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Introduction

بسم الله الرحمن الرحيم

We are pleased to introduce this book to show the philosophy on which the academic content has been prepared. This philosophy aims at:

1 Developing and integrating the knowledgeable unit in Math, combining the concepts and relating all the school mathematical curricula to each other.

2 Providing learners with the data, concepts, and plans to solve problems.

3 Consolidate the national criteria and the educational levels in Egypt through:

A) Determining what the learner should learn and why.

B) Determining the learning outcomes accurately. Outcomes have seriously focused on the following: learning Math remains an endless objective that the learners do their best to learn it all their lifetime. Learners should like to learn Math. Learners are to be able to work individually or in teamwork. Learners should be active, patient, assiduous and innovative. Learners should finally be able to communicate mathematically.

4 Suggesting new methodologies for teaching through (teacher guide).

5 Suggesting various activities that suit the content to help the learner choose the most proper activities for him/her.

6 Considering Math and the human contributions internationally and nationally and identifying the contributions of the achievements of Arab, Muslim and foreign scientists.

In the light of what previously mentioned, the following details have been considered:

★ This book contains three domains: algebra, relations and functions, calculus and trigonometry. The book has been divided into related and integrated units. Each unit has an introduction illustrating the learning outcomes, the unit planning guide, and the related key terms. In addition, the unit is divided into lessons where each lesson shows the objective of learning it through the title You will learn. Each lesson starts with the main idea of the lesson content. It is taken into consideration to introduce the content gradually from easy to hard. The lesson includes some activities, which relate Math to other school subjects and the practical life. These activities suit the students' different abilities, consider the individual differences throughout Discover the error to correct some common mistakes of the students, confirm the principle of working together and integrate with the topic. Furthermore, this book contains some issues related to the surrounding environment and how to deal with.

★ Each lesson contains examples starting gradually from easy to hard and containing various levels of thoughts accompanied with some exercises titled Try to solve. Each lesson ends in Exercises that contain various problems related to the concepts and skills that the students learned through the lesson.

Last but not least. We wish we had done our best to accomplish this work for the benefits of our dear youngsters and our dearest Egypt.

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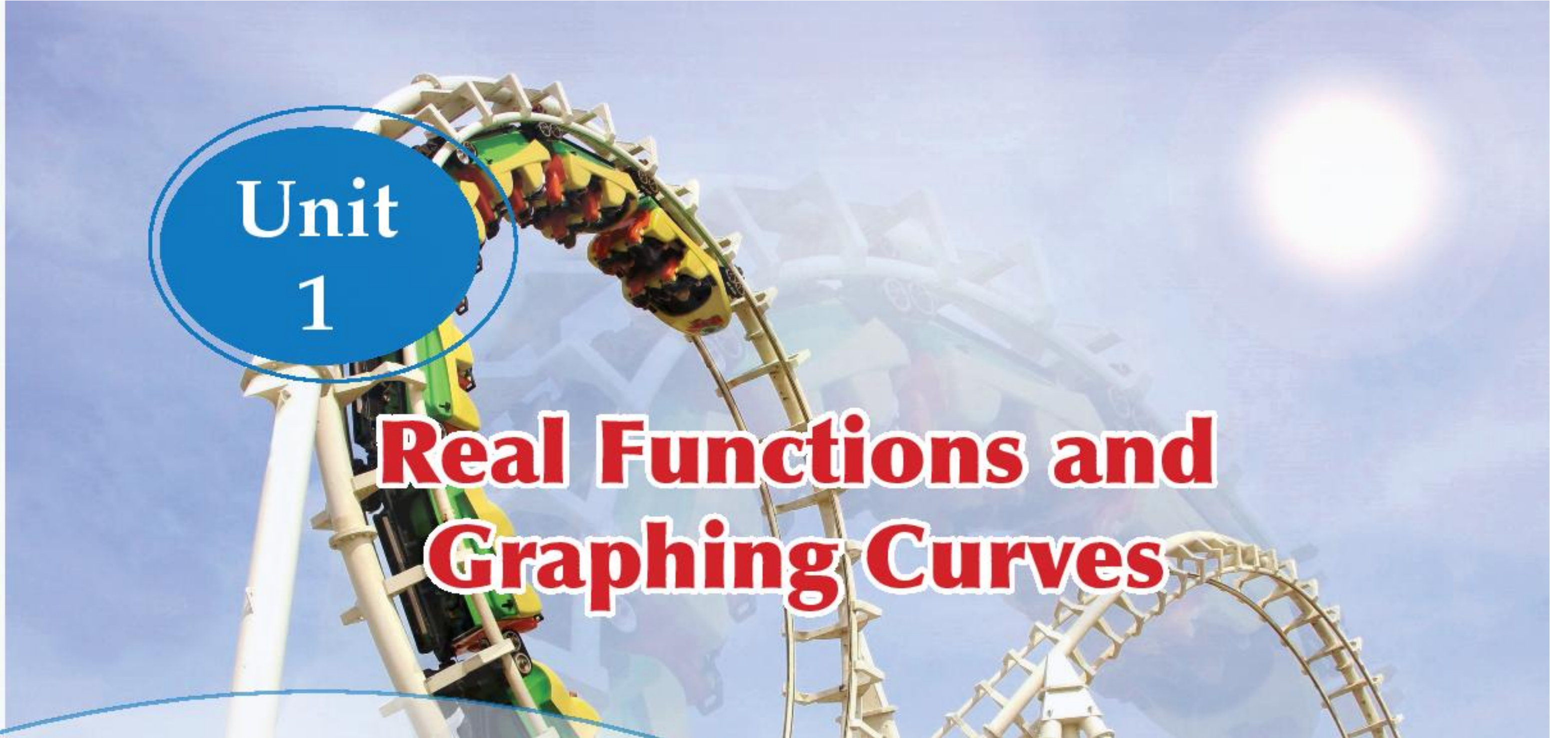


Unit 1

Real Functions and Graphing Curves

Unit introduction

Functions have various types and they have important applications in different life domains such as astronomy, medicine, economy, seismography, geology and demography. We use the functions to calculate the variables of the weather and to predict the expected weather conditions for a period of time in the future or to identify a malfunction position in the heart using the graphs which the electrocardiogram device record. Furthermore, functions are used to fulfill the best profit through studying the two functions of profit and cost or the effect of the age categories on census. Functions are also used in athletic medicine to identify the ideal weight (weight = length (cm) -100) or to calculate the ratio of fats in the body.

In general, functions are commonly used in the industry to study the effect of the different variables on the production quality.

The swiss scientist Leonard Euler (1707 - 1783) is considered one of the most prominent of the eighteenth century in mathematics and physics. He had been credited with using the symbol $y = f(x)$ to express the function . He had considered that the function is a correlation between the elements of two sets with a relation that allows to calculate a variable value related to Y for another independent one. He had converted all the trigonometric ratios which ancient Egyptians, Babylonians and Arabs had excelled into trigonometric functions. In this unit, you are going to learn different forms of the real functions, their behaviour and their graphical representation using the geometrical transformations and graphical programs and to use the real functions in solving life and mathematical problems in different fields.

Unit outcomes

By the end of the unit , the student should be able to :

- Identify the concept of the real function.
- Determine the domain, co-domain and range of the real functions.
- Deduce the monotony of the real functions of a real variable (increasing functions - decreasing functions - constant functions).
- Identify the type of the real function whether it is odd or even.
- identify polynomial functions.
- Graph the curves of (quadratic function - modulus functions - cubic function - rational function) and deduce the properties of each.
- Deduce the effect of the following transformations: $f(x \pm a) \pm b$ and a $f(x \pm b) \pm c$ on the previous functions.
- Apply the previous transformation on graphing the curves of the the real function.
- Solve equations in the form of : $|ax + b| = c$, $|ax + b| = |fx + c|$.
- Solve inequalities in the form of: $|ax + b| < c$ and $|ax + b| \le c$, $|ax + b| \geqslant c$ and $|ax + b| \leqslant c$
- Use the real functions to solve math and life problems in different fields.
- Relate what they learned about the effect of the previous transformations on the trigonometric functions in the form of activities.

Key terms

- Real Function
- Domain
- Co-domain
- Range
- Vertical Line
- Piecewise-Defind Function
- Even Function
- Odd Function
- Monotony of Function
- Increasing Function
- Decreasing Function
- Constant Function
- polynomial Function
- Absolute Value Function
- Rational Function
- Asymptote
- Transformation
- Translation
- Reflection
- Stretching
- Graphical Solution

Unit Lessons

Lesson (1 - 1): Real functions.

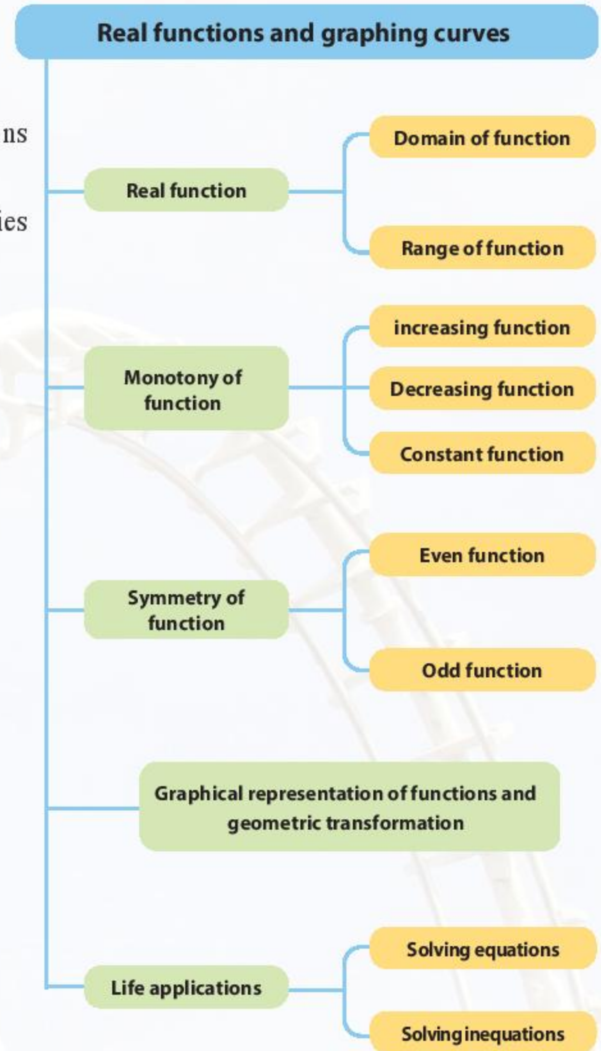
Lesson (1 - 2): Monotony of functions.

Lesson (1 - 3): Even and odd functions.

Lesson (1 - 4): Graphical representation of functions and geometric transformation.

Lesson (1 - 5): Solving the equations and inequalities of absolute value.

Unit planning guide



Materials

Computer set with graphic programs-graphic calculator-Scientific calculator

Unit 1

1 - 1 Real functions

You will learn

- The concept of the real function
- Vertical line test
- The piecewise defined function (defined with more than a rule).
- Identifying the domain and range of the real function.
- Operations on the functions.

Key terms

- Function
- Domain
- Co-domain
- Range
- Arrow diagram
- Cartesian diagram
- Vertical line
- Piecewise defined function
- Rule of the function.

Materials

- Scientific calculator
- Computer with graphic programs

Definition 1

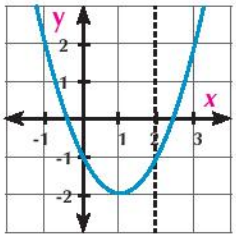
Real function:

The function f is called a real function if each of its domain and co-domain are the set of the real numbers $\mathbb{R}$ or a subset of it.

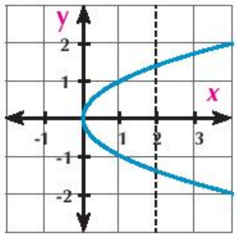
Learn

The vertical line test

If the vertical line is found at each element of the domain elements , it passes through an only point of the points representing the relation, the relation was a function from $X \longrightarrow Y$



une fonction



n'est pas une fonction

Example **Identifying the relation representing a function**

(1) In each of the following figures, show whether y represents a function in x or not .

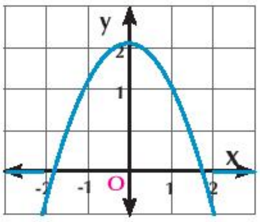


Figure (1)

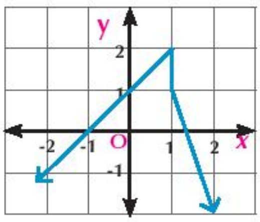


Figure (2)

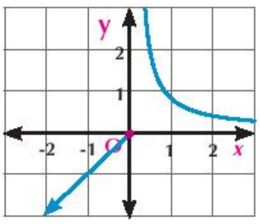


Figure (3)

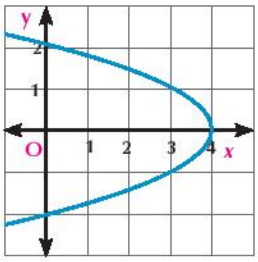


Figure (4)

Solution

Figure (1) represents a function

Figure (2) does not represent a function because the vertical line passing through point (1, 0) intersects the graph at infinite number of points.

Figure (3) represents a function.

Figure (4) does not represent a function because there is a vertical line intersects the curve at more than a point.

Try to solve

1) Show which of the following relations represents a function from X $\longrightarrow$ Y. Why?

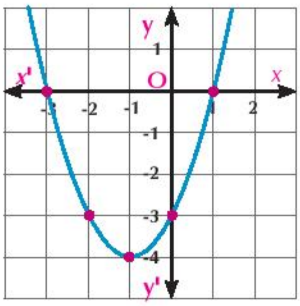


Figure (1)

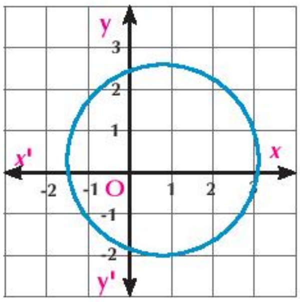


Figure (2)

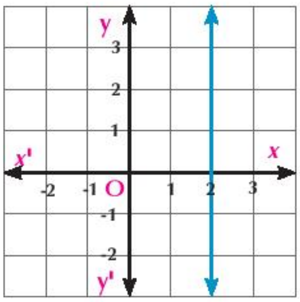


Figure (3)

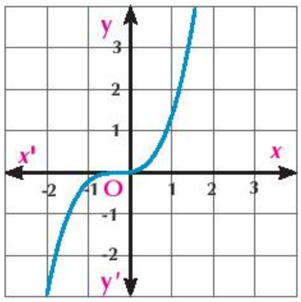


Figure (4)

Example Identifying the range of the function.

2) a) If f: [1, 5] $\longrightarrow$ R where $f(x) = x + 1$

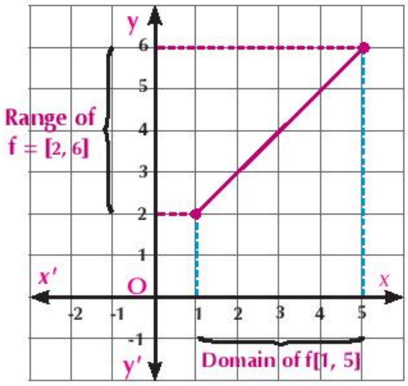
Graph the function f and deduce the range of the function

Solution

a) Function f is a linear function whose domain is [1, 5]. It is represented by a line segment whose two ends are (1, f(1)) and (5, f(5)) i.e. the two points (1, 2) and (5, 6).

The range of function f = [2, 6]

It is the set of y coordinates for all the points which belong to the curve of the function.



Try to solve

2) a) If f: [1 , ∞ [$\longrightarrow$ ℝ, where $f(x) = 1 - x$

Graph the function f, and deduce its range.

b) If g :]- ∞, -1[$\longrightarrow$ ℝ, where $g(x) = 1 - x$

Graph the function g and deduce its range.

Piecewise - defined function

Learn

The piecewise defined function is a real function where each subset of its domain has a different definition base.

Try to solve

3 Check your answer using the function above in **Work together** , then calculate the gas monthly consumption for the following quantities:

a 15 cubic metres b 40 cubic metres c 54 cubic metres

Graphing piecewise- defined function :

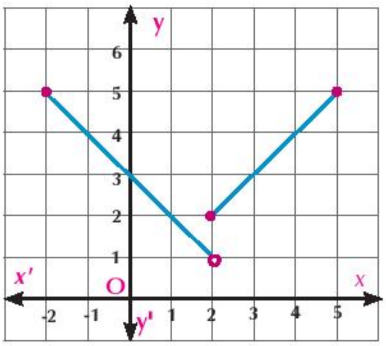
Example

3 If $f(x) = \begin{cases} 3 - x & \text{when} \quad -2 \leqslant x < 2 \\ x & \text{when} \quad 2 \leqslant x \leqslant 5 \end{cases}$

Graph and deduce the domain and range of the function.

Solution

The function f is defined on two intervals and $f(x)$ is identified by two rules:



First rule : $f_1(x) = 3 - x$ when $-2 \leqslant x < 2$ i.e. on interval $[-2, 2[$.

It is up for a linear function represented by a line segment whose two ends are the two points (-2, 5) and (2, 1) by placing an open circle at point (2, 1) because $2 \notin [-2, 2[$ as shown in the figure opposite.

Second rule : $f_2(x) = x$ when $2 \leqslant x \leqslant 5$ i.e on interval [2, 5]

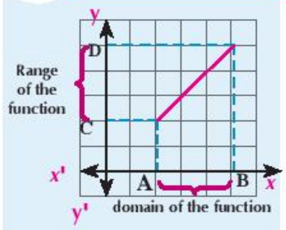
It is up for a linear function represented by a line segment whose two ends are the two points (2, 2) and (5, 5) and the domain of the function $f = [-2, 2[\cup [2, 5] = [-2, 5]$

From the graph, we deduce :

the domain of the function $f = [-2, 5]$

the range of the function $f =]1, 5]$

Notice



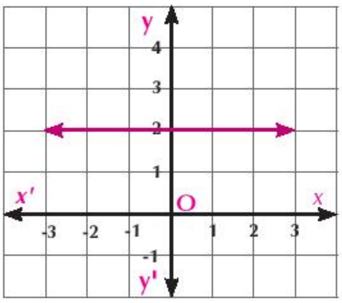
in the graph representing the function f, the domain of the function = [a, b] the range of the function = [c ,d]

Try to solve

4 If $f(x) = \begin{cases} x - 1 & \text{when} \quad -2 \leqslant x < 0 \\ x + 1 & \text{when} \quad x > 0 \end{cases}$

Graph the function and deduce its domain and range.

5 Deduce the domain and range of each function in the following graphs.

a

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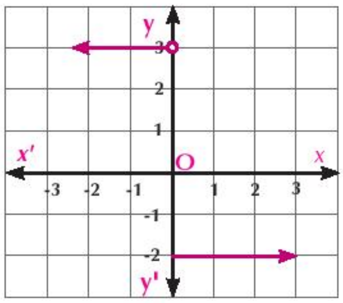
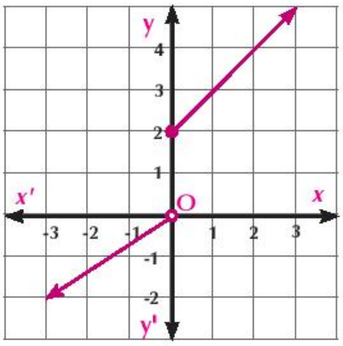
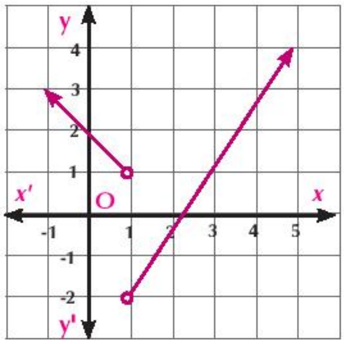

c

d


Identifying the domain of the real functions and the operations on them

The domain of the function is identified from its definition base or its graph.

Example **Determining Domain of the function**

4 Determine the domain of the following real functions defined by the next rules:

a $f_1(x) = \frac{x+3}{x^2 - 9}$ b $f_2(x) = \sqrt{x - 3}$

c $f_3(x) = \sqrt[3]{x - 5}$

Remember

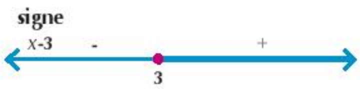
The domain of the polynomial function is the set of real numbers when it is not defined on a subset of it.

Solution

a The function f_1 is not defined when the denominator = 0, so we place $x^2 - 9 = 0$ i.e. $x = \pm 3$

Thus, the domain of the function f_1 is $\mathbb{R} - \{-3, 3\}$

b The domain of the function f_2 is all the values of x which make the value of the number inside the root positive or zero. I.e. the values of x which satisfy $x - 3 \geqslant 0$



$\because\ x - 3 \geqslant 0 \qquad \therefore x \geqslant 3$, $\quad \therefore$ the domain of the function $f_2 = [3, \infty[$

c $f_3(x) = \sqrt[3]{x - 5}$, index of the root is an odd number and the domain of $f_3 = \mathbb{R}$

Notice :

If $f(x) = \sqrt[n]{g(x)}$ where $n \in \mathbb{Z}^+$ and $n > 1$, then $g(x)$ is polynomial

First: When n is an odd number , the domain of the function $f = \mathbb{R}$

Second: When n is an even number , the domain of the function f is the set of the values of x such that: $g(x) \geq 0$

Try to solve

6 Determine the domain of the following real functions defined by the next rules:

a $f_1(x) = \dfrac{2x+3}{x^2 - 3x + 2}$

b $f_2(x) = \sqrt{x - 2}$

c $f_3(x) = \sqrt[3]{x - 5}$

Critical thinking: If the domain of the function f where $f(x) = \dfrac{2}{x^2 - 6x + k}$ is $\mathbb{R} - \{3\}$, find the value of K.

