(x) = \dfrac{x - 2}{x^2 - 5x + 6}$

c) $f(x) = \dfrac{x^3 + 1}{\sin x}$

d) $f(x) = (x + 1)\cos x$

Rembebre that

If f_1, f_2 are two functions each is continous on R then each of the following functions is also continous:

1- $f_1 \pm f_2$ on R

2- $f_1 \times f_2$ on R

3- $\dfrac{f_1}{f_2}$ on R - set of zeros of f_2.

Activity

(7) **Connected with chemistry**

If the reaction rate of a chemical experiment is given by the function f where $f(x) = \dfrac{0.6\,x}{x+12}$ where x is concentration of the solution.

Search on the internet about chemical experiments can be represented by this function

a) Represent f graphically using a graph program.

b) Discuss the continuity of f.

Example

(6) Show that the function f where $f(x) = \sqrt{x^2 + x + 1}$ is continuous on R

Solution

$\because$ since $x^2 + x + 1$ is positive for all values of $x \in \mathbf{R}$

(The discriminant = $b^2 - 4ac = 1 - 4 = -3 <$ zero)

or $x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4}$ **Sum of two squares**

$\therefore\ x^2 + x + 1$ is positive for all values of $x \in \mathbf{R}$

$\therefore f(x) = (\sqrt{x^2 + x + 1})$ defined for all values of $x \in \mathbf{R}$

For all $a \in \mathbf{R}$ we find that $f(a) = \sqrt{a^2 + a + 1}$

$$\lim_{x \to a} f(x) = \lim_{x \to a} (\sqrt{x^2 + x + 1}) = \sqrt{a^2 + a + 1}$$

$\therefore f(a) = \lim\limits_{x \to a} f(x)$ for all $a \in \mathbf{R}$

$\therefore f(x)$ is continuous on $\mathbf{R}$

Try to solve

(8) Discuss the continuity of the function f where $f(x) = \sqrt{x - 2}$ on it's domain.

Exercises 3 - 6

Discuss the continuity of each of the following functions at the indicated points:

1) $f(x) = \begin{cases} x^2 + 2, & x \leqslant 1 \\ 3x, & x > 1 \end{cases}$ at $x = 1$

2) $f(x) = \begin{cases} x^2 - 1, & x \leqslant 2 \\ 2x - 1, & x > 2 \end{cases}$ at $x = 2$

3) $f(x) = \begin{cases} x^2 - 3x + 2, & x \leqslant 3 \\ x^2 - 2x - 1, & x > 3 \end{cases}$ at $x = 3$

4) $f(x) = \begin{cases} x + 4, & x < -2 \\ 1, & -2 \leqslant x < -1 \\ 2x + 3, & x \geqslant -1 \end{cases}$ at $x = -2$, at $x = -1$

5) $f(x) = \begin{cases} \dfrac{\sin(x - 2)}{x^2 - 4}, & x < 2 \\ 1 - \dfrac{3}{x^2}, & x \geqslant 2 \end{cases}$ at $x = 2$

6) $f(x) = \begin{cases} \dfrac{1 - \cos x}{x}, & x > 0 \\ 2\sin x, & x \leqslant 0 \end{cases}$ at $x = 0$

Discuss the continuity of each of the following functions on $\mathbb{R}$:

7) $f(x) = x^3 - 2x^2 + 1$

8) $f(x) \dfrac{-x}{x^2 + 1}$

9) $f(x) = \dfrac{3x + 2}{x^2 - 2x}$

10) $f(x) = \dfrac{2x - 3}{x^2 - 2x - 15}$

11) $f(x) = \dfrac{x}{|x| - 2}$

12) $f(x) = \dfrac{|x + 2|}{(x + 2)^2}$

13) $f(x) = \sin x - 3\cos(x + 1)$

14) $f(x) = x^2 + \cos^2 x$

15) $f(x) = x^3 \sin 2x$

16) $f(x) = \tan^2 x - 1$

17) $f(x) = \dfrac{\sin^2 x + \cos x}{x^2 - 9}$

18) $f(x) = \dfrac{\tan x}{x^2 - 9}$

Discuss the continuity of each of the following functions on the indicated intervals:

19) $f(x) = \begin{cases} \dfrac{x\tan x + \sin^2 3x}{5x^2}, & -\dfrac{\pi}{4} < x < 0 \\ 2\cos 2x, & 0 \leqslant x < \dfrac{\pi}{4} \end{cases}$ on the interval $]-\dfrac{\pi}{4}, \dfrac{\pi}{4}[$

20) $f(x) = \begin{cases} \dfrac{x^6 - 1}{x^3 - 1}, & -4 < x < 1 \\ 3x - 1, & 1 \leqslant x < 4 \\ x^2, & 4 \leqslant x < 6 \end{cases}$ on the interval $]-4, 6[$

Find the values of a in each of the following:

21) $f(x) = \dfrac{x + 3}{x^2 + ax + 9}$ is continuous on $\mathbb{R}$

22) $f(x) = \begin{cases} \dfrac{(x+3)^4 - 81}{x} & \text{when } x \neq 0 \\ a & \text{when } x = 0 \end{cases}$ is continuous on $\mathbb{R}$

Find the values of b, c in each of the following:

23) $f(x) = \begin{cases} x + 1 & , \quad 1 < x < 3 \\ x^2 + b\,x + c & , \quad x \in \mathbb{R} -]1, 3[\end{cases}$ is continuous on $\mathbb{R}$

24) $f(x) = \begin{cases} x + 2b & , \quad x < -2 \\ 3\,b\,x + c & , \quad -2 \leqslant x \leqslant 1 \\ 3x - 2b & , \quad x > 1 \end{cases}$ is continuous on $\mathbb{R}$

Redefine each of the following functions to be continuous on the indicated point (if possible):

25) $f(x) = \begin{cases} x^2 + 1 & , \quad x > 2 \\ \dfrac{x^2 - 4}{x - 2} & , \quad x < 2 \end{cases}$ at $x = 2$

26) $f(x) = \begin{cases} \dfrac{3x + 1 - \cos x}{5x} & , \quad x > 0 \\ \dfrac{3}{5}\cos x & , \quad x < 0 \end{cases}$ at $x = 0$

27) $f(x) = \begin{cases} \dfrac{(x-3)^{50} + (x-3)}{x - 3} & , \quad x > 3 \\ \cos(3 - x) & , \quad x < 3 \end{cases}$ at $x = 3$