Activity

Operations on functions

If f_1 and f_2 are two functions whose two domains are m_1 and m_2 respectively, then:

1 $(f_1 \pm f_2)(x) = f_1(x) \pm f_2(x)$, domain of $(f_1 \pm f_2)$ is $m_1 \cap m_2$

2 $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$, domain of $(f_1 \cdot f_2)$ is $m_1 \cap m_2$

3 $(\dfrac{f_1}{f_2})(x) = \dfrac{f_1(x)}{f_2(x)}$ where $f_2(x) \neq 0$ domain of $(\dfrac{f_1}{f_2})$ is $(m_1 \cap m_2) - Z(f_2)$
where $Z(f_2)$ is the set of zeros of f_2

We notice that: in all the previous cases , the domain of the new function equals the intersection of the two domains of f_1 and f_2 except for the values which make $f_2(x) = 0$ in the division operation.

If $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ where $f_1(x) = 3x - 1$
$f_2 : [-2, 3] \longrightarrow \mathbb{R}$ where $f_2(x) = x - 3$

First: Find the rule and domain for each of the following functions:

a $(f_1 + f_2)$ b $(f_1 - f_2)$ c $(f_1 \cdot f_2)$ d $(\dfrac{f_1}{f_2})$

Second: Calculate the numerical value of each (if possible) :

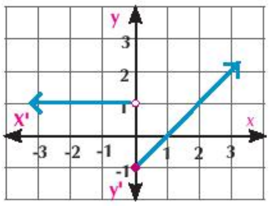
a $(f_1 + f_2)(3)$ b $(f_1 - f_2)(-3)$ c $(f_1 \cdot f_2)(-2)$

d $(f_1 \cdot f_2)(2)$ e $(\dfrac{f_1}{f_2})(4)$ f $(\dfrac{f_1}{f_2})(-1)$

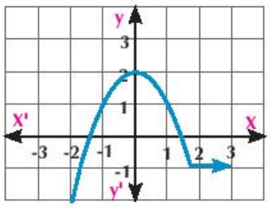
Exercises 1 - 1

Choose the correct answer:

1 The relation shown in the following graphical figures which does not represent a function is :



a



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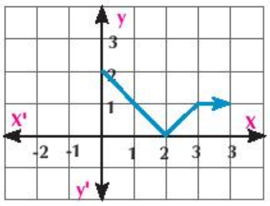
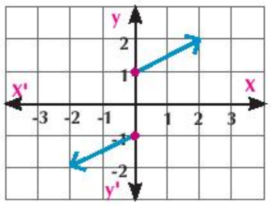


c



d

Answer the following:

2 If $f : X \longrightarrow \mathbb{R}$ and $X = \{1, 2, -2, -3\}$

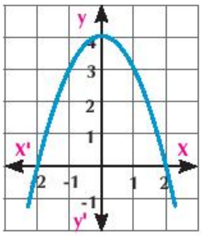
Find the range of the function if $f(x) = 5x - 3$

3 If $g : \{1, 2, 3, 4, 5\} \longrightarrow \mathbb{Z}^+$ where $g(x) = 4x - 3$

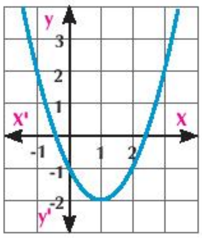
a Write down the range of the function **b** If $g(k) = 17$, find the value of k

4 From the graph , deduce the domain and range of the function in each of the following:

a



b



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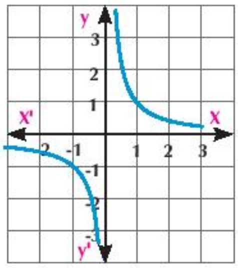

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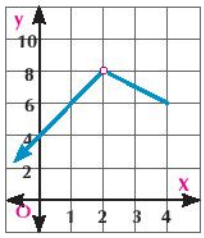


5 Determine the domain of the function f where $f(x) = \begin{cases} x - 1 & \text{when } 2 < x \leqslant 4 \\ -1 & \text{when } -2 \leqslant x \leqslant 2 \end{cases}$

Graph the function, then deduce its range.

6 Graph the function f where :

$$f(x) = \begin{cases} x + 3 & \text{when } x > 2 \\ 2x - 1 & \text{when } x < 2 \end{cases}$$

then deduce its range.

7 If $f(x) = \begin{cases} 2x + 3 & \text{when } -2 \leqslant x < 0 \\ 1 - x & \text{when } 0 \leqslant x \leqslant 4 \end{cases}$

Graph the function f and deduce its range

8. If $f(x) = \begin{cases} x + 1 & \text{when } -3 \leqslant x < 0 \\ x + 2 & \text{when } 0 \leqslant x \leqslant 3 \end{cases}$

Graph the function f and deduce its range.

9. If $f(x) = \begin{cases} -4x + 3 & \text{when } x < 3 \\ -x^3 & \text{when } 3 \leqslant x \leqslant 8 \\ 3x^2 + 1 & \text{when } x > 8 \end{cases}$

Find :

a) $f(2)$ b) $f(3)$ c) $f(10)$

10. **Trade:** The function f, where :

$$f(x) = \begin{cases} \frac{5}{2}x & \text{When } 0 \leqslant x \leqslant 5000 \\ 2^x + 2500 & \text{When } 5000 < x \leqslant 15000 \\ \frac{3^x}{2} + 10000 & \text{When } 15000 < x \leqslant 60000 \end{cases}$$

represents the amount of money charged by a company to distribute an electrical appliance in L.E. If x represents the number of distributed appliances, find :

a) $f(5000)$ b) $f(10000)$ c) $f(50000)$

11. Determine the domain for each of the following real functions defined by the following rules:

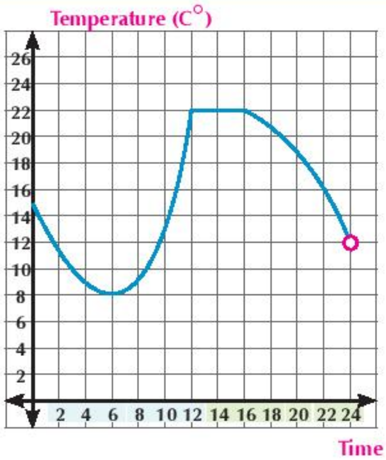
a) $f(x) = \frac{x + 3}{x^2 - 5x + 6}$ b) $f(x) = \frac{x + 1}{x^3 + 1}$

c) $f(x) = \sqrt{X - 2}$ d) $f(x) = \sqrt{4 - x^2}$

Monotonty of Functions 1 - 2

Think and discuss

The opposite graph shows the temperatures recorded in Cairo On a day. Observe the change of degrees over the time, then find using the graph:



a) The periods when the temperature decreases.

b) The periods when the temperature increases.

c) The period when the temperature is constant. Characteristics of the curves help us know the behaviour of the function f and identify whether the function $f(x)$ is increasing , decreasing or constant. It is called the monotony of the function.

You will learn

- Monotony of functions.
- Using graphical program such as (GeoGebra) to graph the function curve.

Key terms

- Monotony
- Increasing function
- Decreasing function
- Constant function

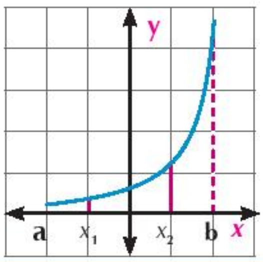
Materials

- Scientific calculator
- Graphic programs

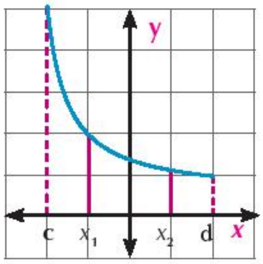
Learn

The function f is
Said to be increasing on the interval]a , b[

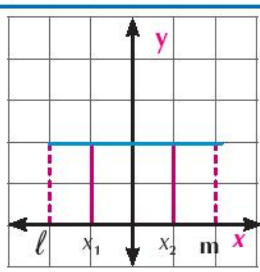
If each of x_1 **and** $x_2 \in$ **]a , b [**
where: $x_2 > x_1$
then: $f(x_2) > f(x_1)$



The function f is
Said to be decreasing on the interval]c , d[
If each of x_1 and $x_2 \in$] c ,d [
where: $x_2 > x_1$
then: $f(x_2) < f(x_1)$

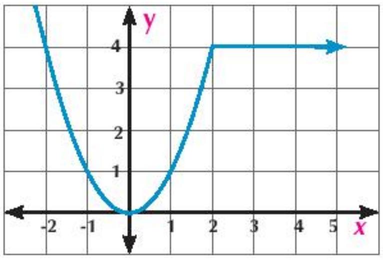


The function f is said to be constant on the interval]ℓ , m[
If each of : x_1 and $x_2 \in$] ℓ , m[
where $x_2 > x_1$
then: $f(x_2) = f(x_1)$



Example

1 Investigate the monotony of the function represented in the figure opposite.

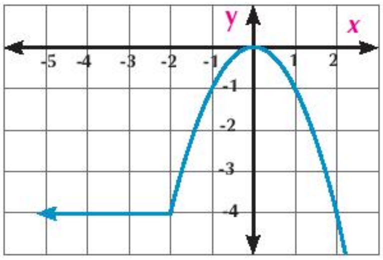


Solution

- the function is decreasing in the interval]-∞, 0[
- the function is increasing in the interval]0, 2[
- the function is constant in the interval]2, ∞[

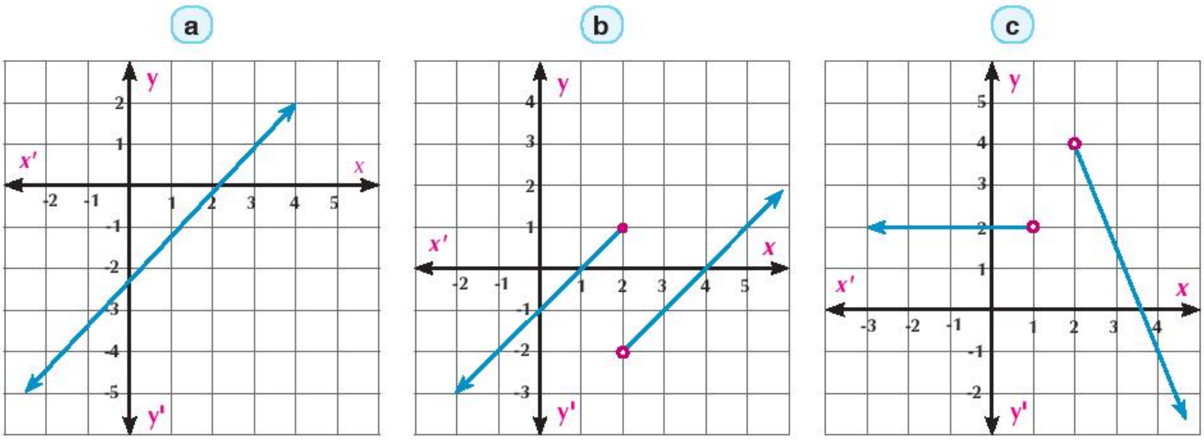
Try to solve

1 **In the opposite figure**:
Investigate the intervals in which the function is increasing, decreasing and constant.



Example

2 Each of the following graphs illustrate the curve of the function $f: X \longrightarrow Y$. Deduce the domain and range of the function , then investigate its monotony



Solution

a The domain of $f = \mathbb{R}$ =] -∞ , ∞ [, range of f =] -∞ , ∞ [the function increases in]-∞ , ∞[

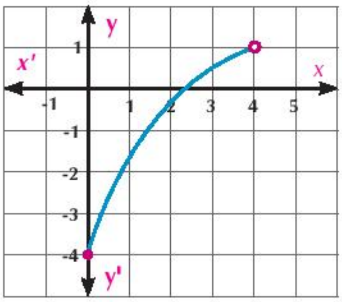
b The domain of f =] -∞ , 2] ∪] 2 , +∞ [=] -∞ , ∞ [, range of $f = \mathbb{R}$
the function increases in]-∞ , 2[and also in] 2 , ∞ [

c The domain of f =] -∞ , 1[∪] 2 , +∞ [, range of f =] - ∞ , 4[
the function is constant in] -∞ , 1[, and decreases in] 2 , ∞ [

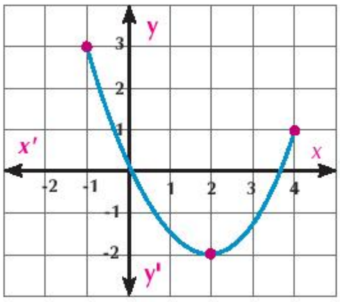
Try to solve

(2) Deduce the domain and range of the function , then investigate its monotony in each of the following graphs:

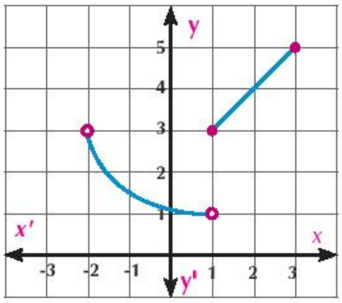
a



b



c



Exercises 1 - 2

(1) Find the range and investigate the monotony of each of the following functions from the following graphs:

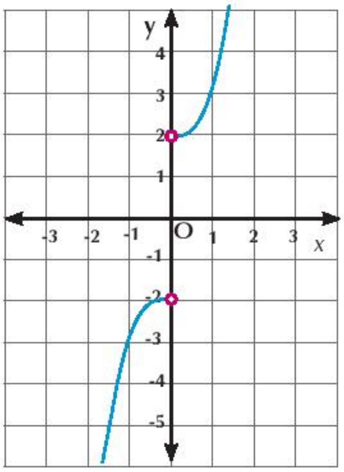


Fig (1)

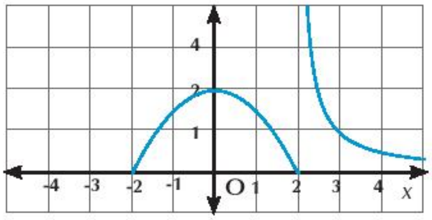


Fig (2)

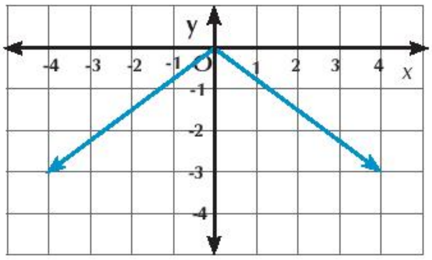


Fig (3)

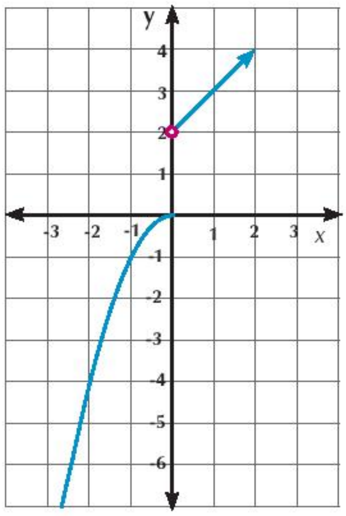
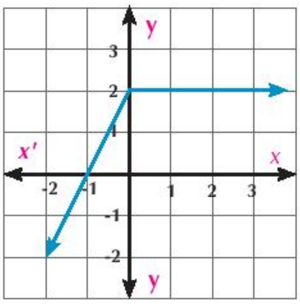


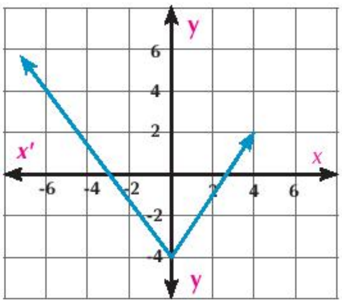
Fig (4)

(2) Determine the domain for each of the represented functions in the following graphs, then write the range and investigate its monotony:

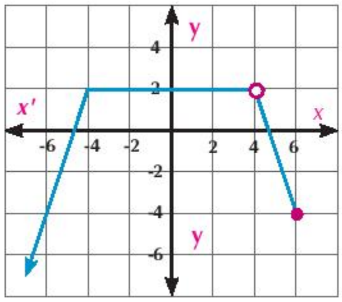
a



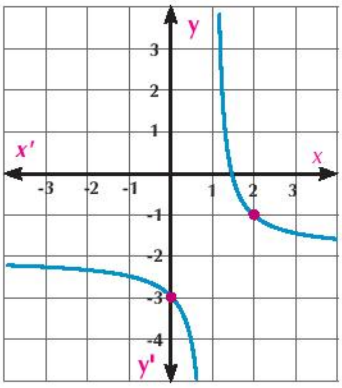
b



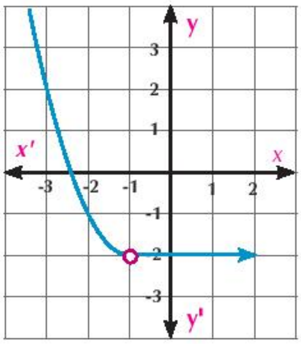
c



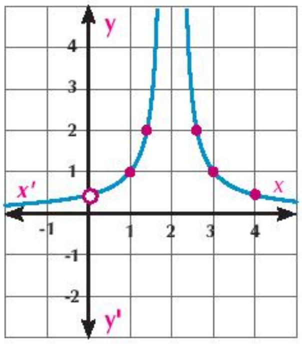
d



e



f



3 If $f: [-2\ ,\ 6] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 4 - x & \text{When} \quad x < 1 \\ x & \text{When} \quad 1 \leqslant x \leqslant 6 \end{cases}$$

Graph the function f, then deduce its range and investigate its monotony.

Even and Odd function 1 - 3

The graph of the function f where $y = f(x)$ may be distinguished by geometrical characteristics that can be easily noticed from the graph. These characteristics can be used to study the functions and their applications. Symmetry around y-axis or around the origin point are of the most popular characteristics.

You will learn

- Symmetry in curves of functions.
- Even functions
- Odd functions

Key terms

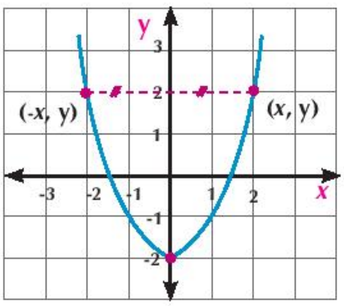
- Symmetry
- even function
- odd function

Materials

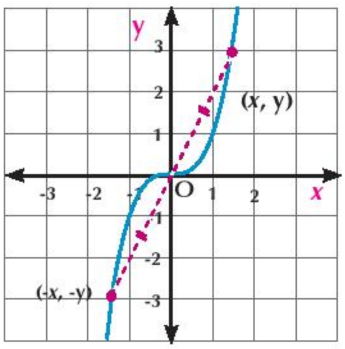
- Scientific calculate
- Graphic programs

Preface

You have previously learned the symmetry around a straight line where the figure can be folded around the straight line to make the two halves of the curve be congruent completely and you have also learned the symmetry around the origin point.



Symmetry around y-axis
Figure (1)



Symmetry around the origin point. Figure (2)

In figure (1):

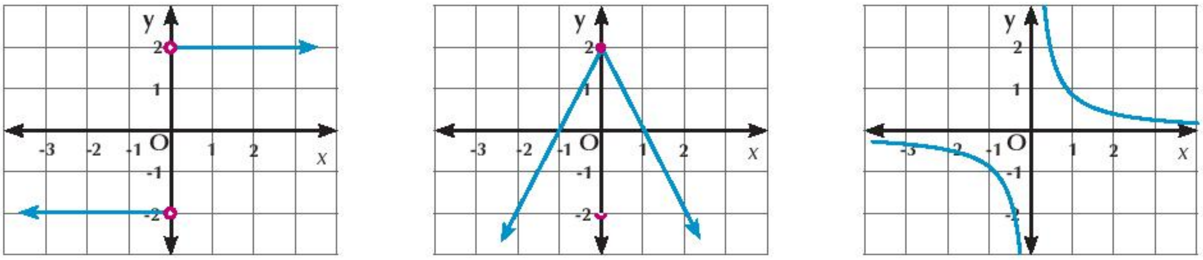
The point $(-x, y)$ which lies on the graph of the function curve is the image of point (x, y) which also lies on the same graph by reflection around y-axis..

In figure (2):

The graph of the relation between x and y shows the symmetry of the curve around the origin point where point $(-x, -y)$ is the image of point (x, y) which lies on the same curve.

Try to solve

1. In the following figures, show which curve is symmetric around y-axis and which is symmetric around the origin point.



(a) (b) (c)

Critical thinking:

Are curves of all functions symmetric around y – axis or around the origin point only? **Explain** .

Even and Odd Functions

Learn

Even function : It is said the function $f: X \longrightarrow Y$ is an even function if $f(-x) = f(x)$, for each x , $-x \in X$ and the curve of the even function is symmetric around y-axis.

Odd function : It is said the function $f: X \longrightarrow Y$ is an odd function if $f(-x) = -f(x)$. for each x , $-x \in X$ and the curve of the odd function is symmetric around the origin point.

Notice : A lot of functions are neither even nor odd.

When you investigate the type of the function whether it is even or odd , the condition of belonging the two elements x and $-x$ to the domain of the function should be satisfied. If this condition is not satisfied, the function is neither even nor odd without finding $f(-x)$

Example

1 Investigate the type of the function f in each of the following and show whether it is odd or even.

a $f(x) = x^2$ b $f(x) = x^3$ c $f(x) = \sqrt{x+3}$ d $f(x) = \cos x$

Solution

a $f(x) = x^2$, and the domain of $f = \mathbb{R}$

for each x and $-x \in \mathbb{R}$, then $f(-x) = (-x)^2 = x^2$

thus: $f(-x) = f(x)$ $\therefore f$ is even function

b $f(x) = x^3$, and the domain of $f = \mathbb{R}$

for each x and $-x \in \mathbb{R}$, then: $f(-x) = (-x)^3 = -x^3$

thus: $f(-x) = -f(x)$ $\therefore f$ is odd function

Important remark :

The function $f: \mathbb{R} \longrightarrow \mathbb{R}$ and $f(x) = ax^n$ where $a \neq 0$ and $n \in \mathbb{Z}^+$ is called exponential function. The function is even when n is an even number and is odd when n is an odd number.

(c) $f(x) = \sqrt{x+3}$, and the domain of f = $[-3 , \infty[$

Notice that $4 \in [-3 , \infty[$ while $-4 \notin [-3 , \infty[$

$\therefore$ The function is neither even nor odd.

(d) $f(x) = \cos x$, and the domain of f = $\mathbb{R}$

$\therefore$ for each $-x$ and $x \in \mathbb{R}$, then $f(-x) = \cos(-x) = \cos x$

I.e. $f(-x) = f(x)$ $\therefore$ f is an even function

Remember

$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$
$\tan(-x) = -\tan x$

Try to solve

(2) Investigate the type of each function in the following functions and show whether it is even, odd or otherwise.

(a) $f(x) = \sin x$ (b) $f(x) = x^2 + \cos x$ (c) $f(x) = x^3 - \sin x$

(d) $f(x) = x^2 \cos x$ (e) $f(x) = x^3 \sin x$ (f) $f(x) = x^3 \cos x$

(g) $f(x) = x^3 + x^2$ (h) $f(x) = \sin x + \cos x$ (i) $f(x) = \sin x \cos x$

What do you infer ?

Important properties :

If each of f_1 and f_2 are even functions and each of g_1 and g_2 are odd functions, then :

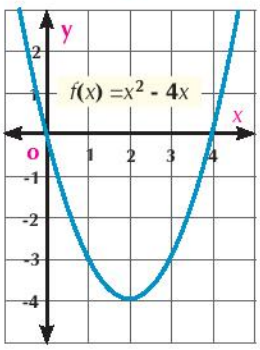
1) $f_1 + f_2$ is even function
2) $g_1 + g_2$ is odd function
3) $f_1 \times f_2$ is even function
4) $g_1 \times g_2$ is even function
5) $f_1 \times g_2$ is odd function
6) $f_1 + g_2$ is neither even nor odd

Use these properties to verify your answers in **Try to solve**.

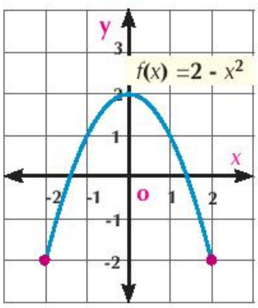
Example

(2) Each of the following graphs illustrates the curve of the function f. Determine whether the function is even, odd or otherwise, then check your answer algebraically.

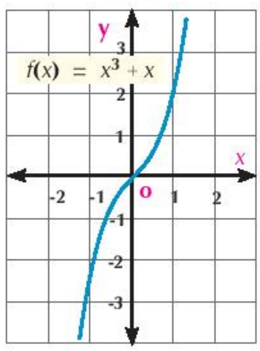
(c)



(b)



(a)



Solution

a $f(x) = x^3 + x$, from the graph of the function f, we notice that :

The domain of $f = \mathbb{R}$ and the curve of the function is symmetric around the origin point. I.e. the function is odd

$\because$ each of $x \in \mathbb{R}$ and $-x \in \mathbb{R}$ $\therefore f(-x) = (-x)^3 + (-x)$

By simplifing : $f(-x) = -x^3 - x$

take off (-1) a common factor $f(-x) = -(x^3 + x)$

$f(-x) = -f(x)$

I.e. the function is odd.

b $f(x) = 2 - x^2$, from the graph of the function f, we notice that:

the domain of $f = [-2, 2]$ and the curve of the function is symmetrical around y-axis . I.e. the function is even

$\because$ each of $x \in [-2, 2]$ and $-x \in [-2, 2]$ $\therefore f(-x) = 2 - (-x)^2$

By simplifying $f(-x) = 2 - x^2$

$f(-x) = f(x)$

I.e. the function is even

c $f(x) = x^2 - 4x$, from the graph of the function f, we notice that :

the domain of $f = \mathbb{R}$ and the curve of the function is neither symmetric around y-axis nor around the origin point. I.e. the function is neither even nor odd.

$x \in \mathbb{R}$:

$\because$ Each of $x \in \mathbb{R}$ and $-x \in \mathbb{R}$ $\therefore f(-x) = (-x)^2 - 4(-x)$

By simplifying $f(-x) = x^2 + 4x \neq f(x)$ $\therefore f$ is not an even function

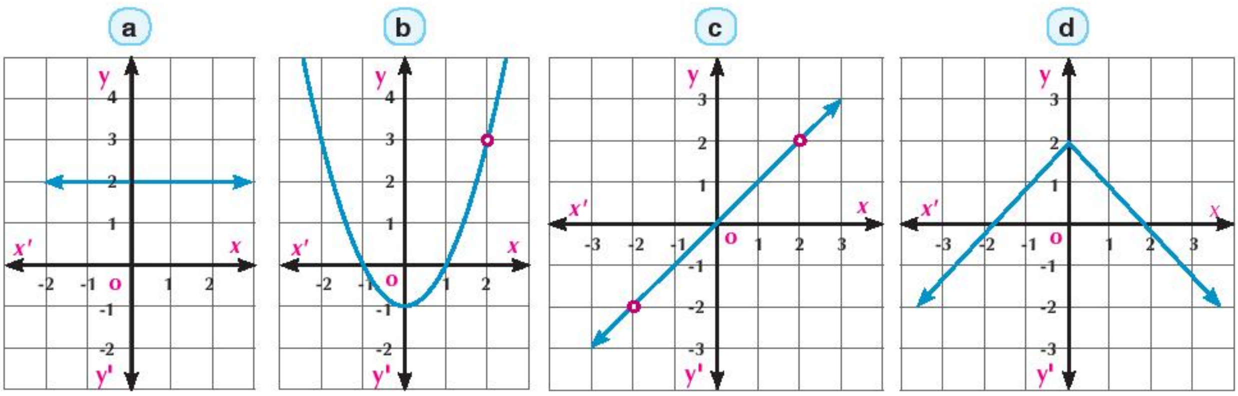
But $-f(x) = -x^2 + 4x$

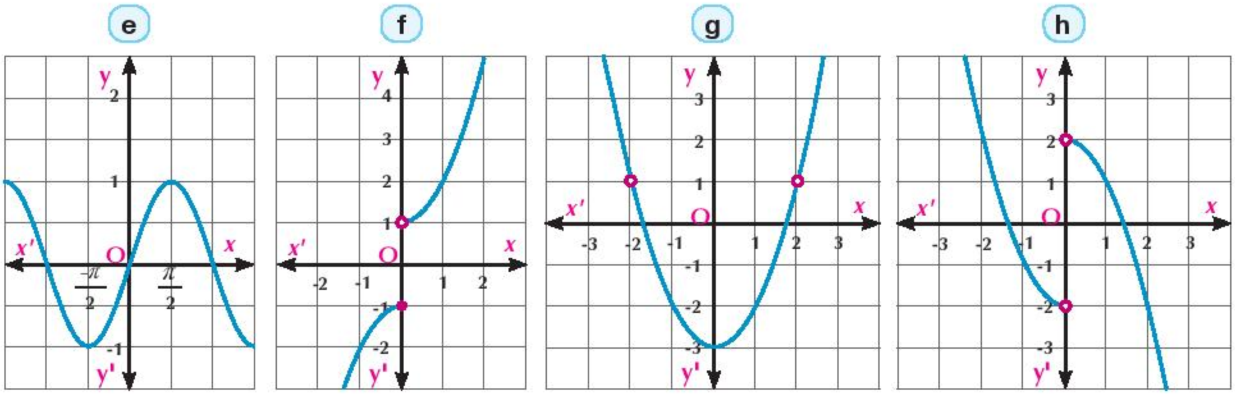
Then $f(-x) \neq -f(x)$ $\therefore f$ is not an odd function

I.e. the function is neither even nor odd.

Try to solve

3) Tell whether each of the functions represented in the following figures is even, odd or otherwise.





Exercises 1 - 3

1) Tell whether the symmetry of the curve is around x-axis, y-axis or origin point, then explain.

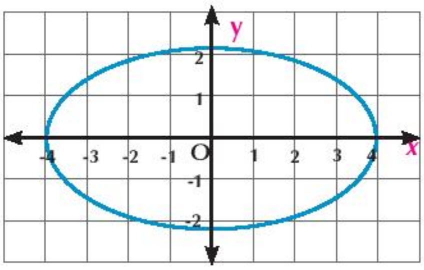


Fig (1)

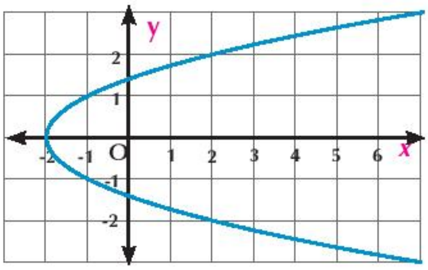


Fig (2)

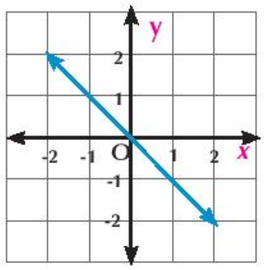


Fig (3)

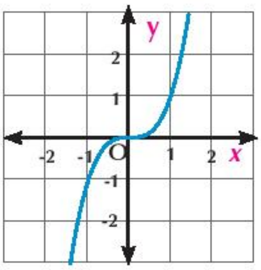


Fig (4)

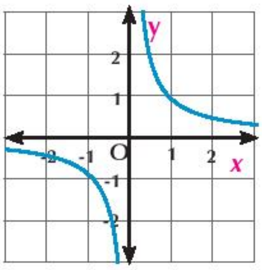


Fig (5)

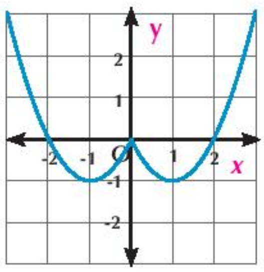


Fig (6)

2) Find the range of each function and tell whether it is even, odd or otherwise.

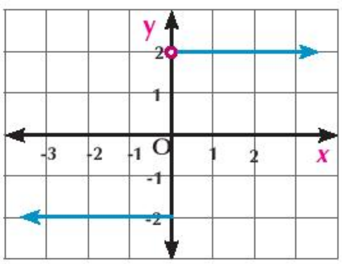


Fig (a)

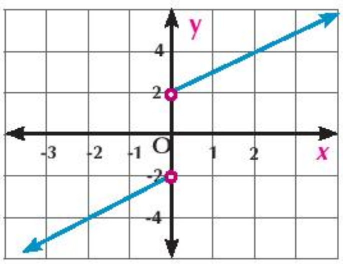


Fig (b)

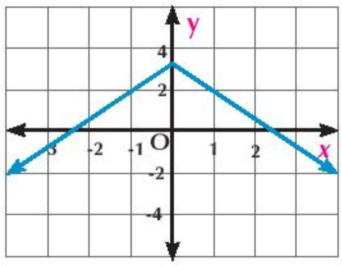


Fig (c)

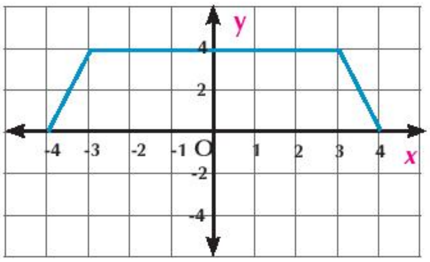


Fig (d)

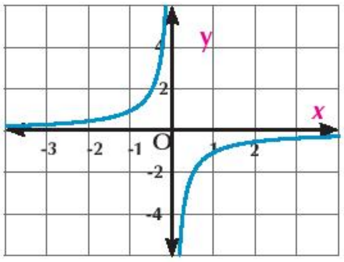


Fig (e)

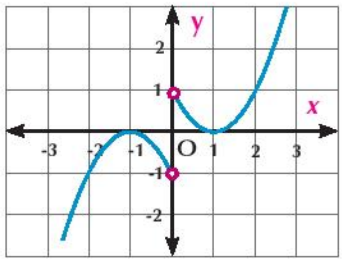


Fig (f)

3 Tell whether the function f is even , odd or otherwise.

a $f(x) = x^4 + x^2 - 1$ **b** $f(x) = 3x - 4x^3$ **c** $f(x) = 5$

d $f(x) = x^2 - 3x$ **e** $f(x) = \frac{x^3 + 2}{x - 3}$ **f** $f(x) = x \cos x$

4 If f_1, f_2 and f_3 are three real functions where $f_1(x) = x^5, f_2(x) = \sin x$ and $f_3(x) = 5x^2$, tell which of the following functions is even, odd or otherwise.

a $f_1 + f_2$ **b** $f_1 + f_3$ **c** $f_1 \times f_2$ **d** $f_3 \times f_2$

5 Answer the following using the next graphs :

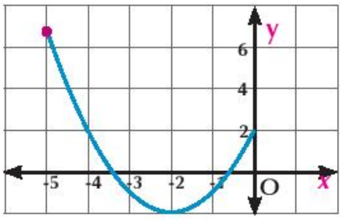


Fig (1)

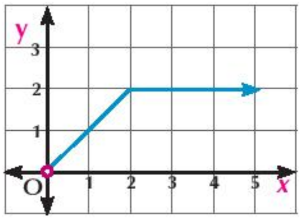


Fig (2)

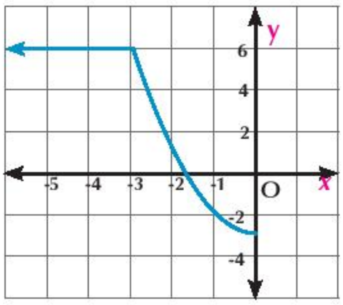


Fig (3)

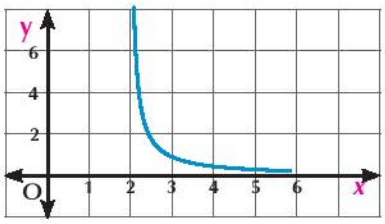


Fig (4)

First: Complete the graph in figures (1) and (3) in your notebook to get an even function on its domain.

Second: Complete the graph in figures(2) and (4) in your notebook to get an odd function on its domain.

Third: Determine the domain and range of the function in each case , then investigate its monotony

Unit 1

Graphical representation and geometrical transformations

1 - 4

Polynomial functions

You have previously learned the polynomial function whose base is in the form: $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_n x n$

where : a_0 , a_1 , a_2 , a_3 , ... , $a_n \in \mathbb{R}$, $a_n \neq 0$ and $n \in \mathbb{N}$

You knew that the domain and co-domain are the set of the real numbers R (or a subset of it). As a result, these functions are called polynomial functions of n degree. The degree of a polynomial is the highest power of the independent variable x.

Notice:

1- If $f(x) = a_0$ and $a_0 \neq 0$ then f is called a constant polynomial function.

2- The polynomial functions of the first degree are called linear functions, of the second degree are called quadratic functions and of the third degree are called cubic functions.

3- Adding or subtracting functions of different powers and constants, we get a polynomial function.

4- Zeros of the polynomial function are the x-coordinates of the intersecting points of its curve with x-axis.

Graphing the curves of the functions.

First: Polynomial functions:

Learn

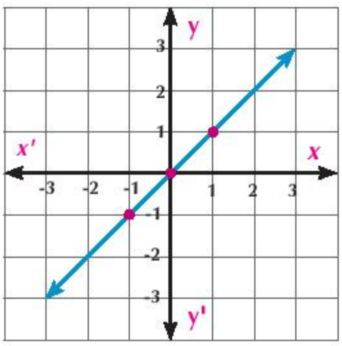
Here is the graphical representation of some polynomial functions :

1) The simplest form of the linear function is :

$f(x) = x$

it is a function f that joins the number itself and a straight line passing through point (0, 0), and its slope = 1 represents it.

(**Check:** the range of $f = \mathbb{R}$, f is odd and f is increasing in $\mathbb{R}$).



You will learn

- The polynomial functions (linear function - quadratic function and cubic function)
- The absolute value function.
- The rational function
- using the geometrical transformation of the function f to graph the curves
  - $y = f(x) + a$
  - $y = f(x + a)$
  - $y = f(x + a) + b$
  - $y = -f(x)$
  - $y = a f(x)$
  - $y = a f(x + b) + c$
- Transformation of some trigonometric functions.

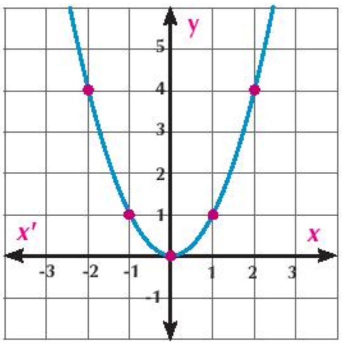
Key terms

- Transformation
- Translation
- Reflection
- Vertical
- Horizontal
- Asymptote

Materials

- Scientific calculator.
- Computer
- Graph program

2) The simplest form of the quadratic function f is :

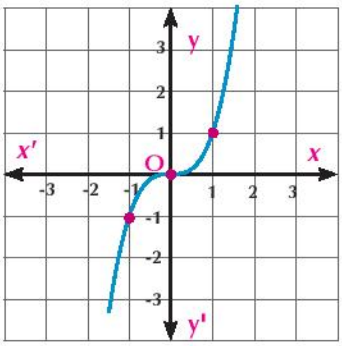


$$f(x) = x^2$$

the quadratic function maps the number to its square. It is represented by an upward open curve, symmetrical about y-axis and the vertix point of the curve is (0, 0).

(**Check:** the range of $f = \mathbb{R}$, f is even, f is decreasing at $]-\infty, 0[$ and increasing at $]0, +\infty[$)

3) The simplest form of the cubic function f is:



$$f(x) = x^3$$

The cubic function maps the number to its cubic.

It is represented by a curve whose symmetrical point is (0, 0).

(**Check:** the range of $f = \mathbb{R}$, f is odd and f is increasing in $\mathbb{R}$)

Example

1 Graph the function f where:

$$f(x) = \begin{cases} x^2 & \text{when } x < 2 \\ 4 & \text{when } x > 2 \end{cases}$$

Solution

1) When $x < 2$ and $f(x) = x^2$

We graph $f(x) = x^2$ for each $x \in]-\infty, 2[$

Place an **open circle** at point **(2 , 4)** as shown in **fig (1)**

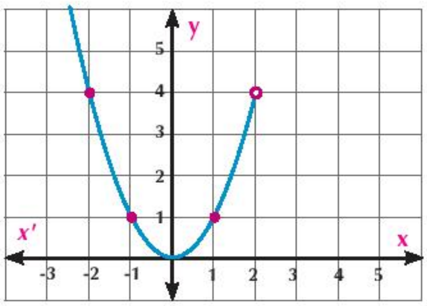


Fig (1)

2) when $x > 2$ and $f(x) = 4$

We graph the constant function $f(x) = 4$ for each $x \in]2,\infty[$ on the same graph **fig (2)**

Notice that the domain of the function $f = \mathbb{R} - \{2\}$ and the range of $f = [0, \infty[$

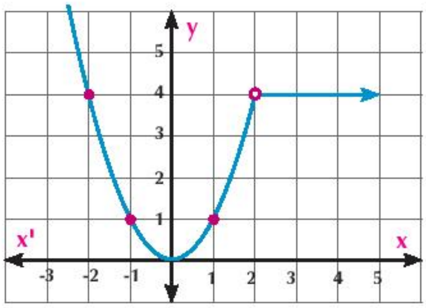


Fig (2)

Try to solve

1. Graph the function f where:

$$f(x)=\begin{cases} x^2 & \text{when } x<0 \\ x & \text{when } x\geq 0 \end{cases}$$

then deduce the range of the function and investigate its monotony.

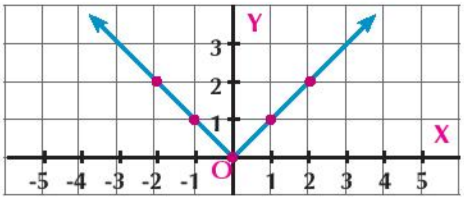
Learn The Absolute Value Function

the simplest form of the absolute value function is:

$$f(x) = |x|,\ x \in \mathbb{R}$$

It is defined as:

$$f(x)=\begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$



Notice: $|-3| = |3| = 3$, $|0| = 0$, $\sqrt{(-2)^2} = \sqrt{2^2} = 2$

i.e.: $|x| \geq 0$, $|-x| = |x|$, $\sqrt{x^2} = |x|$

The function f is represented by two rays starting at point (0, 0), the slope of the first = 1 and the slope of the other = −1.

(**Check:** the range of f = [0 , ∞[, f is even, f is decreasing in]-∞ , 0[and is increasing in]0, ∞ [ )

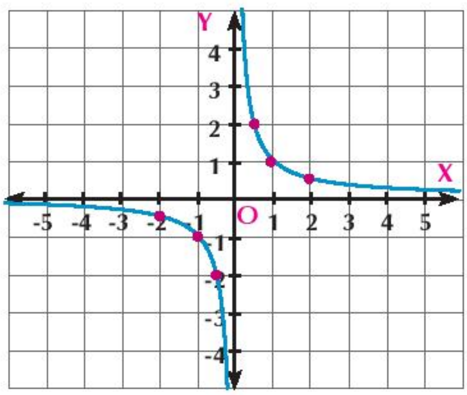
Learn Rational Function

the simplest form of the rational function is:

$$f(x) = \frac{1}{x},\ x \in \mathbb{R} - \{0\}$$

the function f maps the number to its multiplicative inverse. It is represented by a curve whose symmetrical point is (0 , 0). The curve consists of two parts one of them lies on first quadrant and the other lies on third quadrant and each part approaches the two axes and doesnot intersect them ($x = 0$ and y = 0 are two approaching lines of the curve)

(**Check:** The range of $f = \mathbb{R} - \{0\}$, f is odd, f **is decreasing in**]-∞, 0[**and is decreasing in**]0, ∞ [ )



Geometrical transformations of the function curves

First : vertical translation of the function curve

Work together

Work with your classmate

1) Graph the curve of the function $f : f(x) = x^2$ using Geogebra.
2) Place the pointer on the curve vertex and drag it vertically upward for one unit. Notice the change of the function rule to express a new function whose rule is:
   $f(x) = x^2 + 1$ as shown in **figure (1).**

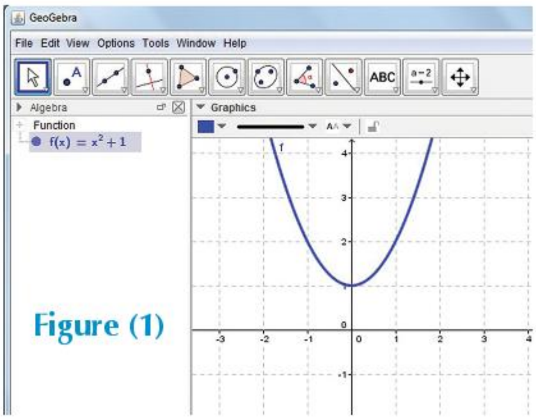


Figure (1)

3) Drag the curve vertex of the function to the points (0 , 2) and (0 , 3) then record your observations each time.