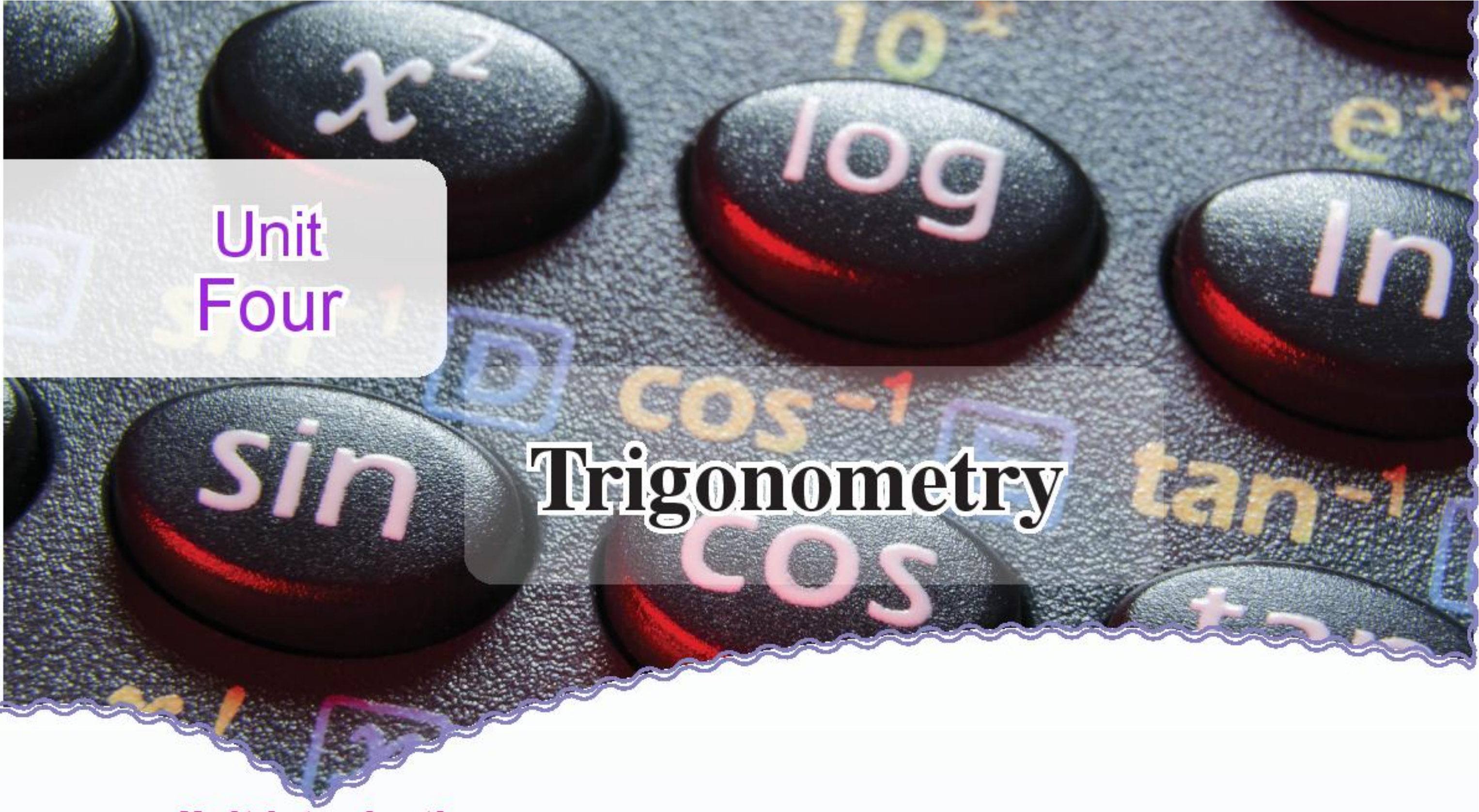
28) $f(x) = \dfrac{x^2 - x - 6}{x - 3}$ at $x = 3$

29) Find the value of c which makes the function f is continuous at $x = c$ where:

$$f(x) = \begin{cases} 2 - x^2 & x \leqslant c \\ x & x > c \end{cases}$$

General Exercises

For more exercises, please visit the website of Ministry of Education.



Unit Four

Trigonometry

Unit introduction

Trigonometry is one of mathematics branches . The ancient Egyptians were the first to work with the rules of trigonometry , they used it to build their pyramids and temples . Our knowledge to trigonometry refers to the Greeks who put the laws of it, and used it to deduce some relations joining the lengths of sides of triangle by the measures of its angles . The Arabs and Muslims scientists contributed in the solutions of trigonometric equations, and they used the tangents secants and their correspondences to measure the angles and distances . They created a way to construct tables of sines for coplanar triangles . we would like to point to the Swiss scientist Leonhard Euler (1707 -1783) who introduced anew expression for the trigonometric functions . he also used a lot of mathematical symbols which enables in the use of advanced mathematical problems studied in schools and universities .

In this unit we will hand laws and relations the sides of triangle by its angles .

Unit objectives

By the end of this unit the student should be able to:

- Deduce the sin rule which states In any triangle the lengths of sides Are proportional to the sines of opposite angles
- Use the sin rule to find lengths of sides of triangle
- Use the sin rule to find the measures of angles (two solutions for unknown angle)
- Deduce the relation between the sin rule,
- The length of radius of circumcircle of triangle Use it to solve different exercises
- Deduce the cosine rule for any triangle
- Use the cosine rule to find length of unknown
- Side of triangle
- Use the cosine rule to find to find the measure Of unknown angle of triangle
- Use the sine , cosine rules to solve triangle given Measures of two angles , length of one side Lengths of two sides, measure of included angle Lengths of the three sides
- Use the calculator to solve exercises , different Activities on sine ,cosine rules

Key terms

- Trigonometry
- Sine Rule
- Acute Angle
- Obtuse Angle
- Right Angle
- Possible Solutions
- Unique Solution
- Shortest Side
- Longest Side
- Area of The Triangle
- The Lengths of The Sides of the Triangle
- Opposite Angle of a Side
- Smallest Angle
- Largest Angle
- Cosine Rule

Lessons of the unit

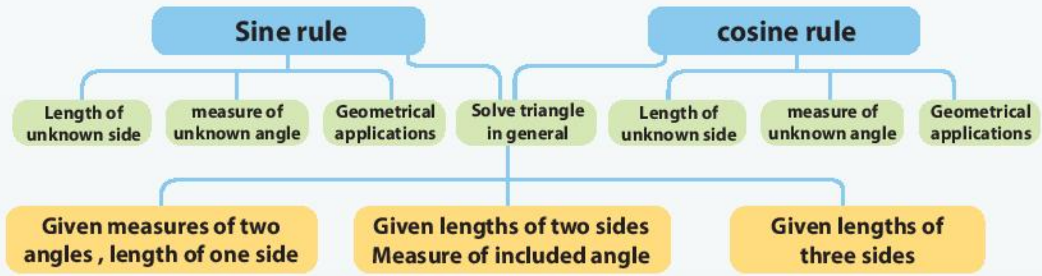
Lesson (4 - 1): The sine rule.

Lesson (4 - 2): The cosine rule.

Materials

Scientific calculator

Chart of the unit



Unit Four

4 - 1 The Sine Rule

You will learn

- The sine rule (law) for any triangle.
- Using the sine rule to solve the triangle.
- Modelate and problem solving using the sine rule.
- The relation between the sine rule and the radius of the circumcircle of any triangle.