4) Drag the curve of $f(x) = x^2$ vertically downward for two units and notice the change of the function rule to express a new function whose rule is : $f(x) = x^2 - 2$ as shown in Figure (2)

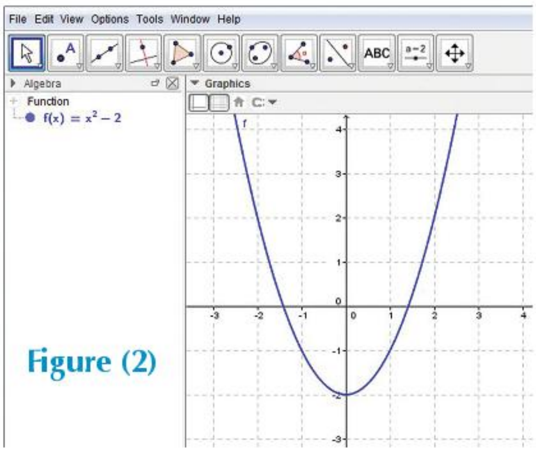


Figure (2)

Think : Show how $f(x) = x^2 - 5$ can be graphed using the curve of $f(x) = x^2$?

Of the previous, we can notice that :

If $f(x) = x^2$, $g(x) = x^2 + 1$ and $h(x) = x^2 - 2$, then :

1) The curve of $g(x)$ is the same curve of $f(x)$ by translation of a magnitude of one unit in the positive direction of y-axis.
2) The curve of $h(x)$ is the same curve of $f(x)$ by translation of a magnitude of two units in the negative direction of y-axis.

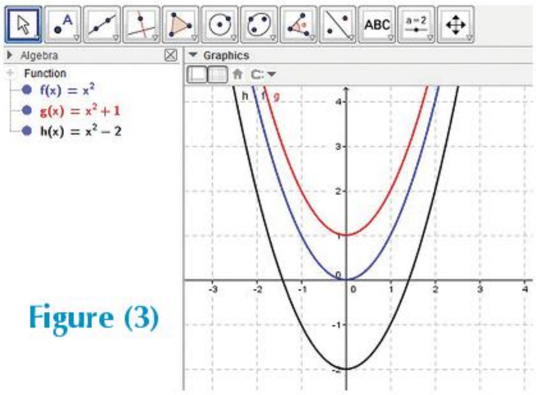


Figure (3)

Critical thinking : Use the curve of $f(x) = x^3$ to show how the following curves can be graphed:

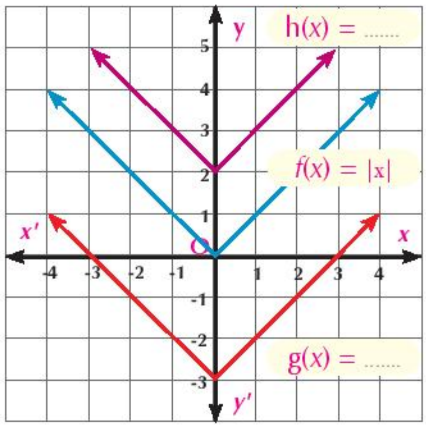
(a) $g(x) = x^3 + 4$ (b) $h(x) = x^3 - 5$

Learn drawing the curve of $y = f(x) + a$

For any function f, the curve of $y = f(x) + a$ is the same curve of $y = f(x)$ by translation of a magnitude of a unit in the direction of $\overrightarrow{oy}$ when $a > 0$ and in the direction of $\overrightarrow{oy'}$ when $a < 0$.

Example

2 The opposite figure shows the curves of the functions f, g and h where each of g and h are an image of the function f by a vertical translation . Write the rules of g and h .



Solution

$\because$ The curve of the function g is the same curve of the function f by translation of a magnitude of three units in the direction of $\overrightarrow{oy'}$

$g(x) = f(x) - 3$ $\quad \because f(x) = |x|$ $\quad \therefore g(x) = |x| - 3$

$\because$ the curve of the function h is the same curve of the function f by translation of a magnitude of two units in the direction of $\overrightarrow{oy}$

$\therefore h(x) = f(x) + 2$ $\quad \because f(x) = |x|$ $\quad \therefore h(x) = |x| + 2$

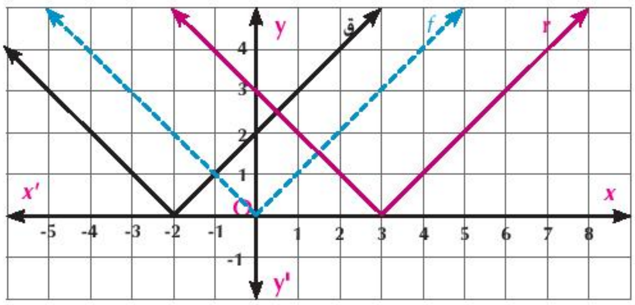
Second : Horizontal translation of the function curve:

Learn

Graphing the curve of $y = f(x + a)$

For any function f, the curve of $y = f(x + a)$ is the same curve of $y = f(x)$ by translation of a magnitude a of units in the direction of $\overrightarrow{ox}$ when $a < 0$ and in the direction of $\overrightarrow{ox'}$ when $a > 0$.

Notice : In the figure opposite : $f(x) = |x|$:



1) The curve of the function g is the same curve of the function f by translation of a magnitude of three units in the direction of $\overrightarrow{ox}$

$\therefore g(x) = |x - 3|$ and the starting point of the two rays is (3, 0)

2) The curve of the function h is the same curve of the function f by translation of a magnitude of two units in the direction of $\overrightarrow{ox'}$

$\therefore h(x) = |x + 2|$ and the starting point of the two rays is (-2 , 0)

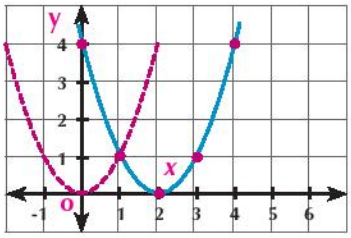
Example

3 Use the curve of the function f where $f(x) = x^2$ to represent each of the two functions g and h where:

a $g(x) = (x - 2)^2$ **b** $h(x) = (x + 3)^2$

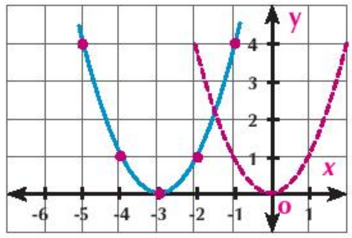
Solution

a



- The curve of $g(x) = (x - 2)^2$ is the same curve of $f(x) = x^2$ by translation of 2 units in the positive direction of x –axis and the vertex point of the curve is (2, 0).

b



- The curve of $h(x) = (x + 3)^2$ is the same curve of $f(x) = x^2$ by translation of 3 units in the negative direction of x –axis and the vertex point of the curve is (-3 , 0).

Try to solve

2 Use the curve of the function f where $f(x) = x^2$ to represent each of the two functions g and h where :

a $g(x) = (x + 4)^2$ b $h(x) = (x - 3)^2$

Critical thinking : If $f(x) = x^2$, show how the curve of the function g where $g(x) = (x - 3)^2 + 2$ can be graphed.

Graphing the curve of $y = f(x + a) + b$

Of the previous , we deduce that : The curve of $y = f(x + a) + b$ is the same curve of $y = f(x)$ by horizontal translation of a magnitude of a of units (in the direction of $\overrightarrow{ox}$ when $a < 0$ and in the direction of $\overrightarrow{ox'}$ when $a > 0$) , then vertical translation of a magnitude of b of units (In the direction of $\overrightarrow{oy}$ when $b > 0$ and in the direction of $\overrightarrow{oy'}$ when $b < 0$).

Try to solve

3 Use the curve of the function f where $f(x) = x^2$ to represent each of the two functions g and h where :

a $g(x) = (x + 2)^2 - 4$ b $h(x) = (3 - x)^2 - 1$

Example **Applying the geometrical transformations on graphing the function curves**

4 Graph the curve of the function g where $g(x) = \frac{1}{x - 1} + 3$, then determine the range of the function and investigate its monotony from the graph

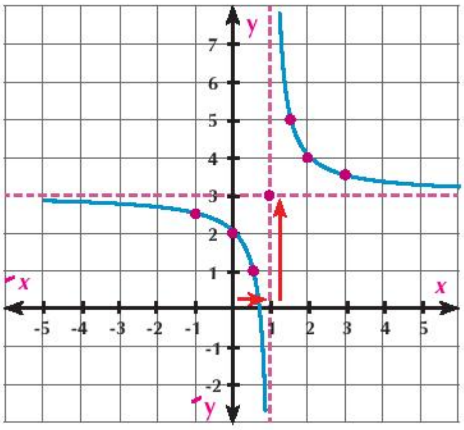
Solution

The curve of the function g is the same curve of the function f where $f(x) = \frac{1}{x}$ by translation of a magnitude of one unit in the direction of $\overrightarrow{ox}$

(a = -1 < 0), then translation of a magnitude of three units in the direction of $\overrightarrow{OY}$ and the point of symmetry of the function curve g is the point (1, 3) and the range of $g = \mathbb{R} - \{3\}$

Monotony of the function g:

g is decreasing in] - ∞, 1[and also decreasing in] 1 , ∞ [



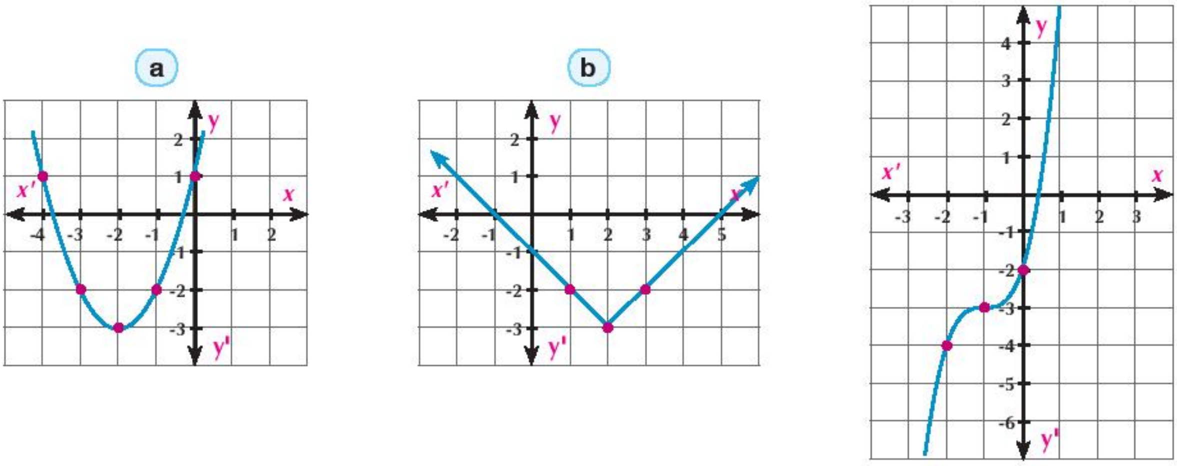
Critical thinking : Can it be said that $f(x) = \frac{1}{x-2} + 3$ is decreasing on its domain ? Explain.

Try to solve

4 Use the curve of the function f where $f(x) = \frac{1}{x}$ and $x \neq 0$ to represent each of the following:

a $g(x) = \frac{1}{x+2} + 1$ b $h(x) = \frac{2x-3}{x-2}$

5 Write down the rule of the function represented graphically by the following graphs:

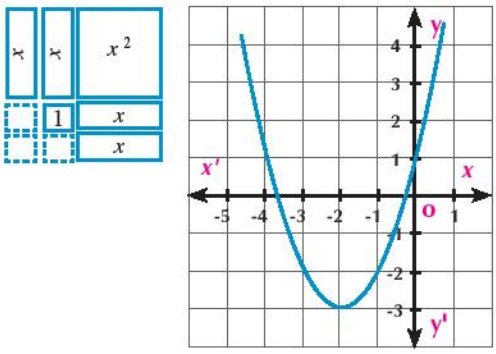


Remark: The curve of $f(x) = x^2 + 4x + 1$ can be graphed using the vertical and horizontal translation of the curve $g(x) = x^2$ as follows.

$f(x) = x^2 + 4x + 1$ by completing the square

$= (x^2 + 4x + 4) - 3$

$= (x + 2)^2 - 3$

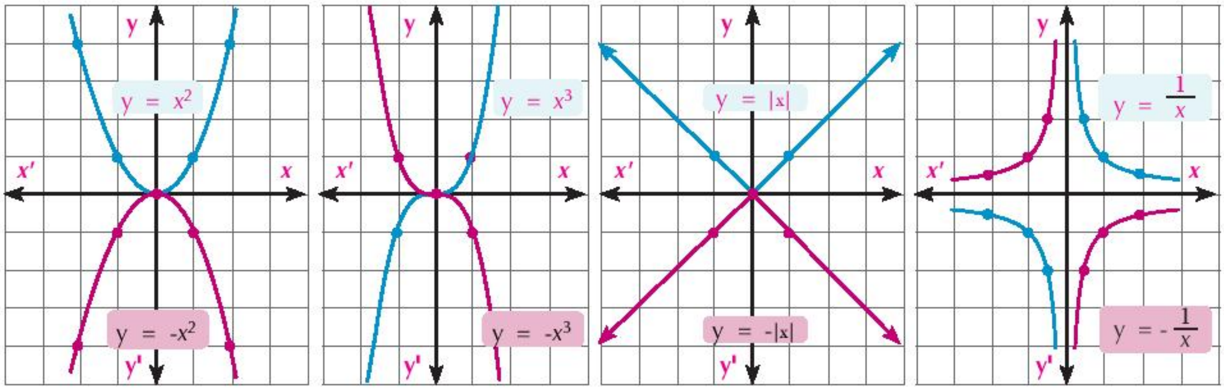


i.e. the curve of the (given) function f is the same curve of the function g where $g(x) = x^2$ by translation of a magnitude of two units in the direction of $\overrightarrow{ox'}$, then three units in the direction of $\overrightarrow{oy'}$. The opposite figure shows that.

Application : Graph the curve of the function $f(x) = x^2 + 6x + 7$ using the vertical and horizontal translation of the function $g(x) = x^2$, then investigate the monotony of the function f.

Third: The reflection of the function curve on x-axis

The following grphs illustrate the reflection of the curves of some standard functions on x-axis.



What do you notice? What do you infer ?

Learn

Graph the curve of $y = -f(x)$

for any function f, the curve of $y = -f(x)$ is the same curve of $y = f(x)$ by reflection on x-axis.

Example

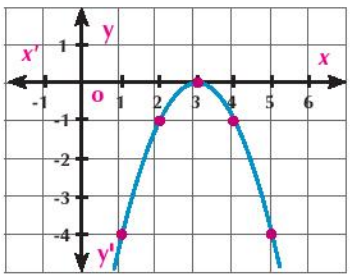
Applying the geometrical transformations on graphing the curves

(5) Use the curves of the standard functions to graph the functions g, h and z where:

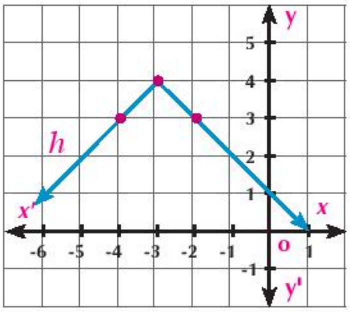
(a) $g(x) = -(x - 3)^2$ (b) $h(x) = 4 - |x+3|$ (c) $z(x) = 2 - \dfrac{1}{x - 3}$

Solution

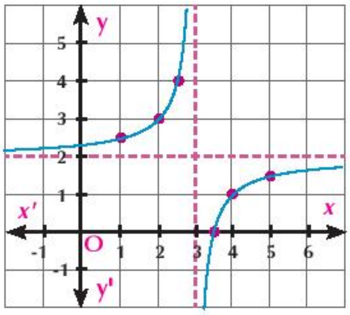
(a) The curve of g(x) is the reflection of the curve of $f(x) = x^2$ on x-axis, then horizontal translation of a magnitude of three units in the direction of $\overrightarrow{ox}$, and the vertex point of the curve is (3 , 0) and the curve is open downward .



b The curve of $h(x)$ is a reflection of the curve of $f(x) = |x|$ on x-axis, then horizontal translation of a magnitude of 3 units in the direction of $\overrightarrow{ox'}$ and vertical translation of a magnitude of four units in the direction of $\overrightarrow{oy}$. The starting point of the two rays is point (-3 , 4) and the curve is open downward.



c The curve of z(x) is a reflection of the curve of $f(x) = \frac{1}{x}$ on x-axis, then horizontal translation of a magnitude of three units in the direction of $\overrightarrow{ox}$ and vertical translation of a magnitude of two units in the direction of $\overrightarrow{oy}$ and the symmetrical point of the curve is (3, 2).



Try to solve

6 Graph the curve of the function g in each of the following where:

a $g(x) = 3 - (x + 1)^2$ **b** $g(x) = -(x - 3)^3$ **c** $g(x) = 3 - |x - 5|$

Check your graph using a graphical program or the graphical calculator.

Exercises 1 - 4

1 Graph the curve of the function f, then determine its range and investigate its monotony

a $f(x) = \begin{cases} |x| & \text{when } x \leqslant 0 \\ x^2 & \text{when } x > 0 \end{cases}$ **b** $f(x) = \begin{cases} 4 & \text{when } x < -2 \\ x^2 & \text{when } x > -2 \end{cases}$

c $f(x) = \begin{cases} x^3 & \text{when } x < 1 \\ 1 & \text{when } x > 1 \end{cases}$

Choose the correct answer:

2 The curve of $g(x) = x^2 + 4$ is the same curve of $f(x) = x^2$ by translation of a magnitude of 4 units in the direction of :

a $\overrightarrow{ox}$ **b** $\overrightarrow{ox'}$ **c** $\overrightarrow{oy}$ **d** $\overrightarrow{oy'}$

3 The curve of the function $g(x) = |x + 3|$ is the same curve of $f(x) = |x|$ by translation of a magnitude of 3 units in the direction of:

a $\overrightarrow{ox}$ **b** $\overrightarrow{ox'}$ **c** $\overrightarrow{oy}$ **d** $\overrightarrow{oy'}$

4. The curve vertex point $f(x) = (2 - x)^2 + 3$ is :

   a) (2, 3) b) (2, -3) c) (-2, 3) d) (-2, -3)

5. The symmetry point of the curve of the function $f(x) = 2 - (x + 1)^3$ is :

   a) (1, 2) b) (-1, 2) c) (2, 1) d) (2, -1)

6. The symmetry point of the curve of the function f where $f(x) = \dfrac{1}{x - 3} + 4$ is :

   a) (3, -4) b) (-3, -4) c) (3, 4) d) (-3, 4)

Answer the following:

7. Use the curve of the function f where $f(x) = x^2$ to represent the following graphically:

   a) $f_1(x) = x^2 - 4$ b) $f_2(x) = (x - 3)^2$ c) $f_3(x) = (x - 1)^2 - 2$

8. Use the curve of the function f where $f(x) = |x|$ to represent the following graphically:

   a) $f_1(x) = |x| + 1$ b) $f_2(x) = |x + 2|$ c) $f_3(x) = |x - 3| - 2$

   ➢ **Then find the coordinates of the intersecting points of the curves with the two coordinate axes.**

9. Use the curve of the function f where $f(x) = x^3$ to represent the following graphically :

   a) $f_1(x) = f(x) - 3$ b) $f_2(x) = f(x - 2)$ c) $f_3(x) = f(x + 3) + 2$

   ➢ **Then determine the symmetry point of the curve of each function.**

10. If the function f where $f(x) = \dfrac{1}{x}$ graph the function h and determine the symmetry point of the function curve :

    a) $g(x) = f(x - 3)$ b) $g(x) = f(x) + 2$ c) $g(x) = f(x - 2) + 2$

11. Use the curve of the function f where $f(x) = x^2$ to represent the following graphically :

    a) $f_1(x) = 4 - x^2$ b) $f_2(x) = -(x - 3)^2$ c) $f_3(x) = 2 - (x + 3)^2$

Unit 1

1 - 5 Solving absolute value equations and inequalities

Real Functions

First: Solving equations

Think and discuss

In one figure, represent the two curves of the two functions f and g where f is a modulus function and g is a constant function graphically. Notice the graph , then answer:

a) How many probable intersecting points are there for the two curves of the two functions together ?

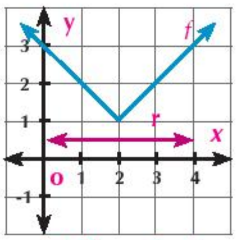
b) If the interesting points of the two curves are found together. Do the ordered pairs satisfy the rule of each function.

c) If the intersecting points of the two curves are found together?

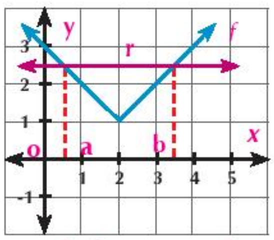
Notice :

1) At the intersecting points (if found) , $f(x) = g(x)$, and vice versa for each x belong to the common domain of both functions.

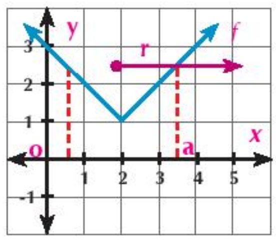
2) For any two functions f and g, the solution set of the equation $f(x) = g(x)$ is the set of x-coordinates of the intersecting points of their two curves as shown in the following figures:



Solution set = ϕ



Solution set = {a, b}



Solution set = {a}

Solve the equation : $|ax + b| = c$

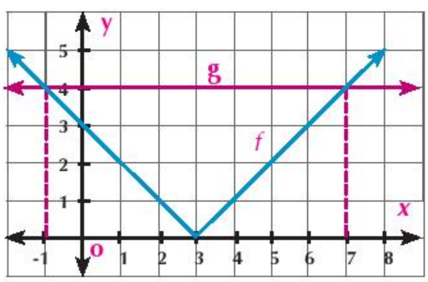
Example

1) Solve the equation : $|x - 3| = 4$ algebraically and graphically.

Solution

Let $f(x) = |x - 3|$ and $g(x) = 4$

1) Graph the curve of the function $f(x) = |x - 3|$ by translating the curve of $f(x) = |x|$ 3 units in the direction of $\overrightarrow{ox}$



You will learn

- Solve the modulus equations graphically.
- Solve the modulus equations algebraically.
- Solve the modulus inequalities graphically.
- Solve the modulus in equalities algebraically.
- Model problems and life applications to solve using the modulus equations and equalities.

Key Terms

- Equation
- Inequality
- Graphical solution

Materials

- Graphica calculator
- Graph paper
- Graphic programs

2) On the same figure , graph $g(x) = 4$ where g is a constant function represented by a straight line parallel to x-axis and passing through point (0, 4)

$\because$ The two points intersect at (-1 , 4) and (7 , 4)

$\therefore$ The solution set is {-1 , 7}

Algebraic solution :

From the definition of the modulus function : $f(x) = \begin{cases} x - 3 & \text{when } x \geqslant 3 \\ -x + 3 & \text{when } x < 3 \end{cases}$

When $x > 3$: $x - 3 = 4$ i.e. : $x = 7 \in]3 , \infty[$

When $x \geqslant 3$: $-x + 3 = -4$ i.e. : $x = -1 \in]-\infty , 3]$

The solution set of the equation is **{ - 1 , 7}** this matches with the graphical solution.

Try to solve

1 Solve each of the following equations graphically and algebraically.

a $|x| - 4 = 0$ **b** $|x| + 1 = 0$ **c** $|x - 7| = 5$

Some Properties of the Absolute Value

Learn

1) $|ab| = |a| \times |b|$ for example :

$|2 \times -3| = |-6| = 6$ and $|2| \times |-3| = 2 \times 3 = 6$

2) $|a + b| \leqslant |a| + |b|$

The equality happens only when **a and b** have the same sign. For example :

$|4 + 5| = |4| + |5| = 9$ et $|-4 - 5| = |-4| + |-5| = 9$

3) $|a - x| = |x - a|$

Notice :

1) If $|x| = a$ then $x = a$ or $x = -a$ for each $a \in \mathbb{R}^+$

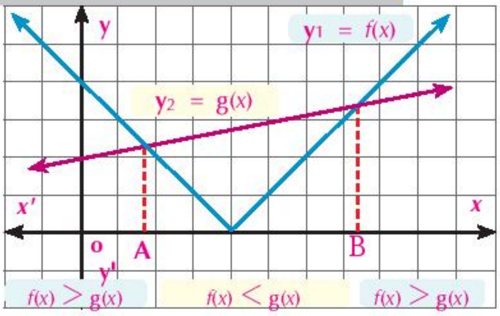
2) If $|a| = |b|$ then $a = b$ or $a = -b$ for each $a \in \mathbb{R}$, $b \in \mathbb{R}$

3) $|x|^2 = |x^2| = x^2$

Second : Solving the inequalities

You have previously learned the inequalities and known that they are mathematical phrases containing one of the symbols : ($<$, $>$, $\leqslant$, $\geqslant$). Solving the inequality means that you find the value or the set of values of the variable which satisfy the inequality and make it true.

Solving the inequalities graphically



The opposite figures shows a curve for each of the two functions f and g where: $y_1 = f(x)$ and $y_2 = g(x)$ and the solution set of the equation $f(x) = g(x)$ is {a , b}

I.e.: $y_1 = y_2$ when x = a or x = b

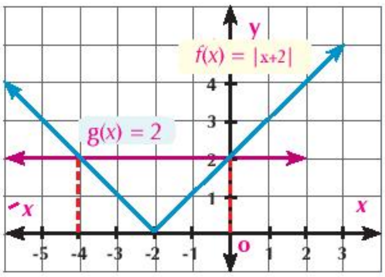
We notice: $y_1 < y_2$ i.e. $f(x) < g(x)$ when $x \in]a, b[$

$y_1 > y_2$ i.e. $f(x) > g(x)$ when $x \in]-\infty, a[\cup]b, \infty[$

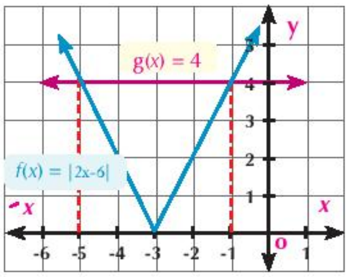
Example

2

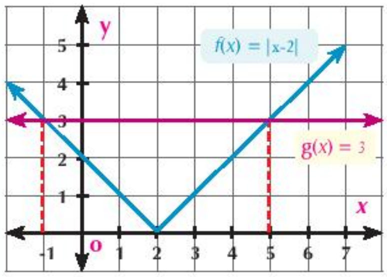
a



b



c



Solution set of inequality	Solution set of inequality	Solution set of inequality
$\lvert x+2 \rvert < 2$	$\lvert 2x+6 \rvert > 4$	$\lvert x-2 \rvert \leqslant 3$
is : $]-4, 0[$	is : $]-\infty, -5] \cup [-1, \infty[$	is : $[-1, 5]$
	i.e.: $\mathbb{R} -]-1, 5[$	

Try to solve

2 Find the solution set of the following inequalities using the graphs in example (7):

a $\lvert x+2 \rvert \leqslant 2$ **b** $\lvert 2x+6 \rvert \leqslant 4$ **c** $\lvert x-2 \rvert > 3$

Solving the inequalities algebraically

Learn

First: if $\lvert x \rvert \leqslant a$ and $a > 0$ **then** $-a \leqslant x \leqslant a$

Second: if $\lvert x \rvert > a$ and $a > 0$ **then** $x > a$ **or** $x \leqslant -a$

Example

3 Find the solution set of the following inequalities in the form of an interval:

a $\lvert x-3 \rvert < 4$

Solution

a $\because |x - 3| < 4$ **i.e..** $-4 < x - 3 < 4$ **Adding 3 to inequality**

$\therefore -4 + 3 < x - 3 + 3 < 4 + 3$ **i.e.** $-1 < x < 7$

$\therefore$ **the solution set** $=]-1 , 7[$

Remember

for a , b and c

if: a < b , b < c then a < c

if: a < b then

a + c < b + c

a c < b c when c > 0

a c > b c when c < 0

Try to solve

3. Find the solution set of the following inequalities in the form of an interval :

a $|x - 7| < 11$ **b** $|3x + 7| \leqslant 8$

Life applications

Example **Meteorology**

4. A meteorological station has recorded the temperature of Cairo on a day. If the temperature has been 32° in difference 7° from its normal rate on that day. What is the expected temperature recorded in Cairo on that day ?

Solution

Let the temperature expected to be recorded in Cairo on that day $= x°$

$\therefore |x - 32| = 7$ i.e. $x - 32 = \pm 7$

then $x = 32 + 7 = 39$ or $x = 32 - 7 = 25$

i.e. the temperature expected to be recorded is 39° or 25°

Exercises (1 - 5)

Complete :

1. The solution set of the equation $|x| = \frac{1}{2}$ is ....
2. The solution set of the equation $|x| + 3 = 0$ is ....
3. The solution set of the inequality $|x - 2| \leqslant 0$ is ....

Choose the proper solution set from the following list for each equation or inequality :

4. $|x - 2| = 3$
5. $|x - 2| < 3$
6. $|x - 2| > -3$
7. $|x - 2| \leqslant 3$
8. $|x - 2| > 3$
9. $|x - 2| = -3$

a. $]-1, 5[$
b. R
c. $\{-1, 5\}$
d. $R - [-1, 5]$
e. ϕ
f. $[-1, 5]$

Find the solution set for each of the following equations algebraically :

10. $|x + 3| = 6$
11. $|2x - 7| = 5$
12. $|3 - 2x| = 7$

Find the solution set for each of the following equations graphically :

13. $|x + 4| = 3$
14. $|2x - 5| = 3$

Find the solution set for each of the following equations graphically:

15. $|x - 1| < 3$
16. $|x - 2| \leqslant 5$
17. $|x + 3| > 2$

Find the solution set for each of the following equations algebraically:

18. $|2x - 1| > 3$
19. $|2x + 3| \leqslant 7$
20. $|3x - 7| \geqslant 2$

Unit Two

Exponents, Logarithms and their Applications

Unit introduction

The concept of logarithms had been used in mathematics early the Seventeenth century by the scientist John Napier as a method to simplify calculations to help navigators, scientists, engineers and others depend on to do their calculations easier by using the calculating ruler and logarithmic tables. They had benefited from the properties of logarithms by substituting the multiplying operations to find the logarithm of the product of tow numbers, by using the property of addition with respect to the property of $\log_a(xy) = \log_a x + \log_a y$. Thanks to the scientist Leonhart Euler. In the eighteenth century, he connected the concept of logarithm with the concept of the exponential function to widen the concept of logarithms and connect with the functions.

Logarithmic scaler can be used in various fields. For example, the decibel is a logarithmic unit used to measure the sound intensity and voltage ratio. Logarithmic scaler is also used to measure the power of hydrogen Ph(it is a logarithmic scaler) to identify the acidity of a solution in chemistry.

Unit objectives

By the end of this unit, the student should be able to:

- Identify the exponential function.
- Identify the graphical representation of the exponential function and deduce its properties.
- Identify the rules of the rational exponents.
- Solve an exponential equation in the from: $a^x = b$.
- Identify the logarithmic equation.
- convert from the exponential form into logarithmic form aglabraically and vice versa.
- I dentify the graphical representation of the logarithmic function in limited intervals and deduce its properties.
- Deduce the relation between the exponential and logarithmic functions graphically.
- Identify the rules of logarithms.
- Solve logarithmic equations.
- Solve problems including applying the rules of logarithms.
- Identify the common logarithms of base 10.
- Find the value of logarithms using the calculator.
- Use the calculator to solve some exponential equations.

Key - terms

- The n^{th} Power
- Base
- Exponent
- n^{th} Root
- Rational – exponent
- Expontential Function
- Exponential Growth
- Exponential Decay
- Domain
- Range
- Reflection
- Logarithm
- Logarithmic Equation
- Logarithmic Function

Unit lessons

Lesson 1: Rational exponents

Lesson 2: Exponential Function and its applications

Lesson 3: Solving exponential equations

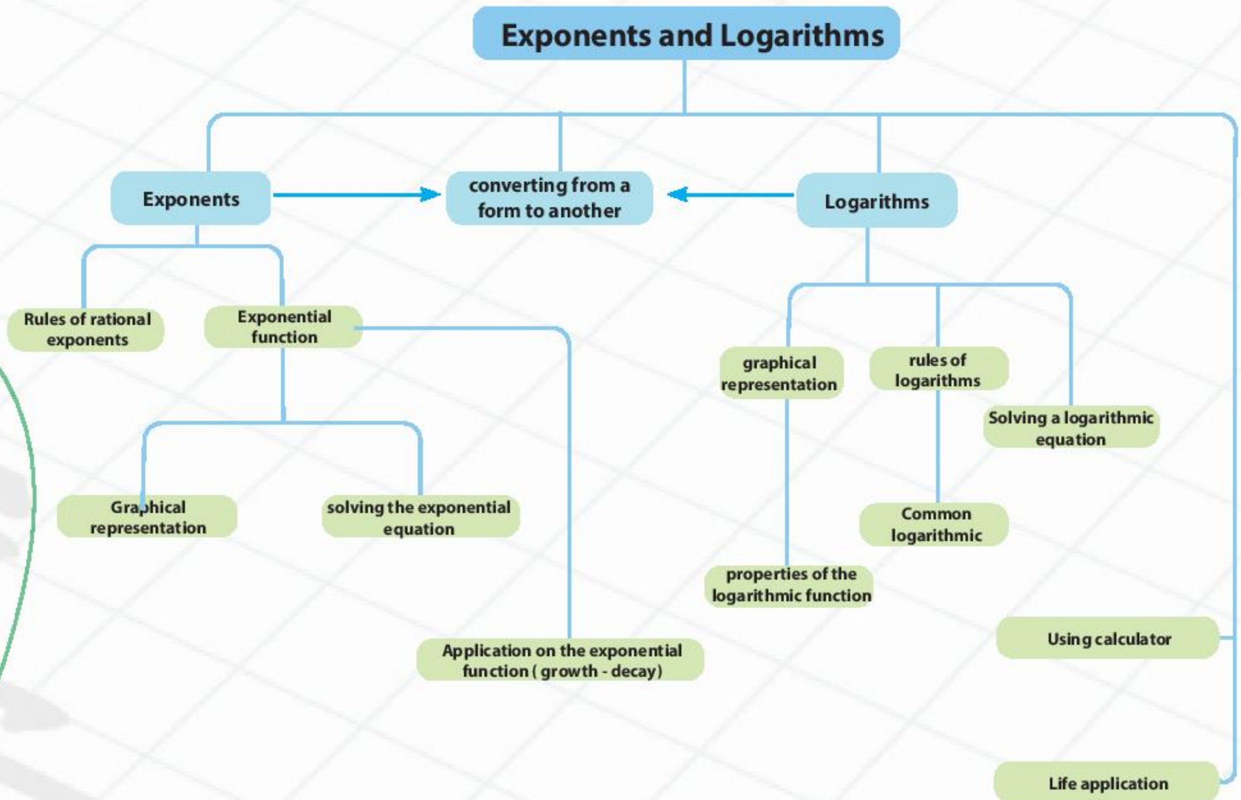
Lesson 4: Logarithmic function and its graphical representation

Lesson 5: Some properties of logarithms

Materials

Scientific calculator – geogebra–graph

Unit planning guide



Unit Two

2 - 1 Rational Exponents

You will learn

- Generalize the rules of exponents.
- n^{th} root.
- Rules of rational exponents.

Key terms

- The n^{th} power
- Base
- Exponent
- n^{th} root
- Rational exponent

Materials

- Scientific calculator
- Graphic programs

Notice

Index of the root $\sqrt[n]{a}$ symbol of the root, number inside the root

Preface

You have previously learned the square roots of a non-negative real number and identified some properties of both cubic and square roots. You have also learned the integers and identified some of their own properties. In this lesson , you are going to learn the rational exponents.

Learn

The n^{th} Root

You have known that the square root of a number is an inverse operation of squaring this number. Similarly, the n^{th} root of a number is an inverse operation to place the n^{th} power of this number.

Example:

1. If $x^3 = 8$ then 2 is the cubic root of 8 i.e $\sqrt[3]{8} = 2$
2. If $x^5 = 32$ then 2 is the fifth root of 32 i.e $\sqrt[5]{32} = 2$
3. If $x^n = a$ then x is the n^{th} root of a i.e $\sqrt[n]{a} = x$

Definition

For any real number, $a \geqslant 0$, $n \in Z^+ - \{1\}$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$

This relation is true when $a < 0$ and n is an odd number > 1

Example:

1

$(16)^{\frac{1}{4}} = \sqrt[4]{16} = 2$ $\qquad$ $(-9)^{\frac{1}{2}} = \sqrt{-9} \notin R$

$-(27)^{\frac{1}{3}} = -\sqrt[3]{27} = -3$ $\qquad$ $(-243)^{\frac{1}{5}} = \sqrt[5]{-243} = -3$

Example

1. If $x^n = a$, find the values of x in R (if found) in each of the following cases:

   a) $n = 5$, $a =$ zero $\qquad$ b) $n = 4$, $a = 81$

   c) $n = 2$, $a = -4$ $\qquad$ d) $n = 3$, $a = -8$

Solution

a **when** $n = 5$, $a = \text{zero}$ then $x^5 = 0$ **then** $x = \sqrt[5]{0} = 0$

b **when** $n = 4$, $a = 81$ then $x^4 = 81$ **then** $x = \pm\sqrt[4]{81} = \pm 3$

c **when** $n = 2$, $a = -4$ then $x^2 = -4$ **then** $x = \pm\sqrt{-4} \notin R$

d **when** $n = 3$, $a = -8$ then $x^3 = -8$ **then** $x = \sqrt[3]{-8} = -2$

from the previous example, we deduce that:

if $x^n = a$, then the values of x, which satisfy the equation become clear in the following table:

n	a	$\sqrt[n]{a}$
$n \in Z^+ - \{1\}$	$a = 0$	$\sqrt[n]{a} = \text{zero}$
positive even integer	$a > 0$	there are tow real roots $\pm \sqrt[n]{a}$
positive even integer	$a < 0$	there arenot real roots.
positive odd integer, $n \neq 1$	$a \in R$	there is only a real root $\sqrt[n]{a}$

Try to solve

1 Find the values of x in each of the following (if possible):

a $x^2 = 36$ b $x^5 = -32$ c $x^3 = 125$

d $x^4 = 1296$ e $x^2 = -49$ f $x^7 = -128$

2 **Critical thinking**: use a numerical exampte to show the difference between the sixth root of a and $\sqrt[6]{a}$.

Definition 3

if $n \in Z^+ - \{1\}$, $m \in Z^+$, $\sqrt[n]{a} \in R$ then: $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Example:

$(16)^{\frac{3}{2}} = (\sqrt{16})^3 = (4)^3 = 64$

$\sqrt[3]{(-125)^2} = (\sqrt[3]{(-125)})^2 = (-5)^2 = 25$

Example

2 Find each of the following in the simplest form:

a $-\sqrt[3]{8 a^6 b^9}$ b $\pm\sqrt{64 (a^2 + 3)^6}$

Solution

a) $-\sqrt[3]{8a^6 b^9} = -\sqrt[3]{(2a^2b^3)^3} = -2a^2b^3$

b) $\pm\sqrt{64(a^2+3)^6} = \pm\sqrt{[8(a^2+3)^3]^2}$

$= \pm 8(a^2+3)^3$

Try to solve

3) Find each of the following in the simplest form:

a) $\sqrt[4]{625a^{12}}$ b) $\sqrt[5]{-243b^5}$ c) $\sqrt[7]{128(a+b)^7}$

Using The Modulus

the modulus of a number is used if the index of the root (n) is an even number, then $\sqrt[n]{a^n} = |a|$, but if the index is an odd number, it is not necessary to use the modulus.

$$\sqrt[n]{x^n} = \begin{cases} |x| & \text{if n is even.} \\ x & \text{if n is odd.} \end{cases}$$

Example

3) Find each of the following in the simplest form :

a) $\sqrt{9x^2}$ b) $\sqrt[3]{-8x^3}$

c) $\sqrt[4]{(2-\sqrt{3})^4}$ d) $\sqrt[6]{(1-\sqrt{7})}$

Notice

the square of any of the two numbers (a) or (-a) is a^2

Solution

a) $\sqrt{9x^2} = \sqrt{(3x)^2} = |3x|$

b) $\sqrt[3]{-8x^3} = \sqrt[3]{(-2x)^3} = -2x$

c) $\sqrt[4]{(2-\sqrt{3})^4} = |2-\sqrt{3}| = 2-\sqrt{3}$ where $2 > \sqrt{3}$

d) $\sqrt[6]{(1-\sqrt{7})} = |1-\sqrt{7}| = \sqrt{7}-1$ where $\sqrt{7} > 1$

Try to solve

4) Find each of the following in the simplest form:

a) $\sqrt[4]{16a^{12}}$ b) $\sqrt[6]{(x-2)^{18}}$

c) $\sqrt[3]{(2-\sqrt{5})^3}$ d) $\sqrt[4]{(2-\sqrt{5})^4}$

Definition 4

if $n \in Z^+ - \{1\}$, $m \in Z^+$, $\sqrt[n]{a} \in R$ then: $a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}$

Example: $7^{-\frac{3}{5}} = \frac{1}{7^{\frac{3}{5}}}$, $\frac{1}{4^{-\frac{2}{3}}} = 4^{\frac{2}{3}}$

Definition 5

if $n \in Z^+ - \{1\}$, $\sqrt[n]{a}$, $\sqrt[n]{b}$ are two real numbers, then:

- $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ where $b \neq$ zero

Example

4 Find each of the following in the simplest form:

a $\frac{\sqrt{8} \times 4^{-1} \times 2^{\frac{3}{2}}}{6^{-2} \times 3^2}$

b $\frac{32^{-\frac{3}{5}} \times 8^{\frac{2}{3}}}{\sqrt[4]{4} \times \sqrt[8]{16^5}}$

Solution

a the expression $= \frac{8^{\frac{1}{2}} \times 4^{-1} \times 2^{-\frac{3}{2}}}{6^{-2} \times 3^2}$ **converting roots into rational exponents.**

$= \frac{(2^3)^{\frac{1}{2}} \times (2^2)^{-1} \times 2^{-\frac{3}{2}}}{(3 \times 2)^{-2} \times 3^2}$ **factorizing each base into its primary factors.**