Key terms

- Trigonometry
- Sine Rule
- Acute Angle
- Obtuse Angle
- Right Angle

Materials

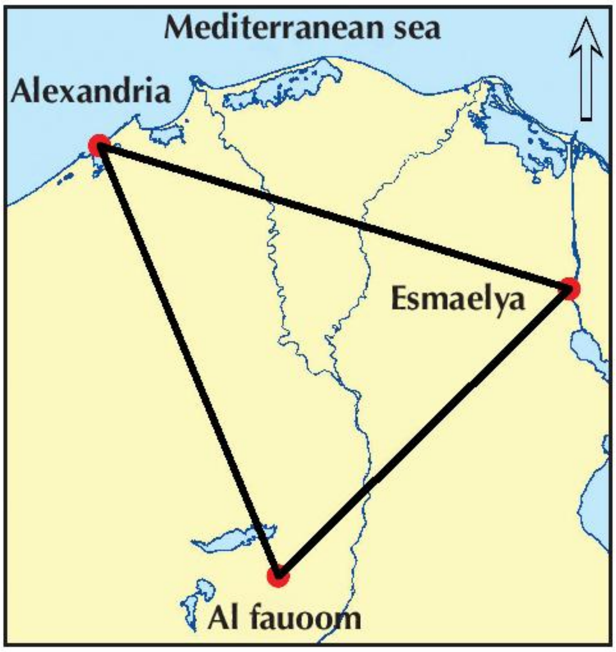
- Scientific calculator

You have learned how to find the length of a side of the right-angled triangle given the lengths of the other two sides or the length of one of its sides and the measure of one of its acute angles. Now you will learn other methods to find the lengths of the sides and the measures of the angles of the triangle in general.

Activity

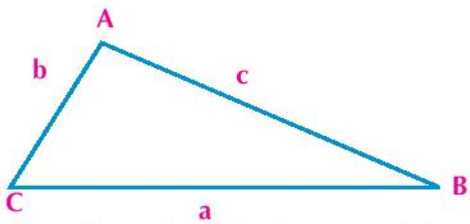
Kareem wanted to measure the distance between Alfaiyum and Ismailia using the data on the given map by taking drawing scale 1 cm : 43 km. Be sure of your measures after you have studied the methods of solving of non right angled triangles.

one of these methods is the sine rule



Learn

The Sine Rule



In triangle ABC we use the small letters a, b and c to denote the lengths of the sides opposite to angles A, B and C respectively. We can use formula of the the area of the triangle to conclude the sine rule which gives the relation between the lengths of the sides of a triangle and sines of the opposite angles as follow.

area of triangle = $\frac{1}{2}$ b c sin A = $\frac{1}{2}$ a c sin B = $\frac{1}{2}$ a b sin C

(different forms of the area of the triangle ABC)

$bc \sin A = ac \sin B = ab \sin C$ **multiplaying by 2**

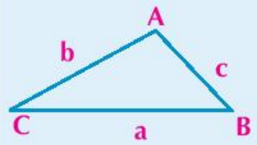
$\frac{bc \sin A}{abc} = \frac{ac \sin B}{abc} = \frac{ab \sin C}{abc}$ **dividing by abc**

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ **by simplifying**

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ **properties of proportion**

Then: In any triangle, the lengths of the sides are proportional to the sines of the opposite angles. This relation is known by the sine rule i.e. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Remember that



area of triangle ABC

$= \frac{1}{2} ab \sin C$

$= \frac{1}{2} ac \sin B$

$= \frac{1}{2} bc \sin a$

Self-learning: Can you prove the sin Rule by other methods? Show that.

Using the sine Rule to find the length of a side of a triangle

Example

(1) Find the length of the longest side in the triangle ABC in which $m(\angle A) = 54° 33'$, $m(\angle B) = 49° 22'$, $a = 124.5$cm

Solution

The longest side is opposite to the greatest angle **(inequality of triangle)**

$m(\angle C) = 180° - [m(\angle A) + m(\angle B)]$

$= 180° - [45° 33' + 49° 22'] = 76° 5'$

$\therefore$ the longest side is c (opposite to the greatest angle $\angle C$)

$\because \frac{a}{\sin A} = \frac{c}{\sin C}$ $\therefore \frac{124.5}{\sin 54° 33'} = \frac{c}{\sin 76° 5'}$

$\therefore c = \frac{124.5 \sin 76° 5'}{\sin 54° 33'} = 148.4$cm

Remember that

In any triangle the longest side is opposite to the biggest angle and the shortest side is opposite to the smallest angle.

Try to solve

(1) Find the length of the shortest side in the triangle ABC in which $m(\angle A) = 43°$, $m(\angle B) = 65°$, $c = 8.4$cm

Solving the triangle using the sine rule

Solving the triangle is to find its unknown elements using the given measurements, in condition that aside length is to be given at least.

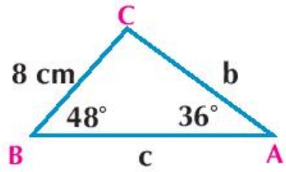
Solving the triangle, given the length of one side and the measures of two angles:

Example

(2) Solve the triangle ABC in which $a = 8$cm, $m(\angle A) = 36°$, $m(\angle B) = 48°$

Solution

$m(\angle C) = 180° - (36° + 48°) = 96°$



$\because \frac{\sin A}{a} = \frac{\sin B}{b}$ $\therefore \frac{\sin 36°}{8} = \frac{\sin 48°}{b}$

$\therefore b = \frac{8 \sin 48°}{\sin 36°} \simeq 10.114$ cm.

Use the calculator with mode of the degree measurement then press the keys from left to right:

start → (8 × SIN 4 8) ÷ SIN 3 6 =

$\because \frac{\sin A}{a} = \frac{\sin C}{c}$ $\therefore \frac{\sin 36°}{8} = \frac{\sin 96°}{c}$ $\therefore c = \frac{8 \sin 96°}{\sin 36°} \simeq 13.535$cm

By using calculator:

start → (8 × SIN 9 6) ÷ SIN 3 6 =

In $\triangle ABC$: $b = 10.114$ cm , $c = 13.535$ cm ,

Try to solve

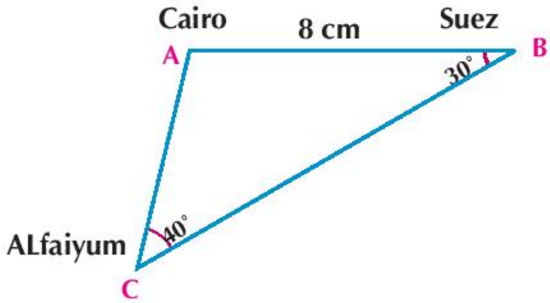
2. Solve the triangle ABC in which $a = 8$cm , $m(\angle A) = 60°$, $m(\angle B) = 40°$

Example

3. **Geography:** The opposite figure shows the positions of three Egyptian towns that form a triangle. If the distance between Cairo and Suez on the map is 8 cm, the measure of the angle at Alfaiyum is 40° and that at Suez is 30°. If the drawing scale is 1cm:16.75 km. Calculate to the nearest Km.

a. The distance between Cairo and Alfaiyum.

b. The distanc between Suez and Alfaiyum.