$= \frac{(2)^{\frac{3}{2}} \times 2^{-2} \times 2^{-\frac{3}{2}}}{3^{-2} \times 2^{-2} \times 3^2}$ **By simplifying**

$= 2^{\frac{3}{2} - 2 - \frac{2}{2} + 2} \times 3^{2 - 2}$

$= 2^{zero} \times 3^{zero} = 1$

b the expression $= \frac{32^{\frac{3}{5}} \times 8^{\frac{2}{3}}}{4^{\frac{5}{8}} 16 \times \frac{1}{4}}$ **converting roots into rational exponents.**

$= \frac{(2^5)^{\frac{2}{3}} (2^3) \times \frac{3}{5}}{(2^2)^{\frac{5}{8}} (2^4) \times \frac{1}{4}}$ **factorizing each base into its primary factors.**

$= \frac{2^3 \times 2^2}{2^{\frac{1}{2}} \times 2^{\frac{5}{2}}}$

$= 2^{3+2-\frac{1}{2}-\frac{5}{2}} = 2^2 = 4$

Try to solve

5 Find each of the following in the simplest form :

a $\frac{\sqrt[3]{243} \times \sqrt{8^{-1}}}{\sqrt{2} \times \sqrt[3]{9}}$

b $\frac{\sqrt[4]{4} \times \sqrt[5]{2}}{\sqrt{2} \times \sqrt[10]{4}}$

Solving the equations:

Example

5) Find the solution set for each of the following equations in R:

a) $x^{\frac{2}{3}} = 9$ b) $(x + 1)^{\frac{3}{4}} = 8$

Solution

a) $\because x^{\frac{2}{3}} = 9$ **By placing the third power of the two sides**

$\therefore (x^{\frac{2}{3}})^3 = 9^3$

$\therefore x^2 = 9^3$ **By taking off the square root of both sides**

$\sqrt{x^2} = \sqrt{39}$ $\therefore |x| = {}^{3}3$ $\therefore x = \pm 27$

$\therefore$ The solution set = {27, -27}

b) $(x + 1)^{\frac{3}{4}} = 8$ **By placing the fourth power of the two sides**

$\therefore (x + 1)^3 = 8^4$

$\therefore (x + 1) = (\sqrt[3]{8})^4$

$\therefore x + 1 = 2^4$ $\therefore x = 15$ $\therefore$ The solution set = {15}

Try to solve

6) Find the solution set for each of the following equations in R:

a) $x^{\frac{5}{2}} = 32$ b) $\sqrt[3]{(x-1)^5} = \frac{1}{32}$

Exercises 2 - 1

1) **Write down each of the following in an exponential form:**

a) $\sqrt[3]{x}$ b) $\sqrt[4]{a^3}$ c) $2\sqrt[3]{n}$

d) $\sqrt[4]{a^2 b^3}$ e) $\sqrt[6]{x^5}$ f) $\frac{\sqrt[3]{x}}{\sqrt[5]{x^2}}$

2) **Write down each of the following in a root form:**

a) $a^{\frac{1}{2}}$ b) $b^{\frac{2}{3}}$ c) $6y^{\frac{3}{4}}$

d) $8b^{\frac{4}{9}}$ e) $(3x)^{-\frac{2}{3}}$ f) $5^{\frac{1}{2}}$

3) **Find the value of each of the following in the simplest form:**

a) $(16)^{\frac{3}{4}}$ b) $(-32)^{\frac{3}{5}}$ c) $27^{-\frac{4}{3}}$

d) $(\frac{1}{8})^{\frac{2}{3}} + (\frac{1}{4})^{\frac{1}{2}}$ e) $\frac{\sqrt[4]{4}}{\sqrt{2}}$ f) $\frac{1}{(2^{-2} \times 4^{\frac{1}{2}} \times 8^{\frac{2}{3}})^{-2}}$

4) **Find in the simplest form:**

a) $(a^{-\frac{2}{3}})^{-3}$ b) $\sqrt[3]{x} \times x^{\frac{1}{2}}$ c) $(3^2 + 4^2)^{\frac{1}{2}}$

d $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}})$ e $(x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}} y^{\frac{1}{3}} + y^{\frac{2}{3}})$

f $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})^2$ g $(1^3 + 2^3 + 3^3)^{\frac{1}{2}}$

5 Reduce each of the following in the simplest form:

a $\sqrt[5]{243} + \sqrt[9]{512}$ b $(\frac{16}{81})^{\frac{1}{3}} \times (\frac{729}{8})^{\frac{1}{2}}$ c $(16)^{\frac{2}{3}} \div (8)^{\frac{3}{2}}$

d $(27)^{\frac{2}{3}} - (64)^{\frac{5}{6}}$ e $\sqrt{0.1} \times \sqrt[3]{0.216} \times \sqrt{2,5}$ f $\frac{8^{\frac{3}{8}} \times 4^{\frac{-3}{16}}}{2^{-\frac{5}{4}}}$

g $(125)^{\frac{2}{3}} \times 81^{\frac{1}{4}} \times (15)^{-1}$ h $\frac{16^{x-\frac{1}{4}} \times 9^{x+\frac{1}{2}}}{8^{x-1} \times 18^{x+2}}$

Choose the correct answer:

6 if $\sqrt[3]{x^2} = 9$, then $x \in$

a $\{27\}$ b $\{27, -27\}$ c $\{1\}$ d ϕ

7 $64^{-\frac{1}{6}} =$

a 2 b -2 c $\frac{1}{2}$ d $-\frac{1}{2}$

8 $\sqrt[3]{x^{-3}} =$

a x^{-1} b $-x$ c $|x^{-1}|$ d $-|x|$

9 $\sqrt[4]{x^4y^8} =$

a $x\,y^2$ b $\pm\; x\,y^2$ c $|x|y^2$ d $x|y^2|$

10 If $x^{\frac{3}{2}} = 8$, then $x =$

a 4 b -4 c $\frac{1}{4}$ d $-\frac{1}{4}$

11 $\frac{6^{-\frac{1}{5}} \times 6^{\frac{3}{5}}}{\sqrt[5]{36}} =$

a 1 b 6 c $\frac{1}{6}$ d $\sqrt{6}$

12 Find the solution set for each of the following equations in R**:**

a $x^{\frac{1}{2}} = 5$ b $x^{\frac{7}{2}} = \frac{1}{128}$ c $\sqrt{x^3} = 27$

d $(x - 5)^{\frac{5}{2}} = 32$ e $3\,x^{-\frac{3}{4}} = \frac{3}{8}$ f $2^{3x-1} = \frac{16}{\sqrt{2}}$

Unit Two

2 - 2 Exponential Function and its applications

You will learn

- Exponential function.
- Representing the exponential function graphically.
- Properties of the exponential function.

Key terms

- Expontential Function
- Exponential Growth
- Exponential Decay

Materials

- Scientific calculator
- Graphic programs

Add to your information

the exponential function $f(x) = a^x$ in case $a > 1$ is called the growth function and is related to many life applicatins such as over population, and the banking compound interst . The exponential function $f(x) = a^x$ in case $0<a<1$ is called the decay function and is relatd to many applications such as the half-life of the radioactive atoms.

Preface

In our daily life, there are a lot of situations that need very accurate calculations such as banking profits, over population, cell reproduction in some organisms and the half-life of the radioactive atoms and so on. These situations require the concept of the exponential function which we are going to learn and investigate some of its properties.

Learn

Exponential Function

Definition 1

if a is a positive real number $a \neq 1$ then the function:

f where $f: R \longleftarrow R^+$, $f(x) = a^x$

is called an exponential function whose base is a

Notice

In algebraic function: the independent variable (x) is the base while the power is a real number.

In exponential function: The independent variable (x) is the power while the base is the real number and does not equal one.

Verbal expression: Explain why the function $f(x) = (-3)^x$ where $x \in R$ is not exponential function.

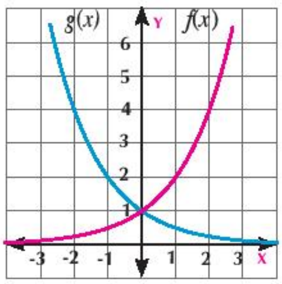
Graphical Representation of the Exponential Function

Example

1. use the values of $x \in [-3, 3]$ to graph in one figure a part of the curve of each of the following tow functions:
$f(x) = 2^x$, $g(x) = (\frac{1}{2})^x$

Solution

x	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x)$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



we can deduce the next properties of the exponential function from the graph:

1 the function f: $f(x) = 2^x$ is increasing on its domain because $(a > 1)$
the function g: $g(x) = (\frac{1}{2})^x$ is decreasing on its domain because $(0 < a < 1)$

2 the range of both functions is R^+

3 the curve of the function f: $f(x) = 2^x$ is the image of the function curve g: $g(x) = (\frac{1}{2})^x$ by reflection on y-axis

Try to solve

1) Use the values of $x \in [-2, 2]$ to graph in one figure the curve of each of the functions $f_1(x) = 2^x$, $f_2(x) = 3^x$ and $f_3(x) = 4^x$.

Example

2) if $f(x) = 3^x$, complete the following :

a) $f(2) =$ .... b) $f(x+2) =$ .... $\times 3^x$ c) $f(x) \times f(-x) =$ ....

Solution

a) $f(2) = 3^2 = 9$ b) $f(x+2) = 3^{x+2} = 3^x \times 3^2 = 9 \times 3^x$

c) $f(x) \times f(-x) = 3^x \times 3^{-x} = 3^{x-x} = 3^{\text{zero}} = 1$

Exercises 2 - 2

1. Graph each of the following functions, then find the domain and range of each and show which of them is increasing and which is decreasing.

   a) $f(x) = 2^x$ b) $f(x) = 3^x$ c) $f(x) = (\frac{1}{2})^x$ d) $f(x) = 2^{-x+1}$

2. **Compete:**

   a) the function $f : f(x) = 2^x$ intersects Y- axis at point ....

   b) the function f: $f(x)=2^{1-x}$ intersects Y -axis at point ....

   c) if the curve of the function f: $f(x) = a^x$ passes through point (1, 3), then a = ...

   d) the curve of the function f: $f(x) = 3^x$ is the image of the curve of the function g: $g(x) = (\frac{1}{3})^x$ by reflection in ....

   e) the function f where $f(x) = a^x$ is decreasing if a $\in$ ....

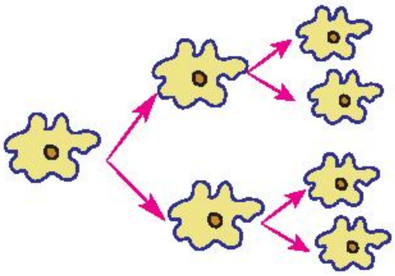
   f) the function f where $f(x) = (2a)^x$ is increasing when a $\in$ ....

Unit Two

2 - 3 Solving exponential equations

Think and discuss

Amiba reproduce by binary fission where one cell is divided into two cells after a certain period of time, then every new cell is divided into two other cells at the same period of time in the same conditions and so on.



1 Find the number of cells resulted from one cell after nine periods of time.

2 Find the number of periods of time required to produce 8192 cells out of this cell.

Learn

Exponential function

if the equation includes one variable in the exponent, it's called an exponential function such as ($2^{x+1} = 8$).

Solving power Equation :

First: if $a^m = a^n$ where $a \notin \{0, 1, -1\}$, **then** $m = n$.

Example

1 Find the solution set for each of the following equations in R:

a $2^{x+3} = 8$ **b** $3^{x-2} = (\frac{1}{27})^x$

Solution

a $\because 2^{x+3} = 8$ $\therefore 2^{x+3} = 2^3$

$\therefore x + 3 = 3$ thus, $x =$ zero

$\therefore$ Solution set = {zero}

b $\because 3^{x-2} = (\frac{1}{27})^x$ $\therefore 3^{x-2} = 3^{-3x}$

$\therefore x - 2 = -3x$ $\therefore x + 3x = 2$

$\therefore 4x = 2$ thus $x = \frac{1}{2}$

$\therefore$ Solution set = $\{\frac{1}{2}\}$

You will learn

- Power function.
- Representing the power functions graphically.
- Properties of power function.

key terms

- Power function
- Graphical solution

Materials

- Scientific calculator
- Graphic programs

Try to solve

(1) Find the solution set for each of the following equations in R:

(a) $5^{x+1} = 25$ (b) $2^{1-x^2} = \frac{1}{8}$

second: **if** $a^m = b^m$ where $a, b \notin \{0, 1, -1\}$،

either: **1-** m = zero

or: **2-** a = b when **m** is an odd number.

3- $a = \pm b$ when **m** is an even number.

Example

(2) Find the solution set for each of the following equations in R:

(a) $3^{x+2} = 7^{x+2}$ (b) $4^{x-2} = 3^{2x-4}$

Solution

(a) $\because 3^{x+2} = 7^{x+2}$

$\therefore x + 2 =$ zero thus $x = -2$

$\therefore$ Solution set = {-2}

(b) $\because 4^{x-2} = 3^{2x-4}$ $\therefore 4^{x-2} = 3^{2(x-2)}$

$\therefore 4^{x-2} = 9^{x-2}$

$\therefore x - 2 =$ zero thus $x = 2$

$\therefore$ Solution set = {2}

Try to solve

(2) Find the solution set for each of the following equations in R:

(a) $5^{x-1} = 4^{x-1}$ (b) $2^{2x-6} = 7^{x-3}$

Example

(3) If $f(x) = 2^{x+1}$, find the value of x which satisfies $f(x) = 32$

Solution

$\because f(x) = 32$ $\therefore 2^{x+1} = 32$

$\therefore 2^{x+1} = 2^5$ $\therefore x + 1 = 5$

$\therefore x = 4$ $\therefore$ Solution set = {4}

Try to solve

(3) If $f(x) = 7^x$, find the value of x which satisfies $f(x + 1) = 49$

Exercises 2 - 3

1 **Complete:**

a) if $5^{x-2} = 1$ then x = ...

b) if $3^{x-2} = 7^{x-2}$ then x = ...

c) if $2^{x+1} = 5^{x+1}$ then 3^{x+1} = ...

d) if $2^{|x|} = 32$ then x = ...

Choose the correct answer

2 if $3^{x-5} = 9$, then x = ...

a) 2 b) 7 c) -3 d) -7

3 if $3^{x} = 9$, then 3^{x+1} = ...

a) 5 b) 15 c) 27 d) 45

4 The number $5^{x+1} + 5^{x}$ is divisible by .... for all the natural values of x.

a) 7 b) 6 c) 13 d) 17

5 if $(\frac{2}{3})^{x-2} = \frac{8}{28}$, then x = ...

a) 2 b) 3 c) 4 d) 5

6 Find the solution set for each of the following equations in R:

a) $3^{x+4} = 9$

b) $2^{x-5} = \frac{1}{32}$

c) $5^{x+2} = 1$

d) $3^{|x|} = 3$

e) $2 \times 3^{x-2} = 54$

f) $7^{x-5} = 3^{x-5}$

g) $2^{3x-6} = 5^{x-2}$

h) $(\frac{3}{2})^{x-2} = \frac{8}{27}$

i) $2^{x} \times 5^{-x} = \frac{4}{25}$

j) $4^{x} = 64$

k) $4^{1-x} = \frac{1}{4}$

l) $(3)^{x-5} = \frac{1}{9}$

7) If $f(x) = 2^x$, find the solution set for each of the following equations:

a) $f(x) = 8$ b) $f(x+1) = \frac{1}{32}$

8) If $f(x) = 3^{x+1}$, find the solution set for each of the following equations:

a) $f(x) = 27$ b) $f(x-1) = \frac{1}{9}$

9) If $f(x) = 7^{x-2}$, find the solution set for each of the following equations:

a) $f(x) = 343$ b) $f(2x) = \frac{1}{49}$

10) **Discover the error:** Mohammed and Karem have solved the equation $2 \times 2^x = 16$ as follows:

Mohammed's Solution

$2 \times 2^x = 16$

$\therefore 4^x = 16$

$\therefore 4^x = 4^2$

$\therefore x = 2$

Karem's solution

$2 \times 2^x = 16$

$\therefore 2^x = \frac{16}{2} = 8$

$\therefore 2^x = 2^3$

$\therefore x = 3$

What is the right solution? Why?

Logarithmic Function and its Graphical Representation

Unit Two

2 - 4

Think and discuss

Check the following exponential functions and try to solve each of them:
if $2^x = 2$, $2^y = 4$, $2^z = 3$, then:

1- $x =$ .... , $y =$ ....

2- The value of z is included between two consecutive integers which are .... and ....

Notice that the value of y cannot be calculated directly such as x and z, so we need to the concept of a new function to calculated the value of y.

Learn

Logarithmic Function

if x and a are two positive numbers where $a \neq 1$, then the logarithmic function $y = \log_a x$ is the inverse function of the exponential function $y = a^x$

Example: **if** $\log_2 32 = 5$, **then** $2^5 = 32$ and vice versa.

Verbal expression:

If point (c, d) ∈ exponential function $y = a^x$, then:

1- point (........, ...) ∈ function $y = \log_a x$.

2- The exponential form $a^c = d$ where $a \in R^+ - \{1\}$ is equivalent to the logarithmic form ....

Example

Converting into the logarithmic form.

(1) convert each of the following into the logarithmic form:

(a) $3^4 = 81$ (b) $25^{-\frac{1}{2}} = \frac{1}{5}$ (c) $10^{-2} = 0{,}01$

Solution

(a) $\log_3 81 = 4$ (b) $\log_{25} \frac{1}{5} = -\frac{1}{2}$ (c) $\log_{10} 0{,}01 = -2$

Verbal expression: Can we convert $(-2)^4 = 16$ into a logarithmic form? Explain.

You will learn

- Definition of the logarithmic function.
- Graphical representation of the logarithmic function.
- Converting from the exponential form into the logarithmic form and vice versa.
- Solving some simple logarithmic equation.

Key terms

- logarithm
- inverse function
- domain
- common logarithm

Materials

- Calculator.
- Computer.

Tip

$\log_a x = y$ is called the logarithmic form while $a^y = x$ is called the equivalent exponential form. Notice that (a) is a bositive base. If $(-3)^4 = 81$, then there is not a logarithmic form equivalent to it.

Try to solve

(1) Express each of the following in a logarithmic form:

(a) $10^3 = 1000$ (b) $8^{\frac{1}{3}} = 2$ (c) $b^x = y$ where $b \in R^+ - \{1\}$

Common Logarithm

it is the logarithm whose base is 10 and written without writing the base. I.e. $\log_{10} 7 = \log 7$ and $\log_{10} 127 = \log 127$. The button [log] in the calculator can be used to find the common logarithm of any number.

Example

(2) convert each of the following into the exponential form:

(a) $\log_2 32 = 5$ (b) $\log 1000 = 3$ (c) $\log_2 1 = \text{zero}$

Solution

(a) $2^5 = 32$ (b) $10^3 = 1000$ (c) $2^{\text{zero}} = 1$

Try to solve

(2) convert each of the following into the exponential form:

(a) $\log_{125} 25 = \frac{2}{3}$ (b) $\log 100 = 2$ (c) $\log_5 5 = 1$

Example Finding the values of logarithmic phrases

(3) Find the value of each:

(a) $\log_5 125$ (b) $\log 0{,}01$

Solution

(a) Let $\log_5 125 = x$ and by converting into the exponential form:

$\therefore 5^x = 125$ $\therefore 5^x = 5^3$ thus $x = 3$

$\therefore \log_5 125 = 3$

(b) Let $\text{Log}0.01 = y$ (a common logarithm whose base is 10) and by converting into the exponential form.

$\therefore 10^y = 0.01$ $\therefore 10^y = 10^{-2}$

thus $y = -2$ $\therefore \log 0.01 = -2$

Try to solve

3 Find the value of each:

a $\log_3 81$ b $\log_{\frac{1}{2}} 32$

Example Solving the equations

4 Find the solution set for each of the following equations in R:

a $\log_2 (x+5) = 3$ b $\log_5 625 = x - 1$ c $\log_x (x+6) = 2$

Solution

a The equation is defined for all the values of $x + 5 >$ zero I.e $x > -5$ (is the domain of defining the equation).

By converting the equation into the exponential form

$\therefore x + 5 = 2^3$ $\therefore x + 5 = 8$

thus $x = 3$

$\therefore 3 \in$ the domain of defining the equation $\therefore$ Solution set = {3}

b The equation is defined for all real values of x and by converting the equation into the exponential form .

$\therefore 5^{x-1} = 625$ $\therefore 5^{x-1} = 5^4$

$\therefore x - 1 = 4$ thus $x = 5$

$\therefore$ Solution set = **{5}**

c The equation is defined for all the values of x which satisfy each of $\begin{cases} x + 6 > \text{zero} \\ x > \text{zero} \\ x \neq 1 \end{cases}$

I.e the domain of defining the equation is]zero , ∞ [- {1}

and by converting the equation into the exponential form:

$x^2 = x + 6$ $x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

either $x = 3$ or $x = -2$

Since $x = -2 \notin$ the domain of defining the equation

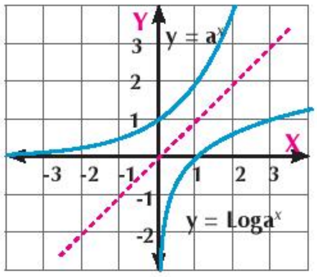
$\therefore$ Solution set = **{3}**

Try to solve

4 Find the solution set for each of the following equations in R:

a $\log_5 (3x-1) = 1$ b $\log_3 27 = x + 2$ c $\log_{(x-1)} 9 = 2$

Graphical Representation of the Logarithmic Function



If $f(x) = a^x$ where $a \in R^+ -\{1\}$, then the inverse function of the function f is called the logarithmic function. I.e. $y = \log_a x$

Relation between the exponential and logarithmic function

The opposite figure represents the exponential function $y = a^x$ and logarithmic function $y = \log_a x$. Study the properties of both functions for domain , range, monotony and symmetry around the straight line $y = x$ in one figure.