Solution

$m(\angle A) = 180° - (30° + 40°) = 110°$

$\because \frac{AC}{\sin 30°} = \frac{BC}{\sin 110°} = \frac{8}{\sin 40°}$

$\therefore AC = \frac{8 \times \sin 30°}{\sin 40°} \simeq 6.22$cm

$\therefore$ the distance between Cairo and Alfaiyum $\simeq 6.22 \times 16.75 \simeq 104$ km

$BC = \frac{8 \times \sin 110°}{\sin 40°} \simeq 11.7$cm

$\therefore$ the distance between Suez and Alfaiyum $\simeq 11.7 \times 16.75 \simeq 196$ km

Geometrical Applications on the Sine Rule

well known problem

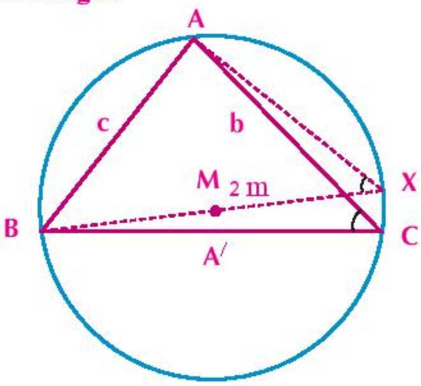
In any triangle ABC: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$

where r is the length of the radius of the circumcircle of triangle ABC

Proof :- **(not required)**

If the circle passes through the vertices of an acute angled triangle

Draw the circumcircle of the acute angled triangle ABC and draw the diameter $\overline{BX}$ and the chord $\overline{XA}$



$\because m(\angle BAX) = 90°$, $m(\angle AXB) = m(\angle ACB)$

$\therefore \sin X = \frac{AB}{BX}$, $\sin C = \frac{AB}{BX}$

$AB = BX \sin C$

$\therefore c = 2r \sin C$ $\qquad \frac{c}{\sin C} = 2r$

Similarly, we can prove that: $\frac{a}{\sin A} = 2r$, $\frac{b}{\sin B} = 2r$

$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$

Self learning: Prove the previous law if the circle passes through vertices of on obtuse angled triangle.

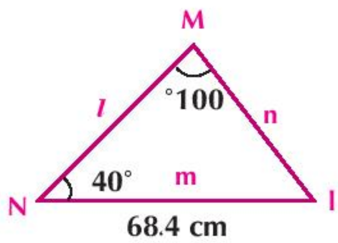
Example

(4) triangle LMN in which , m = 68.4cm , m(∠M) = 100° , m(∠N) = 40° find:

(a) l (b) The length of the radius of circumcircle of the triangle LMN

(c) The surface area of the triangle LMN

Solution



$m(\angle L) = 180° - (100° + 40°) = 40°$

$\frac{l}{\sin 40°} = \frac{68.4}{\sin 100°}$ **(sin rule)**

$l = \frac{68.4}{\sin 100°} \times \sin 40° \simeq 44.64 cm$ **required (1)**

$\because \frac{m}{\sin M} = 2r$ $\qquad \therefore \frac{68.4}{\sin 100°} = 2r$

i.e. $r = \frac{68.4}{2 \sin 100°} \simeq 34.72$ cm **required (2)**

S.A of triangle LMN $= \frac{1}{2} m\, l \sin N = \frac{1}{2} \times 68.4 \times 44.64 \sin 40° = 981.1\ cm^2$

Try to solve

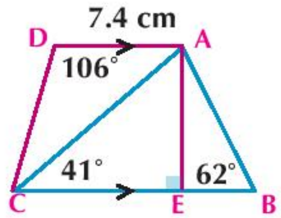
(3) ABC is a triangle in which a = 25cm , m(∠B) = 35° 18' ,
m(∠C) = 103° 42' Find its area and the length of the radius of its circumcircle

Example

(5) ABCD is a trapezium in which $\overline{AD} \parallel \overline{BC}$, AD = 7.4cm , m(∠B) = 62° , m(∠D) = 106°,
m(∠ACB) = 41°. find

1st: the length of each $\overline{AC}$, $\overline{BC}$ **2nd:** The surface area of the trapezium ABCD.

Solution



In the triangle ACD

$\because m(\angle DAC) = m(\angle ACB) = 41°$ **(alternate)**

, $m(\angle ACD) = 180° - (41° + 106°) = 33°$

$\because \frac{AC}{\sin 106°} = \frac{7.4}{\sin 33°}$ $\quad \therefore AC = \frac{7.4 \times \sin 106°}{\sin 33°} = 13.06cm$

in triangle ABC $\quad m(\angle BAC) = 180° - (62° + 41°) = 77°$

$\because \frac{BC}{\sin 77°} = \frac{13.06}{\sin 62°}$ $\quad \therefore BC = \frac{13.06 \times \sin 77°}{\sin 62°} = 14.41cm$

$\because \frac{AE}{AC} = \sin 41°$ $\quad \therefore AE = 13.06 \times \sin 41° = 8.568cm$

$\because$ **S.A of trapezium** = **length of middle base × height.**

$$= \frac{AD + BC}{2} \times AE = (\frac{7.4 + 14.41}{2}) \times 8.568 = 92.434cm^2 \simeq 92cm^2$$

Try to solve

4. ABCD is a quadrilateral in which, CD = 100cm , $m(\angle BCA) = 36°$, $m(\angle BDA) = 55°$, $m(\angle BCD) = 85°$, $m(\angle CDA) = 87°$, Find the length of $\overline{BD}$, $\overline{AC}$ to the nearest cm.

Exercises 4 - 1

Complete each of the following:

1. In any triangle, the lengths of the sides are proportional to ....
2. In triangle ABC if 2 sin A = 3 sin B = 4 sin C then a : b : c = ...
3. ABC is an equilateral triangle the length of its side is $10\sqrt{3}$ cm, then the length of the diameter of its circumcircle = ...
4. ABC is a triangle in which $m(\angle A) = 60°$, $m(\angle C) = 40°$, c = 8.4cm then a = ... cm
5. In triangle ABC, $\frac{2b}{\sin B} = $ ....r

Choose the correct answer:-

6. The length of the radius of the circumcircle of triangle ABC in which $m(\angle A) = 30°$, a = 10cm is ....

   A 10cm B 20cm A 5cm D 40cm

7. If the length of the radius of the circumcircle of the triangle ABC is 4cm , $m(\angle A) = 30°$ then a =

   A 4cm B 2cm A $4\sqrt{3}$ cm D $\frac{1}{16}$ cm

8. In triangle ABC , the expression 2r sin A equals ....

   A a B b A c D area of($\triangle ABC$)

9. If r is the length of the radius of the circumcircle of triangle XYZ , then $\frac{y}{2 \sin Y} = $ ....

   A r B 2r A $\frac{1}{2}$ r D 4r

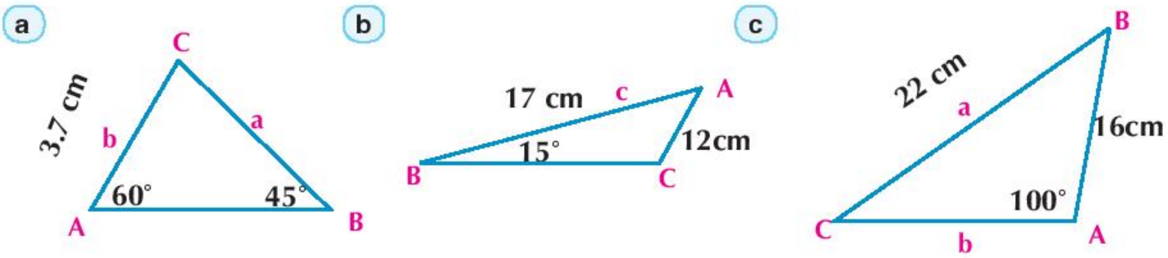
10. In triangle LMN, m(∠L) = 30°, MN = 7cm then the length of the diameter of the circumcircle of triangle LMN is ....

   A) 7cm B) 3.5cm A) 14cm D) $\frac{14}{\sqrt{3}}$cm

11. In triangle XYZ, if 3 sin X = 4 sin Y = 2 sin Z , then x : y : z = ...

   A) 2 : 3 : 4 B) 6 : 4 : 3 A) 3 : 4 : 6 D) 4 : 3 : 6

12. **Solve each of the following triangles**:



13. ABC is a triangle in which m(∠A) = 60°, m(∠B) = 45°, prove that: a : b : c = $\sqrt{6}$: 2 : $\sqrt{3}$ + 1

14. ABCD is a parallelogram in which AB = 19.77cm the diagonals $\overline{AC}$, $\overline{BD}$ make with the side $\overline{AB}$ angles of measures 36° 22', 44° 58', Find the lengths of the diagonals.

15. ABCD is a trapezium in which $\overline{AD}$ // $\overline{BC}$, AD = 10.7cm, m(∠D) = 100°, m(∠B) = 61°19', m(∠CAD) = 33° 50' , Find the length of each of $\overline{AC}$, $\overline{BC}$

16. ABCD is a quadrilateral in which m(∠BCD) = 85°, m(∠CDA) = 87°, m(∠BCA) = 36°, m(∠BDA) = 55°, CD = 1000 meter Find to the nearest meter the length of $\overline{BD}$, $\overline{AC}$.

17. ABC is a triangle in which sin C = 0.35 , c = 14cm, find the area of the circumcircle of triangle ABC in terms of π

18. ABC is a triangle in which a = 58 cm, m(∠B) = 38°, (∠C) = 62° find the length of the perpendicular from A to $\overline{BC}$

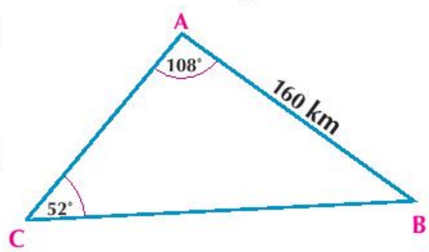
19. ABC is a triangle in which m(∠A) = 60°, m(∠B) = 45°, if a + b = ($\sqrt{6}$ + 2) cm find each of a, b

20. ABC is a triangle inscribed in a circle of diameter length 20 cm. If m(∠A) = 42°, m(∠B) = 74° 48', Find the lengths of the sides of the triangle ABC

21. ABC is a triangle in which c = 19cm, m(∠A) = 112°, m(∠B) = 33°, Find to the nearest hundredth the length of each of b and the radius of the circumcircle of the triangle.

22. **Geography:** The opposite figure represents the positions of three towns A, B and C Find to the nearest km:

   a) Distance between A , C b) Distance between B and C



23. **Creative thinking:**

   a) In triangle ABC prove that: $\frac{3a - 4b}{3 \sin A - 4 \sin B} = \frac{c}{\sin C}$

   b) If Δ is the S.A of the triangle ABC prove that area of $\Delta = a^2 \left(\frac{\sin B \sin C}{2 \sin A}\right)$

Unit Four

4 - 2 The Cosine Rule

You will learn

- The cosine rule of any triangle.
- Using the cosine rule to solve the triangle .
- Modelate and solving mathematical and life problems using the cosine rule.

Key terms

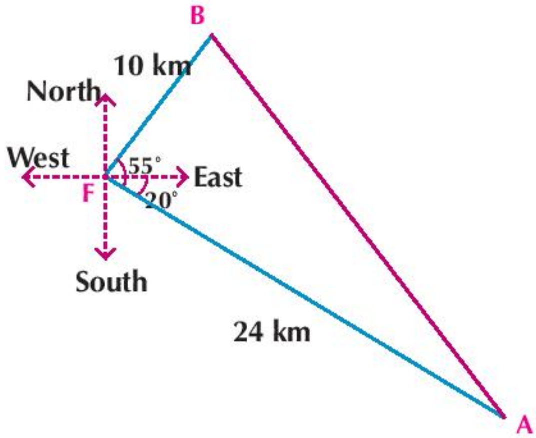
- Cosine Rule
- Acute Angle
- Obtuse Angle
- Right Angle

Materials

- Scientific Calculator

Think and discuss

Two ships A , B move at the same moment from a port. The first in direction 20° south of east, for 24 km while the second moved in direction 55° north of east for 10 km in the same time. Calculate the distance between them at the end of this time.



By using the suitable drawing scale to find the length of $\overline{AB}$.

Are you able to use the sine rule to find the length of $\overline{AB}$?

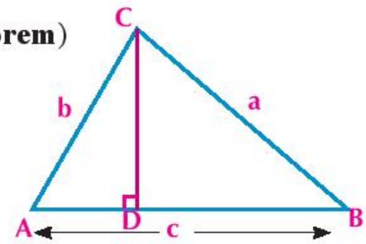
Can you deduce another rule to find the length of $\overline{AB}$ in terms of the lengths of $\overline{FA}$, $\overline{FB}$ And the measure of the included angle between them. explain that.

Learn

The Cosine Rule

$\triangle$ BDC is a right angled at D:

$(BC)^2 = (CD)^2 + (BD)^2$ (**pythagoras theorem**)



then:

$a^2 = (b \sin A)^2 + (c - b \cos A)^2$

$= b^2 \sin^2 A + b^2 \cos^2 A + c^2 - 2bc \cos A$

(**Expanding brackets**)

$= b^2 (\sin^2 A + \cos^2 A) + c^2 - 2bc \cos A$ (b^2 **common factor**)

$= b^2 + c^2 - 2bc \cos A$ (**simplifying**)

then:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Remember

Pythagoras identity

$\sin^2 A + \cos^2 A = 1$

Critical thinking: Is the previous law is true when $\triangle$ ABC is right angled at A? Explain your answer.