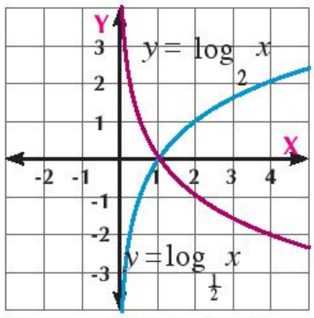
Example

1 Graph the curve of each of the two functions $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ in one figure.

Solution

we choose values of x the powers of the number 2 (the base) $\{2^{-2}, 2^{-1}, 2^0, 2^1, 2^2\}$

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$	-2	-1	zero	1	2
$\log_{\frac{1}{2}} x$	2	1	zero	-1	-2



From the graph, you can deduce the following properties for the curve of the logarithmic function:

the domain = R+ , and the range = R

the function $y = \log_a x$ is increasing for each $a > 1$ and is decreasing for each $0 < a < 1$

Exercise 2 - 4

1 **Complete:**

a) The exponential form equivalent to the form $\log_3 27 = 3$ is ....

b) The logarithmic form equivalent to the form $3^{\text{zero}} = 1$ is ....

c) $\log 0{,}001 =$ ....

d) $\log_2 1 =$ ....

e) If $\log_x 4 = 2$, then $x =$ ....

f) if $\log_2 128 = x + 1$, then $x =$ ....

g) The domain of the function f: $f(x) = \log_2 x$ is..........

h) The function f where $f(x) = \log_a x$ is decreasing for each a $\in$ ....

i) The curve of the function f where $f(x) = \log_2 x$ passes through point (8, ...)

j) If Log3 = x and Log5 =y, then log 15 = ... (in terms of x,y)

2 Find the solution set for each of the following equations in R:

a) $\log_3 (x - 1) = 2$

b) $\log_5 (x + 2) = 3$

c) $\log_x 9 = \frac{2}{3}$

d) $\log_{x+1} 8 = \frac{3}{4}$

e) $\log_x (x + 2) = 2$

f) $\log_x 9 = 2$

3 Find the value of the following without using the calculator.

a) $\log_5 1$

b) $\log_7 7$

c) $\log_3 9$

d) $\log_6 3 + \log_6 2$

4 use the calculator to find the value of the following:

a) $\log 15$

b) $\log_2 27$

c) $4\log 7 - 5\log 13$

Unit Two

2 - 5 Some Properties of Logarithms

You will learn

- using some properties of logarithms.
- Solving logarithmic equations.
- Using the calculator to solve the exponential functions.
- Life applications on logarithms.

Key terms

- logarithmic equation

Materials

- Scientific calculator
- Computer with graphic programs

In the previous lesson, you learned the concept of a logarithm and how to represent the logarithmic function graphically. Now, we are going to list some properties of logarithms to help you simplify the logarithmic expressions or solve the equations containing a logarithm.

Learn

Some Properties of Logarithms

if $a \in R^+ - \{1\}$, x , $y \in R^+$, then

1- $\log_a a = 1$

For example, $\log_3 3 = 1$, $\log 10 = 1$

2- $\log_a 1 = \text{zero}$

For example, $\log_5 1 = \text{zero}$, $\log 1 = \text{zero}$

Try to prove **1** and **2** from the difinition of logarithms

3- Multiplication property in logarithms:

$\log_a xy = \log_a x + \log_a y$ where x and $y \in R^+$

To prove the correctness of this property:

place $b = \log_a x$ and $c = \log_a y$

From the difinition of logarithms, we find that:

$x = a^b$, $y = a^c$

then $xy = a^b \times a^c$ **I.e** $xy = a^{b+c}$

By converting this form into the logarithmic form, then:

$\log_a xy = b + c$

By substituting the two values of b and c, **then** $\log_a xy = \log_a x + \log_a y$

Example

1. Find the value of $\log_{34} 2 + \log_{34} 17$ without using the calculator.

Solution

the expression= $\log_{34}(2 \times 17)$ **use property (3)**

$= \log_{34} 34$

$= 1$ **use property (1)**

4- Division property in logarithms:

$\log_a \frac{x}{y} = \log_a x - \log_a y$ (try to prove the correctness of this relation yourself)

Example

(2) Find the value of log 50 - log 5 without using the calculator.

Solution

the expression = $\log \frac{50}{5}$ **use the division property**

$= \log 10 = 1$ **use property (1)**

Try to solve

(1) Find the value of $\log_2 7 - \log_2 3{,}5$ without using the calculator.

5- Property of power logarithm:

$\log_a x^n = n \log_a x$ where $x > 0$ (try to prove the correctness of the relation yourself)

Example

(3) Find the value of $\log_5 125$ without using the calculator.

Solution

the expression= $\log_5 5^3$

$= 3 \log_5 5$ **use the property of power**

$= 3 \times 1 = 3$ **use property (1)**

Notice that: $\log_a (\frac{1}{x}) = - \log_a x$ where $x \in R^+$

6 - Property of changing the base

$\log_y x = \frac{\log_a x}{\log_a y}$ **and proving the correctness of this property**

By placing: $z = \log_y x$

$y^z = x$ **by converting into the exponential form**

$z \log_a y = \log_a x$ **the logarithm of the two sides is taken off for the base a**

then $z = \dfrac{\log_a x}{\log_a y}$ **I.e:** $\log_y x = \dfrac{\log_a x}{\log_a y}$

Example

(4) **Reduce to the simplest form** $\log_7 16 \times \log_2 49$

Solution

the expression $= \dfrac{\log 16}{\log 7} \times \dfrac{\log 49}{\log 2}$ **use property (6)**

$= \dfrac{\log 2^4}{\log 7} \times \dfrac{\log 7^2}{\log 2}$

$= \dfrac{4 \log 2}{\log 7} \times \dfrac{2 \log 7}{\log 2}$ **use property (5)**

$= 4 \times 2 = 8$

Try to solve

(2) Find the solution of the example above by changing the base into another number but not 10.

7 - Property of the multiplicative inverse.

$\log_b a = \dfrac{1}{\log_a b}$ **I.e** both $\log_b a$ and $\log_a b$ are multiplicative inverse of each other (try to prove the correctness of this relation).

Example

(5) Find the value of $\dfrac{1}{\log_3 15} + \dfrac{1}{\log_5 15}$ without using the calculator.

Solution

the expression $= \log_{15} 3 + \log_{15} 5$ **use property (7)**

$= \log_{15} (3 \times 5)$ **use property (3)**

$= \log_{15} 15 = 1$ **use property (1)**

Try to solve

(3) Find the value of $\dfrac{1}{\log_2 30} + \dfrac{1}{\log_3 30} + \dfrac{1}{\log_5 30}$ without using the calculator.

Simplifying the Logaritmic Experssions

Example

(6) Reduce $\log 0{,}009 - \log\frac{27}{16} + 3\log\frac{5}{2} - \log\frac{1}{12}$ to the simplest form

Solution

the expression $= \log\frac{9}{1000} - \log\frac{27}{16} + \log(\frac{5}{2})^3 - \log\frac{1}{12}$ **property (5)**

$= \log(\frac{9}{1000} \times \frac{16}{27} \times \frac{125}{8} \times \frac{12}{1})$ **property (3), (4)**

$= \log 1 = \text{zero}$ **property (2)**

Try to solve

(4) Reduce $4\log\sqrt{3} - \log\frac{2}{5}1 - \log\frac{9}{7} - \log\frac{1}{2}$ to the simplest form.

Solving Logarithmic Equations

Example

(7) Find the solution set for each of the following equations in R.

(a) $\log_2 x + \log_2 (x + 1) = 1$ (b) $\log_2 x + \log_4 x = 3$

Solution

(a) The function is defined for each $x >$ zero and $x + 1 >$ zero

I.e. $x >$ zero **(domain of defining the equation)**

$\therefore \log_2 x(x + 1) = 1$ **use property (3)**

$\therefore x(x+1) = 2^1$ **converting from the logarithmic form into the exponential form**

$\therefore x^2 + x - 2 = \text{zero}$ $\therefore (x + 2)(x - 1) = \text{zero}$

Either $x = -2$ or $x = 1$, and $x = -2 \notin$ the domain of defining the equation

$\therefore$ Solution set $= \{1\}$

(b) The function is defined for each $x >$ zero **(domain of defining the equation)**

$\therefore \log_2 x + \frac{\log_2 x}{\log_2 4} = 3$ **property (6)**

$\log_2 x + \frac{\log_2 x}{2} = 3$ **multiply by 2**

$\therefore 2\log_2 x + \log_2 x = 6$ $\therefore 3\log_2 x = 6$ $\therefore \log_2 x = 2$

$\therefore x = 4$ **(converting from the logarithmic form into the exponential form)**

where $x = 4 \in$ the domain of defining the equation $\therefore$ Solution set $= \{4\}$

Try to solve

(8) Find the solution set for each of the following equations in R:

(a) $\log(2x + 1) - \log(3x - 1) = 1$ (b) $\log_2 x = \log_x 2$

Solving the Exponential functions by Using Logarithms

Example

(9) Find the solution set for each of the following equations in R rounding the sum to the nearest two decimals :

(a) $2x^{=7}$ (b) $3\,x^{+1} = 5x^{-2}$

Solution

(a) $2x^{=7}$ **take off the logarithm of two sides**

$\therefore \log 2^x = \log 7$ $\therefore\ x\log 2 = \log 7$ $\therefore\ x = \frac{\log 7}{\log 2}$

use the calculator respectively as follows:

⊞ log 7 (▽ log 2 (= 2.807354922

$\therefore x \simeq 2{,}81$ $\therefore$ The solution set = $\{2.81\}$

(check your answer using the calculator) 2 $x^{\blacksquare}$ ans = 7

Try to solve

(5) Find the solution set of each of the equations to the nearest two decimals:

(a) $7^x = 2$

Exercises 2 - 5

choose the correct answer:

(1) $\log_2 8 =$

(a) 4 (b) 3 (c) 16 (d) 10

(2) $\text{Log}2 + \text{Log}5 =$

(a) 1 (b) log 7 (c) Log2,5 (d) 10

(3) $\log_5 \sqrt{5} =$

(a) 2 (b) 5 (c) $\frac{1}{2}$ (d) -1

(4) if $\log 3 = x$, $\log 4 = y$, then $\log 12 =$

(a) $x + y$ (b) $x\,y$ (c) $x - y$ (d) $\log x + \log y$

(5) $2\log_6 2 + 2\log_6 3 =$

(a) 6 (b) 36 (c) 2 (d) 12

(6) $\log_2 5 \times \log_5 2 =$

(a) 1 (b) 10 (c) $\frac{5}{2}$ (d) zero

7) $\log_5 2 \times \log_3 5 \times \log_2 3 =$

a) 30 b) 1 c) zero d) log 30

8) Express each of the following in terms of log x and log $(x + 1)$

a) $\log x\,(x+1)$ b) $\log \frac{x}{x+1}$ c) $\log\sqrt{x}\ (x+1)^2$

9) Reduce to the simplest form:

a) $\log_6 54 - \log_6 9$ b) $\log_6 2 + \log_6 3$ c) $\log_2 12 + \log_2 \frac{2}{3}$

d) $\log 48 + \text{Log}125 - \log 6$ e) $\frac{1 - \text{Log}2}{\text{Log}125}$ f) $\frac{\text{Log}49 + 3\log 7}{\log 7}$

g) $\log_2 16 + \log_3 \sqrt{3} + \log 0{,}1$

h) $\frac{1}{2}\log_3 a + \frac{1}{2}\log_3 b + 2\log_3 c - \log_3 \sqrt{ab} - \log_3 3c^2$

10) Find the solution set for each of the following equations in R:

a) $\log_2 x + \log_2 (x+2) = 3$ b) $\log x + \log (x-3) = 1$ c) $\log_5 x - \log_5 2 = 2$

d) $\log (x+3) - \log 3 = \log x$ e) $\frac{1}{\log_2 x} + \frac{1}{\log_3 x} = 2$ f) $\log x - \frac{3}{\text{Log}x} = 2$

11) Prove that $\log_b a \times \log_c b \times \log_d c \times \log_a d = 1$, then calculate the value of $\log_2 3 \times \log_3 5 \times \log_5 16$

12) Find the value of x in each of the following, then round the sum to the nearest decimal.

a) $3^x = 7$ b) $5^{x-1} = 2$ c) $4 \times 7^{x-2} = 1$ d) $2^{x-3} = 3^{x+1}$

Unit Three

Limits

Unit introduction

calculus is one of the modern branches of mathematics concerning with the studying of limits, continuity, differentation, integration and the infinte series. It is the science used to study the variation in the functions and their factorization.

Calculus deals with numerous geometrical life application, trade and different science. Calculus is basically used to study the behaviour of the function and the change in it and to solve the problems which algebra and other sciences cannot deal with.

Unit objectives

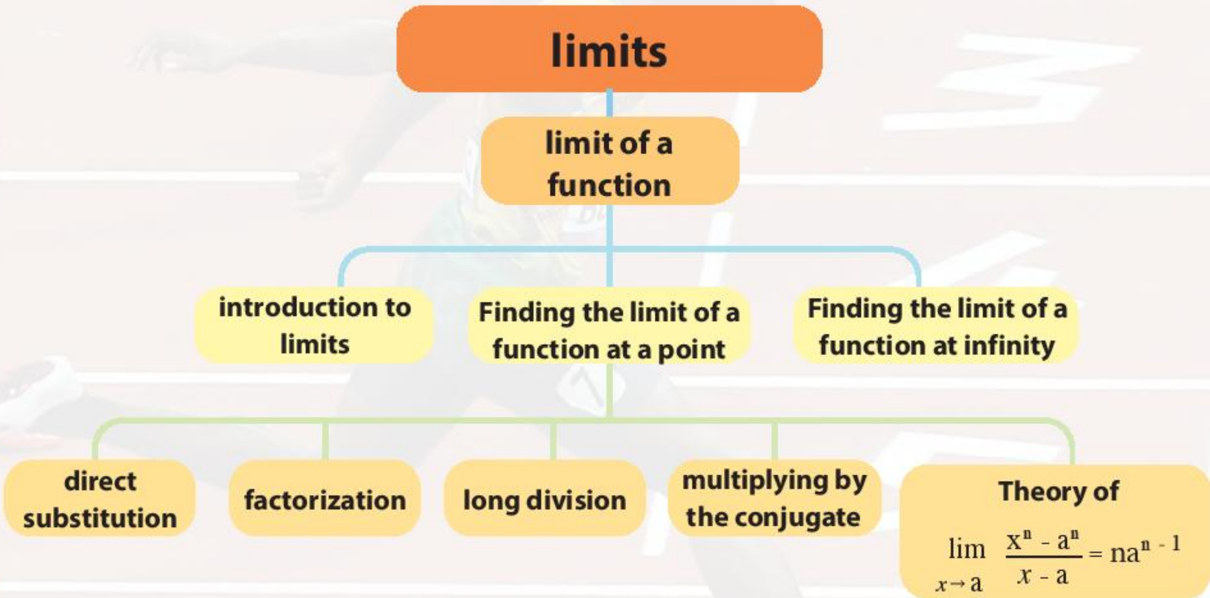
By the end of this unit, the student should be able to:

- Identify an introduction about limits.
- Identify some unspecified quantities like: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$
- Determine a method to find the limit of a function, by direct substitution, factorization, long division, and multiply by conjugates:
- Find the limit of a function using the rule $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = n\, a^{n-1}$
- Conclude the limit of a function using the rule: $\lim\limits_{x \to a} \frac{x^n - a^n}{x^m - a^m} = \frac{n}{m}\, a^{n-m}$
- Find the limit of a function at the infinity algebraically and graphically.
- Use the graphic calculators to check the limit of the function and the value of the limit.
- Identify various applications on the basic concepts of the function limits.

Key terms

- Unspecified Quantity
- Undefined
- Limit of a Function
- direct Substitution
- Conjugate
- Polynomial Function
- Limit of a Function at Infinity

Unit planning guide



Unit Lessons

lesson (3-1): Introduction to limits .

lesson (3-2): Finding the limit of a function algebraically.

lesson (3-3): Limit of a function at infinity.

Materials

Scientific calculator - Computer - Graphical programs for PC

Unit Three

3 - 1 Introduction to Limits of Functions

You will learn

- Unspecified quantities
- Limit of a function at a point.

Key terms

- Unspecified quantity
- Undefined
- Value of a function
- Limit of a function

Matrials

- Scientific calculator.
- Graphical programs for computer

The concept of the function limit at a point is one of the basic concepts in calculus. This concept basicly depends on the behavior of the function at all of its definition points. To study such a behavior, we should identify the types of quantities in the set of real number.

Think and discuss

Find the sum of the following operation, if possible:

1. 3×5
2. $28 \div 4$
3. $4 - 9$
4. $7 \div 0$
5. $0 \div 0$
6. $\infty + 3$
7. $\infty \div \infty$
8. $\infty - \infty$

Remember

∞ is a symbol that indicates an unspecified quantity greater than any real number that can be imagined.

Unspecified Quantities

Learn

In **Think and discuss** above ,we found that some sums of the operations are identified completely such as 1, 2, 3 while others can't be identified such as other operation.

Notice: $7 \div 0$ is undefined where the division by zero does not make a sense.

Now, you can't determine the sum of the operation $0 \div 0$ where there are an infinite numbers of numbers if multiplied by zero, then the product will be zero. Therefore, $\frac{0}{0}$ is unspecified quantity:

$\frac{\infty}{\infty}$, $\infty - \infty$ and $0 \times \infty$ are from the unspecified quantities. **(why?)**

Add to your information

Mathematical operations are performed on the set of real numbers and the two symbols ∞ and $-\infty$ as follows:

1. $\infty + a = \infty$
2. $-\infty + a = -\infty$
3. $\frac{a}{\infty} = \frac{a}{-\infty} = 0$
4. $\infty \times a = \begin{cases} -\infty, & \text{if } a < 0 \\ \infty, & \text{if } a > 0 \end{cases}$
5. $-\infty \times a = \begin{cases} -\infty, & \text{if } a > 0 \\ \infty, & \text{if } a < 0 \end{cases}$

Example

1. Find the sums of the following operations (if possible):

a) $4 + \infty$ b) $3 - \infty$ c) $0 \div 3$ d) $-5 \div 0$

e) $\infty + \infty$ f) $0 \div 0$ g) $5 \times \infty$ h) $-6 \times -\infty$

Solution

a) ∞ b) $-\infty$ c) 0 d) undefined

e) 0 f) unspecified quantity g) ∞ h) ∞

Try to solve

1. Find the results of the following operations (if possible):

a) $0 \div (-2)$ b) $7 \div 0$ c) $9 \div \infty$ d) $\infty \times 0$

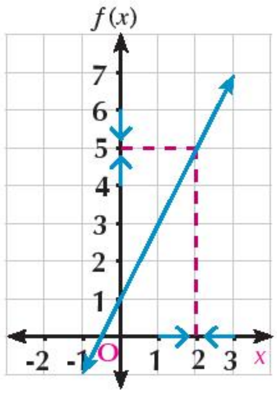
e) $(-7) \times \infty$ f) $(-\infty) + 12$ g) $\infty + \infty$ h) $\infty \div \infty$

The limit of function at a point :

In the following figure: the graphical line of the function f defined in R according **to the rule** $f(x) = 2x + 1$. Complete the following tables , then answer the following questions:

x	$f(x)$
2.1	5.2
2.01	5.02
2.001	5.002
2.0001	5.0002
....	....
↓	↓
▯	▯
$x > 2$ x gets near to 2 in the right direction	

x	$f(x)$
1.9	4.8
1.99	4.98
1.999	4.998
1.9999	4.9998
....	....
↓	↓
▯	▯
$x < 2$ x gets near to 2 in the left direction	



Note:

- What is the value which $f(x)$ gets near to when x gets near to 2 in the right direction?
- What is the value which $f(x)$ gets near to when x gets near to 2 in the left direction?

When x gets near to the number (2) from right and left, $f(x)$ gets near to the number (5). We express that mathematically as follows: $\lim_{x \to 2} (2x + 1) = 5$

Definition 1

If the value of the function f gets near to a unique value L when x gets near to a form left side and right side, then the limit of $f(x)$ equals L. It is **written symbolically** as: $\lim\limits_{x \leftarrow a} f(x) = L$

and read as: Limit of $f(x)$ when x gets near to a equals L

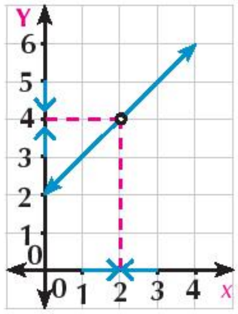
Example

(2) If $f(x) = \frac{x^2 - 4}{x - 2}$, study the values of $f(x)$ when x gets near to 2.

Solution

x	$f(x)$
2,1	4,1
2,01	4,01
2,001	4,001
....	....
↓	↓
2	4
$x > 2$	

x	$f(x)$
1,9	3,9
1,99	3,99
1,999	3,999
....	....
↓	↓
2	4
$x < 2$	



From the graph and data shown in the table above, we find that $f(x) \longrightarrow 4$ when $x \longrightarrow 2$ from the right and left directions $\therefore \lim\limits_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$

Notice:

1- The hole in the graph illustrates a case of unspecified cases of $\frac{0}{0}$ when $x = 2$ (i.e. the function is undefined when $x = 2$)

2- The presence of a limit to the function when $x \longrightarrow 2$ does not necessarily mean the function is defined when $x = 2$.