The cosine rule provides that :

In any triangle ABC:

$a^2 = b^2 + c^2 - 2\,b\,c\cos A$, $\cos A = \dfrac{b^2+c^2-a^2}{2\,b\,c}$

$b^2 = c^2 + a^2 - 2\,c\,a\cos B$, $\cos B = \dfrac{c^2+a^2-b^2}{2\,c\,a}$

$c^2 = a^2 + b^2 - 2\,a\,b\cos C$, $\cos C = \dfrac{a^2+b^2-c^2}{2\,a\,b}$

Hint

It is better to write the laws of cosine to take the sides of the traingle a, b, c in the same way around, so if one formula is known we can deduce the other forms.

Activity

(a) using the calculator to find the length of unknown side of triangle by cosine rule

One of the engineers tried to find the distance between two positions not easy to reach them by using the measuring distances set . He found that his distance from the first point A was 160 m, and from the second C was 220m , m($\angle$BAC) = 54°. **using these data. Find the distance between the two points to the nearest km.**

1 - Determine accurately the data collected by the engineer.

2 - Determine the required.

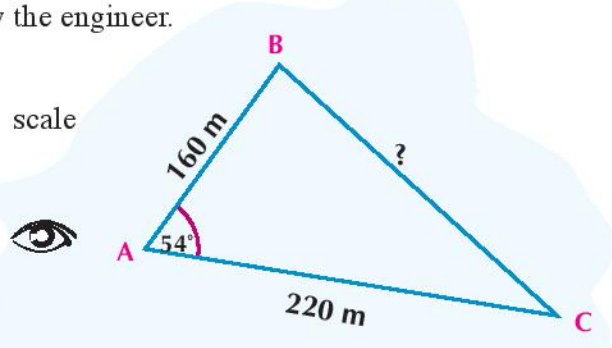
3 - Represent these data by suitable drawing scale using geometrical sets .

4 - Measure $\overline{BC}$ in cm.

5 - Find the real distance between B, C in km.

6 - Can you use the cosine rule to calculate the distance between B , C? Show that.

7 - Compare between the result you got BC using the geometrical measures, and the cosine rule.



Remember

Real length = Drawing length ÷ scale

From the previous activity:

1 - The drawing scale is 1cm to 20 km

2 - By measuring: The length of $\overline{BC}$ = 9 cm in drawing.

3 - The real length of $\overline{BC} \simeq 9 \times 20 \simeq 180$ km

4 - Using the cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$

Then $a^2 = (160)^2 + (220)^2 - 2 \times 160 \times 220 \cos 54° \simeq 32619.9$

Then a $\simeq$ 180.6 km.

5 - The results will be more accurate when the drawing is accurate. We prefere to use laws to give true results.

6 - Using a calculator to find the result:

start ↓

[1] [6] [0] [x^2] [+] [2] [2] [0] [x^2] [−] [2] [×] [1] [6] [0] [×] [2] [2] [0] [×] [cos] [5] [4] [(] [=] [√] [=]

Application on activity: Find the length of the third side (to the nearest $\frac{1}{100}$) in $\triangle$ ABC in which:

a a = 4.36 cm , b = 3.84 cm, m ($\angle$ C) = 101°

b b = 2 cm , c = 5 cm, m ($\angle$A) = 60°

Finding the measure of an angle of a triangle given the lengths of the three sides :

Example

1 Find the measure of the greatest angle of the triangle ABC in which a = 4.6 , b = 3.2 , c = 2.8

Solution

$\because$ The greatest angle is opposite to the longest side

$\therefore$ $\angle$A is the greatest angle

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(3.2)^2 + (2.8)^2 - (4.6)^2}{2 \times 3.2 \times 2.8}$$

by using the calculator

start ↓

[3.2] [x^2] [+] [2.8] [x^2] [−] [4.6] [x^2] [=] [÷] [(] [2] [×] [3.2] [×] [2.8] [)] [=] [Shift] [cos] [=] [°′″]

The cosine is negative , then $\angle$ A is an obtuse angle

$\therefore$ m ($\angle$ A) = 99° 53' 49"

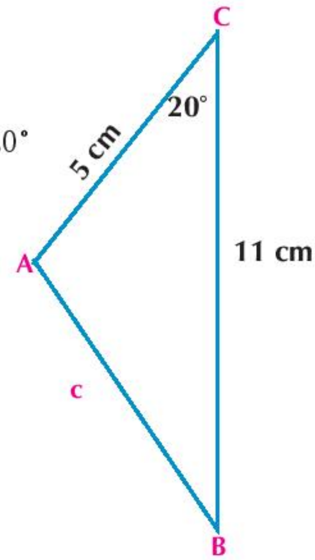
Try to solve

1 Find the measure of the greatest angle of the triangle ABC in which a = 11 cm, b = 10 cm , c = 8 cm

Using the cosine rule for solving the triangle:

The cosine rule allows us to solve the triangle given the lengths of two sides and the measure of the included angle.

First: Solving the Triangle Given the Lengths of Two Sides and the Measure of the Included Angle



Example

(2) Solve the triangle ABC in which a = 11cm , b= 5cm , m (∠ C) = 20°

Solution

We have to find c , m (∠ A) , m (∠ B)

∵ $c^2 = a^2 + b^2 - 2\,ab\cos C$ **(cosine rule)**

$= (11)^2 + (5)^2 - 2 \times 11 \times 5 \cos 20°$

$c = \sqrt{(11)^2 + (5)^5 - 2 \times 11 \times 5 \cos 20°} \simeq 6.529$ cm

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{(5)^2 + (6.529)^2 - (11)^2}{2\,(5)\,(6.529)} \simeq -0.817$

m (∠ A) = 144° 49'

m (∠ B) = 180° - [m (∠ A) + m (∠ C)]

= 180° - [144.786° + 20°] = 15.214°

Remark When you find the measure of an angle of a triangle given that the lengths of two sides of the triangle and the measure of the included angle, it is preferred to use the cosine law rather than the sine law because:

1- In the case of using the **sine law:**

- The sine of the acute and obtuse angles is always positive.

2- In the case of using the **cosine law:**

- The cosine of the obtuse angle is negative.
- The cosine of the acute angle is positive.
- The cosine law allows to solve triangle given lengths of the three sides.

  Given that the sum of lengths of two sides is greater than the length of the 3rd side.

Hint

You can use the sine rule to find m (∠ A), m (∠ B) after finding c, but the beneft coming from using cosin rule is to distinguish between the acute and obtuse angles.

Try to solve

(2) Solve the triangle ABC in which a = 24.6 cm , c= 14.2 cm, m (∠ B) = 42° 18'

Second: Solving the triangle given the length of the three sides:

Example

(3) Solve the triangle ABC in which a = 9cm , b = 7cm , c= 5cm.

Solution

the required is finding the measures of the angles A , B and C

$\because \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{7^2 + 5^2 - 9^2}{2 \times 7 \times 5} = -0.1$

$m(\angle A) \simeq 95° 44' 21''$

$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{5^2 + 9^2 - 7^2}{2 \times 5 \times 9} \simeq 0.633$

$m(\angle B) \simeq 50° 42' 13''$

$m(\angle C) = 180° - [m(\angle A) + m(\angle B)] = 33° 33' 26''$

Try to solve

(3) Solve the triangle ABC in which a = 12.2cm, b = 18.4cm, c = 21.1cm

Geometric Applications on the Cosine Rule

Example

(4) ABC is a triangle in which a = 63cm , b - c = 27 cm, the perimeter of the triangle is 140 cm, Find each of b, c and the measure of the smallest angle of the triangle also find the area of the triangle to the nearest centimeter square.

Solution

$\because a + b + c = 140$ **(perimeter of the triangle)** , a = 63

$\therefore b + c = 140 - 63$ **then** $b + c = 77$ **(1)**

$\because b - c = 27$ **(given)** **(2)**

by adding (1) , (2) :

$2b = 104$ **then** $b = 52$ cm

substituting in (1) $c = 25$cm

We see that c is the shortest side of triangle ABC

$\therefore \angle C$ is the smallest angle of triangle ABC

$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(63)^2 + (52)^2 - (25)^2}{2 \times 63 \times 52} = 0.9230769$

$\therefore m(\angle C) = 22° 37'$

the surface area of the triangle ABC $= \frac{1}{2} ab \sin C$

$= \frac{1}{2} \times 63 \times 52 \times \sin 22° 37' \simeq 630 cm^2$

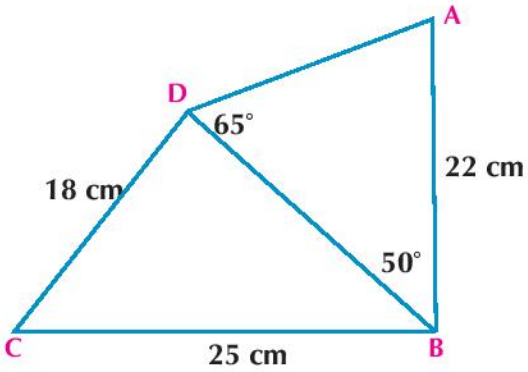
Try to solve

4) $\triangle ABC$ in which $b = 4cm$, $a + c = 11cm$, $a - c = 1cm$, prove that $m(\angle A) = 2m(\angle B)$, then find the perimeter and the area of the triangle ABC, round the area to the nearest centimeter square.

Example

5) ABCD is a quadrilateral in which AB = 22cm, $m(\angle ADB) = 65°$, $m(\angle DBA) = 50°$, BC = 25cm, DC = 18cm, Find: $m(\angle CBD)$, $m(\angle BCD)$

Solution



In $\triangle ABD$

$m(\angle A) = 180° - (50° + 65°)$
$= 180° - 115° = 65°$

$\because m(\angle A) = m(\angle D) = 65°$

$\therefore AB = BD = 22cm$

In $\triangle DBC$

$$\cos(\angle DBC) = \frac{(BD)^2 + (BC)^2 - (DC)^2}{2(BD)(BC)} = \frac{(22)^2 + (25)^2 - (18)^2}{2 \times 22 \times 25} \simeq 0.7137$$

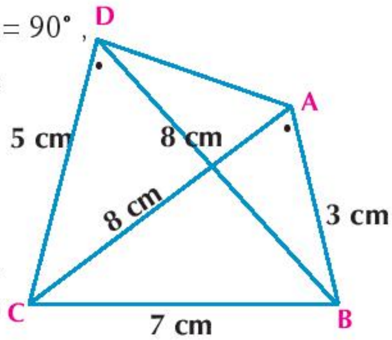
$\therefore m(\angle DBC) \simeq 44° 28' 6''$

$$\cos(\angle BCD) = \frac{(BC)^2 + (CD)^2 - (BD)^2}{2(BC)(CD)} = \frac{(25)^2 + (18)^2 - (22)^2}{2 \times 25 \times 18} \simeq 0.5167$$

$\therefore m(\angle BCD) \simeq 58° 53' 28''$

Try to solve

5) ABCD is a quadrilateral in which $m(\angle DAB) = m(\angle DBC) = 90°$, BD = 10cm, AD = 8cm, $m(\angle DCB) = 30°$, Find AC to the nearest centimeter.



Example

6) ABCD is a quadrilateral in which AB = 3cm, AC = 8cm, BC = 7cm, CD = 5cm, BD = 8cm, prove that the shape ABCD is a cyclic quadrilateral.

Solution

In $\triangle ABC$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(8)^2 + (3)^2 - (7)^2}{2 \times 8 \times 3} = \frac{1}{2}$$

$\therefore m(\angle A) = 60°$ **(1)**

In $\triangle BDC$

$$\cos D = \frac{(CD)^2 + (DB)^2 - (BC)^2}{2(CD)(DB)} = \frac{(5)^2 + (8)^2 - (7)^2}{2 \times 5 \times 8} = \frac{1}{2}$$

$\therefore$ m ($\angle$ D) = 60° **(2)**

$\because$ m ($\angle$ BAC) = m ($\angle$ BDC) and they are on the same base $\overline{BC}$ and on the same side of it , then the shape ABCD is a cyclic quadrilateral

Try to solve

6) ABCD is a quadrilateral in which AB = 9cm, BC = 5cm, CD = 8cm , DA = 9cm, AC = 11cm. prove that the shape ABCD is a cyclic quadrilateral.

Exercises 4 - 2

Complete the following:

1) .... is used to solve triangle given lengths of two sides and the measure of the included angle

2) .... is used to solve triangle given measures of two angles and the length of one side

3) In triangle LMN: $l^2 = m^2 + n^2$ - .... , cos L = $\frac{m^2 + n^2 - ......}{......}$

4) In triangle ABC , the lengths of its sides are 13, 17, 15 then the measure of the greatest angle is ....°

5) XYZ is a triangle , the lengths of its sides are 5.7cm , 7.4cm, 4.3cm then the measure of the smallest angle is ....°

6) $\triangle$ XYZ having x = 10 cm , y = 6cm , m ($\angle$ Z) = 60° then z = ...

7) In triangle LKM, $k^2 + m^2 - l^2$ = ...

Choose the correct answer:

8) The measure of the greatest angle of the triangle whose sides lengths are 3, 5, 7 is:

a) 150° b) 120° c) 60° d) 30°

9) In triangle LMN the expression $\frac{l^2 + m^2 - n^2}{2\,l\,m}$ equals:

a) cos L b) cos M c) cos N d) otherwise

10) In triangle XYZY $y^2 + z^2 - x^2 = 2\,yz \times$ .....

a) cos X b) sin Z c) cos Z d) sin X

11) In triangle ABC , a : b : c = 3 : 2 : 2 then cos A equals

a) $\frac{1}{8}$ b) $-\frac{1}{8}$ c) $\frac{1}{4}$ d) $\frac{3}{4}$

Answer the following questions:

12) **In triangle ABC if:**

a) a = 5cm, b = 7cm, c = 8cm prove that m($\angle$B) = 60°

b) a = 3cm, b = 5cm, c = 7cm prove that m($\angle$C) = 120°

c) $a = 13cm,\ b = 7cm,\ c = 13cm$ find $m(\angle C)$

d) $a = 13cm,\ b = 8cm,\ c = 7cm$ find $m(\angle A)$

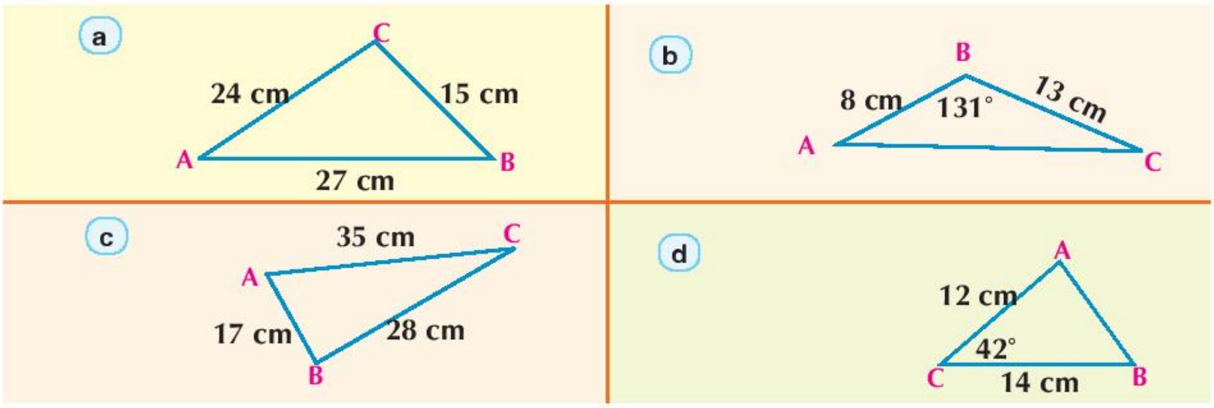
e) $a = 10cm,\ b = 17cm,\ c = 21cm$ find the measure of the smallest angle

f) $a = 5cm,\ b = 6cm,\ c = 7cm$ find the measure of the biggest angle

g) $a = 17cm,\ b = 11cm,\ m(\angle C) = 42°$ find c to the nearest hundredth

h) $b = 16,\ c = 14,\ m(\angle A) = 72°$ find a to the nearest hundredth

13) In the exercises (a - d) solve the triangle ABC:



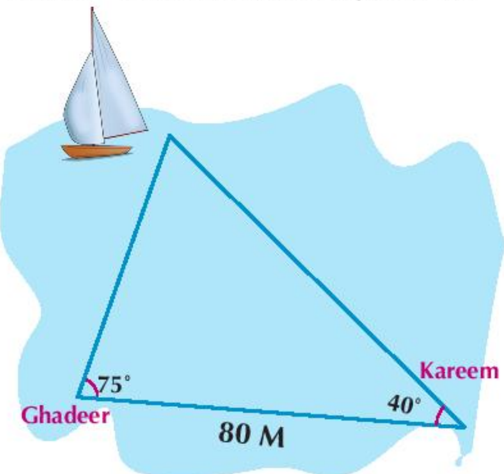
Geometric applications

14) ABCD is a parallelogram in which the length of two adjacent sides are 18 cm , 26 cm , and the measure of the angle between them is 39°. Find the length of the shortest diagonal to the nearest hundredth.

15) ABCD is a quadrilateral in which AB = 9 cm, BC = 5cm, CD = 8cm, AD = 9cm , AC = 11cm, prove that the shape ABCD is a cyclic quadrilateral.

16) ABCD is a parallelogram in which AB = 9 cm, BC = 13 cm, AC = 20 cm, find the length of $\overline{BD}$

17) ABC is a triangle of perimeter 70 cm, a = 26cm , $m(\angle A) = 60°$, Find its surface area.



18) **Maritime navigation:** Kareem and Ghadeer stand on one of the sides river. How far is Kareem from the boat to the nearest m?

19) **Agriculture:** A farmer wanted to fence a triangular piece of land lengths of two sides of it 98 m, 64 m , and the measure of the included angle between them is 52° What is the length of that fence?

(20) **Theoretical proof:** ABC is a triangle in which D is the midpoint of $\overline{BC}$, prove that:

$(AB)^2 + (AC)^2 = 2\,(AD)^2 + 2\,(BD)^2$

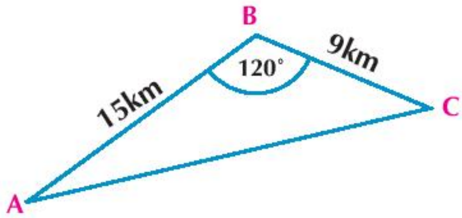
and if AB = 5cm , AC = 8cm , BC = 12 cm find AD.

(21) **Theoretical proof**: (For pioneers) ABC is a triangle in which: $(a + b + c)(a + b - c) = k\,ab$

prove that: $K \in\,]0\,,4[$, then find m($\angle$C) when K = 1

Life application:

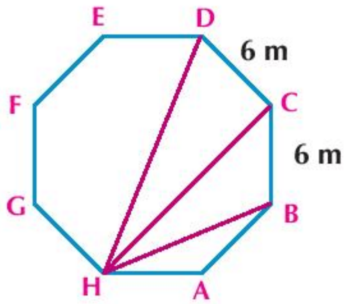
(22) **distances :** Kareem wanted to cover the distance from city A to city C passing by city B Using his motor bike with uniform speed 36 km/h, then returns from city C to city A With uniform speed 42 km/h. **Find:**



(a) The distance between city C , city A

(b) The total time in minutes for the whole journey .

(23) **Architectural design:** Architect designed a building at the form of regular octagon , the length of its side is 6 meter Find the lengths of the diagonals $\overline{HB}$, $\overline{HC}$, $\overline{HD}$.



(24) **Discover the error :** In triangle ABC, if a = 7 cm, b = 10 cm, c = 5 cm, m($\angle$A) = 41.62°. Find m($\angle$B)

Zeeyaad solution

$\because \dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

$\therefore \dfrac{10}{\sin B} = \dfrac{7}{\sin 41.62}$

$\therefore \sin B = \dfrac{10 \sin 41.62}{7} \simeq 0.9488$

$\therefore m(\angle B) = 71.59°$

Kareem solution

$\because \cos B = \dfrac{a^2 + c^2 - b^2}{2\,ac}$

$\therefore \cos B = \dfrac{(7)^2 + (5)^2 - (10)^2}{2 \times 7 \times 5} \simeq -0.3714$

$\therefore m(\angle B) \simeq 111.8°$

General Exercises

For more exercises, please visit the website of Ministry of Education.