Try to solve

(2) If $f(x) = \frac{x^2 - 1}{x + 1}$, then study the values of $f(x)$ when x gets near to (- 1)

Example

(3) Find the $\lim\limits_{x \to 3} f(x)$ in each of the following figures:

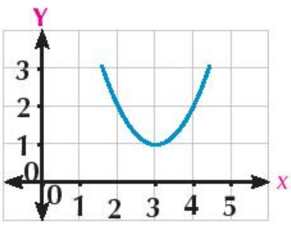


fig (1)

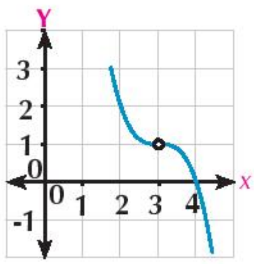


fig (2)

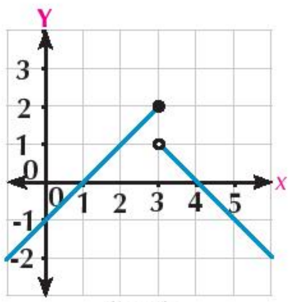


fig (3)

Solution

Fig (1) $\lim\limits_{x \to 3} f(x) = 1$

Fig (2) $\lim\limits_{x \to 3} f(x) = 1$ (Notice that the function is not defined when $x = 3$)

Fig (3) $\lim\limits_{x \to 3} f(x)$ is not existed

Try to solve

3) Find the $\lim\limits_{x \to 1} f(x)$ in each of the following figures:

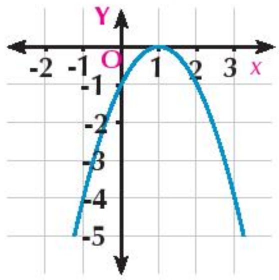


Fig (1)

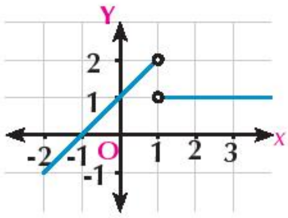


Fig (2)

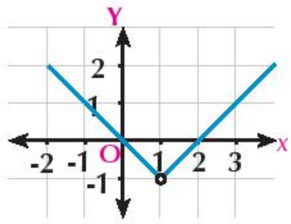


Fig (3)

From previous examples, we conclude that:

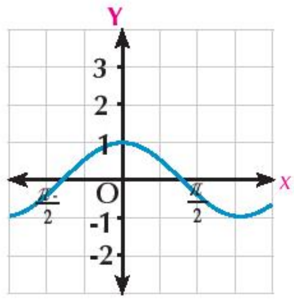
The presence of a limit to the function when $x \longrightarrow$ a does not necessarily mean the function is defined when x = a and vice versa if the function is defined when x = a, that does not necessarily mean the function has a limit when x = a.

Verbal expression: express by your manner the difference between the value of a function at a point and limit of that function at the same point.

Exercises (3 - 1)

First: exercises on finding the limit graphically:

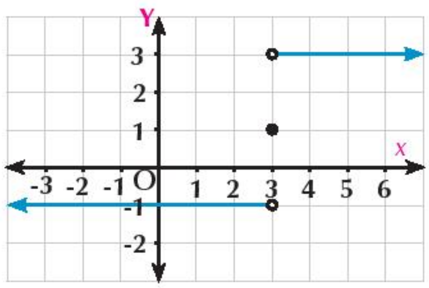
1) From the graph, find:



a) $\lim\limits_{x \to 0} f(x)$

b) $f(0)$

2) From the opposite graph, find the following if possible:



a) $\lim\limits_{x \to 3} f(x)$

b) $f(3)$

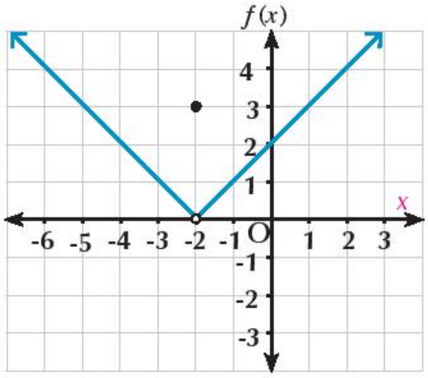
3 From the opposite graph, find:

a $\lim_{x \to -2} f(x)$

b $f(-2)$

c $\lim_{x \to 0} f(x)$

d $f(0)$

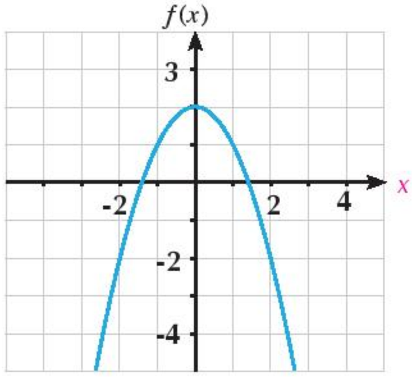


4 The opposite graph illustrates the function $f(x) = 2 - x^2$

Find:

a $\lim_{x \to 0} (2 - x^2)$

b $f(0)$



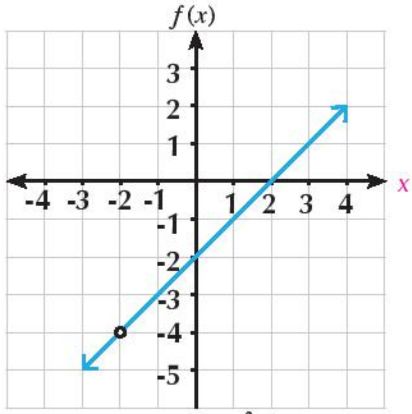
$f(x) = 2 - x^2$

5 The opposite graph illustrates the function $f(x) = \dfrac{x^2 - 4}{x + 2}$

Find:

a $\lim_{x \to -2} f(x)$

b $f(-2)$



$f(x) = \dfrac{x^2 - 4}{x + 2}$

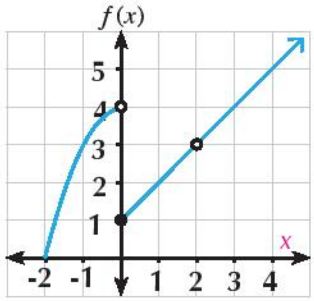
6 From the opposite graph, find:

a $f(0)$

b $\lim_{x \to 0} f(x)$

c $f(2)$

d $\lim_{x \to 2} f(x)$



Second: Finding the limit of a function algebraically:

7) Complete the following table and conclude $\lim\limits_{x \to 2} f(x)$ where $f(x) = 5x + 4$

x	1,9	1,99	1,999	⟶	2	⟵	2,001	2,01	2,1
$f(x)$				⟶	▯	⟵			

8) Complete the following table and conclude $\lim\limits_{x \to -1} (3x + 1)$

x	– 0,9	– 0,99	– 0,999	⟶	– 1	⟵	– 1,001	– 1,01	– 1,1
$f(x)$				⟶	؟	⟵			

9) Complete the following table and conclude $\lim\limits_{x \to -1} \frac{x^2 - 1}{x + 1}$

x	– 0,9	– 0,99	– 0,999	⟶	– 1	⟵	– 1,001	– 1,01	– 1,1
$f(x)$				⟶	؟	⟵			

10) Complete the following table and conclude $\lim\limits_{x \to 2} \frac{x - 2}{x^2 - 4}$

x	1,9	1,99	1,999	⟶	2	⟵	2,001	2,01	2,1
$f(x)$				⟶	؟	⟵			

Unit Three

3 - 2 Finding the Limit of a Function Algebraically

You will learn

- Limit of a polynomial function.
- Some theories of limits.
- Using long division in finding the value of the function limit.
- Using theory of: $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim\limits_{x \to a} \frac{x^n - a^n}{x^m - a^n} = \frac{n}{m} a^{n-m}$

Key terms

- Limit of a function
- Polynomial function
- direct substitution
- synthetic division
- conjugate

Matrials

- Scientific calculators.
- Graphic program for Computer

In this lesson, we are going to identify some methods and theories to help us calculate the limit of a function at a point directly without a need for establishing tables and find the limit numerically or graph the curve to find the limit graphically.

Activity

If $f_1(x) = x^2 + 1$ and $f_2(x) = 2x + 3$, find:

1 - $f_1(1)$ and $\lim\limits_{x \to 1} f_1(x)$ (what do you notice?)

2 - $f_2(0)$ and $\lim\limits_{x \to 0} f_2(x)$ (what do you notice?)

Learn

Limit of a Polynomial Function

theory 1

➤ **If** $f(x)$ is a polynomial function and $a \in R$

then: $\lim\limits_{x \to a} f(x) = f(a)$

Example

1 Find the limit for each of the following functions:

a $\lim\limits_{x \to 2} (x^2 - 3x + 5)$ **b** $\lim\limits_{x \to 3} (-4)$

Solution

a $\lim\limits_{x \to 2} (x^2 - 3x + 5)$

$= 4 - 6 + 5 = 3$ **(direct substitution)**

b $\lim\limits_{x \to 3} (-4) = -4$ **notice that** $f(x) = -4$ is constant for all the values of $x \in R$

Try to solve

1 Find the limit for each of the following functions:

a $\lim\limits_{x \to 1} (2x - 5)$ **b** $\lim\limits_{x \to -2} (3x^2 + x - 4)$

Theory 2

If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} X(x) = m$

then:

1- $\lim_{x \to a} K f(x) = K.L$ where $K \in R$

2- $\lim_{x \to a} [f(x) \pm X(x)] = l \pm m$

3- $\lim_{x \to a} f(x) . X(x) = L.M$

4- $\lim_{x \to a} \frac{f(x)}{X(x)} = \frac{L}{M}$ if $M \neq 0$

5- $\lim_{x \to a} (f(x))^n = L^n$ where $L^n \in R$

Example

(2) Find each of the following limits:

a $\lim_{x \to -1} \frac{3x + 7}{x^2 + 2x - 5}$

b $\lim_{x \to \frac{\pi}{4}} \frac{\tan x}{x}$

Solution

a $\lim_{x \to -1} \frac{3x + 7}{x^2 + 2x - 5} = \frac{\lim_{x \to -1} (3x + 7)}{\lim_{x \to -1} (x^2 + 2x - 5)} = \frac{3 \times -1 + 7}{(-1)^2 + 2(-1) - 5} = \frac{4}{-6} = \frac{-2}{3}$

b $\lim_{x \to \frac{\pi}{4}} \frac{\tan x}{x} = \frac{\lim_{x \to \frac{\pi}{4}} \tan x}{\lim_{x \to \frac{\pi}{4}} x} = \frac{1}{\frac{\pi}{4}} = \frac{4}{\pi}$

Try to solve

(2) Calculate the following limits:

a $\lim_{x \to 2} \frac{x^2 - 3}{2x + 1}$

b $\lim_{x \to \pi} x \cos x$

Theory 3

➢ **If** $f(x) = X(x)$ $x \in \mathbb{R} - \{ a \}$

and $\lim_{x \to a} X(x) = l$ **then** $\lim_{x \to a} f(x) = l$

Example

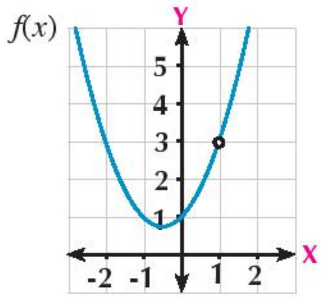
(3) Find: $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

Solution

We notice that $f(x) = \frac{x^3 - 1}{x - 1}$ is unspecified when $x = 1$

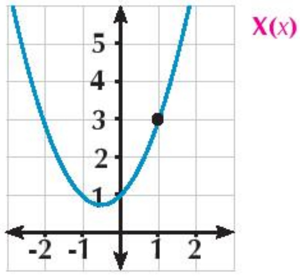
by factorizing and dividing by similar non-zero factors, then

$f(x)$ can be written as:



$f(x) = \frac{(x-1)(x^2+x+1)}{(x-1)} = x^2+x+1$

$= k(x)$



So far, we find that $f(x) = k(x)$ for each $x \neq 1$

Since $\lim\limits_{x\to 1} k(x) = 3$ (polynomial)

According to the previous theory, we conclude that $\lim\limits_{x\to 1} f(x) = 3$

$\therefore \lim\limits_{x\to 1} \frac{x^3-1}{x-1} = 3$

Example

Using conjugate:

(4) Find the following limits:

a) $\lim\limits_{x\to 4} \frac{\sqrt{x-3}-1}{x-4}$

b) $\lim\limits_{x\to 5} \frac{x^2-5x}{\sqrt{x+4}-3}$

Solution

Notice that: $f(x) = \frac{\sqrt{x-3}-1}{x-4}$ is unspecified when $x = 4$

Therefore, we search for a method to get rid of the factor $(x - 4)$ in both the numerator and the denominator.

$$\lim_{x\to 4} \frac{\sqrt{x-3}-1}{x-4} \times \frac{\sqrt{x-3}+1}{\sqrt{x-3}+1} = \lim_{x\to 4} \frac{x-3-1}{(x-4)(\sqrt{x-3}+1)}$$

$$= \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x-3}+1)}$$

$$= \lim_{x\to 4} \frac{1}{\sqrt{x-3}+1} = \frac{1}{2}$$

b) $$\lim_{x\to 5} \frac{x^2-5x}{\sqrt{x+4}-3} = \lim_{x\to 5} \frac{x^2-5x}{\sqrt{x+4}-3} \times \frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}$$

$$= \lim_{x\to 5} \frac{x(x-5)(\sqrt{x+4}+3)}{x+4-9} = \lim_{x\to 5} \frac{x(x-5)(\sqrt{x+4}+3)}{(x-5)}$$

$$= \lim_{x\to 5} x(\sqrt{x+4}+3) = 5(3+3) = 30$$

Try to solve

(3) Find the following limits:

a) $\lim\limits_{x\to 5} \frac{\sqrt{x-1}-2}{x-5}$

b) $\lim\limits_{x\to -1} \frac{x+1}{\sqrt{x+5}-2}$

Theory 4

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n\,a^{n-1}$$

Example

5 $\lim_{x \to 1} \frac{x^{19} - 1}{x - 1} = \lim_{x \to 1} \frac{x^{19} - 1^{19}}{x - 1} = 19 \times 1^{18} = 19$

Corollaries

Corollaries on the theory:

1- $\lim_{x \to 0} \frac{(x + a)^n - a^n}{x} = n\,a^{n-1}$

2- $\lim_{x \to a} \frac{x^n - a^n}{x^m - a^m} = \frac{n}{m} a^{n-m}$

Example

6 Find:

a $\lim_{x \to 0} \frac{(x + 5)^4 - 625}{x}$

b $\lim_{x \to 2} \frac{x^5 - 32}{x^2 - 4}$

c $\lim_{x \to 0} \frac{(x + 1)^{11} - 1}{x}$

d $\lim_{x \to 2} \frac{(x - 4)^5 + 32}{x - 2}$

Solution

a $= \lim_{x \to 0} \frac{(x + 5)^4 - 5^4}{x} = 4 \times 5^3 = 500$

b $= \lim_{x \to 2} \frac{x^5 - 2^5}{x^2 - 2^2} = \frac{5}{2} \times 2^3 = 20$

c $\lim_{x \to 0} \frac{(x + 1)^{11} - 1^{11}}{x} = 11 \times 1^{11-1} = 11$

d $\lim_{x \to 2} \frac{(x - 4)^5 + 32}{x - 2} = \lim_{x \to 2} \frac{(x - 4)^5 - (-2)^5}{(x - 4) - (-2)}$

$= 5\,(-2)^4 = 80$

Try to solve

4 Find:

a $\lim_{x \to -5} \frac{x^4 - 625}{x + 5}$

b $\lim_{h \to 0} \frac{(h + 3)^4 - 81}{h}$

c $\lim_{x \to 0} \frac{\sqrt[3]{x + 1} - 1}{x}$

Exercises (3 - 2)

Complete the following:

1. $\lim_{x \to 2} (3x + 1) =$ ....

2. $\lim_{x \to 1} \frac{x - 1}{x + 1} =$ ....

3. $\lim_{x \to 0} \frac{x^2 - x}{x} =$ ....

4. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} =$ ....

5. $\lim_{x \to a} \frac{x^5 - a^5}{x - a} =$ ....

6. $\lim_{x \to 2} \frac{x^3 - 8}{x - 2} =$ ....

7. $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} =$ ....

8. $\lim_{x \to 2} \frac{x^4 - 16}{x - 2} =$ ....

9. $\lim_{x \to 2} \frac{x^5 - 32}{x^3 - 8} =$ ....

10. $\lim_{x \to 1} (\frac{x^2 - 1}{x - 1})^5 =$ ....

11. $\lim_{x \to 2} \frac{x^2 - x - 2}{2x - 4} =$ ....

12. $\lim_{x \to -1} \frac{x^7 + 1}{x^5 + 1} =$ ....

Choose the right answer:

13. $\lim_{x \to 0} \frac{x^2 - 1}{x}$ equals:

 a) 0 b) 1
 c) 2 d) has no limit

14. $\lim_{x \to -1} \frac{x^2 + x}{x + 1}$ equals:

 a) -1 b) 0
 c) 1 d) 3

15. $\lim_{x \to 2} \frac{2x^2 - 8}{x - 2}$ equals:

 a) 2 b) 4
 c) 6 d) 8

16. $\lim_{x \to \frac{\pi}{2}} \frac{\sin x}{x}$ equals:

 a) 1 b) $\frac{\pi}{2}$
 c) $\frac{2}{\pi}$ d) has no limit

17. $\lim_{x \to \frac{\pi}{4}} \frac{\tan x}{x}$ equals:

 a) $\frac{\pi}{2}$ b) 1
 c) $\frac{4}{\pi}$ d) has no limit

Find the value for each of the following limits (if found)

18) $\lim_{x \to 3} (x^2 - 3x + 2)$

19) $\lim_{x \to -2} \frac{x^2 + 1}{x - 3}$

20) $\lim_{x \to \frac{\pi}{2}} (2x - \text{Sin } x)$

21) $\lim_{x \to \pi} \frac{\text{Cos } 2x}{x}$

22) $\lim_{x \to -1} \frac{x + 1}{x^3 + 1}$

23) $\lim_{x \to 9} \frac{9 - x}{x^2 - 81}$

24) $\lim_{x \to 4} \frac{x^2 + 4}{x - 4}$

25) $\lim_{x \to -1} \frac{x^2 - 1}{x^2 + x}$

26) $\lim_{x \to 4} \frac{4x^2 - 64}{x - 4}$

27) $\lim_{x \to 5} \frac{x^2 - 25x}{x - 5}$

28) $\lim_{x \to -1} \frac{x^2 - x - 2}{x + 1}$

29) $\lim_{x \to -1} \frac{5x^2 + 5}{3x^2 - 3}$

30) $\lim_{x \to -2} \frac{x^3 + 8}{x^2 - 4}$

31) $\lim_{x \to \frac{3}{2}} \frac{2x^2 - x - 3}{4x^2 - 9}$

32) $\lim_{x \to 2} \frac{2x^2 - x - 6}{x^2 - x - 2}$

33) $\lim_{x \to -3} \frac{2x^2 + 5x - 3}{x^2 + x - 6}$

34) $\lim_{x \to 0} \frac{(2x - 1)^2 - 1}{5x}$

35) $\lim_{x \to 0} \frac{(x + 1)^3 - 1}{x}$

36) $\lim_{x \to 1} \left(\frac{x^2}{x + 1} - \frac{3x + 4}{x + 1}\right)$

37) $\lim_{x \to 1} \frac{x^3 + x^2 - 2}{x - 1}$

38) $\lim_{x \to 1} \frac{x^3 - 5x^2 - x}{x^4 + 2x}$

39) $\lim_{x \to 1} \frac{x^3 + x - 2}{x^2 - 1}$

40) $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$

41) $\lim_{x \to 3} \frac{\sqrt{4x - 3} - 3}{x - 3}$

42) $\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$

43) $\lim_{x \to 0} \frac{\sqrt{9x + 16} - 4}{x}$

44) $\lim_{x \to 1} \frac{x^7 - 1}{x - 1}$

45) $\lim_{x \to 2} \frac{x^4 - 16}{x - 2}$

46) $\lim_{x \to 4} \frac{x^3 - 64}{x - 4}$

47) $\lim_{x \to 3} \frac{x^5 - 243}{x - 3}$

48) $\lim_{x \to 2} \frac{x^7 - 128}{2x - 4}$

49) $\lim_{x \to 2} \frac{x^6 - 64}{x^5 - 32}$

50) $\lim_{x \to 4} \frac{2x^3 - 128}{x^2 - 16}$

Unit Three

3 - 3 Limit of a Function at Infinity

You will learn

- Limit of a function at infinity.
- Find the limit of a function at infinity using algebraic solution.
- Find the limit of a function at infinity using graphical solution.

Key - term

- Limit of a function at infinity

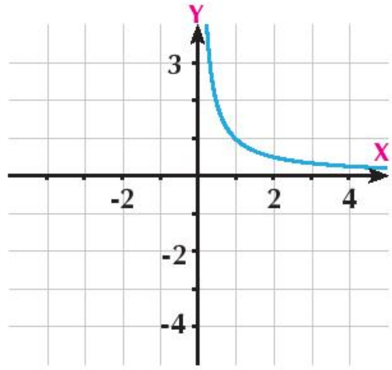
Matrials

- Scientific calculator .
- Pc graphical programs.

In our life and practical applications, we enormously need to know the behavior of the function $f(x)$ when $x \longrightarrow \infty$. The following activity shows that.

Activity

Use a computer program to graph the function f where: $f(x) = \frac{1}{x}$ and $x > 0$. What do you notice in the curve if the positive values of x increase to infinity? From the figure, we notice:



➢ The more the values of x increase and gets near to infinity, the more the values of $f(x)$ get near to zero.

Therefore, we say that the limit of $f(x)$ equals zero when x gets near to infinity.

Learn

Limit of a Function at Infinity

Theory 1

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Corollary

$$\lim_{x \to \infty} \frac{a}{x^n} = 0 \quad \{\text{where } n \in \mathbb{R}^+ \text{ and a is a constant}\}$$

Basic rules:

➢ $\lim_{x \to \infty} c = c$, where c is a constant

➢ if n is a positive number greater than 1, then $\lim_{x \to \infty} x^n = \infty$

Notice that the theory (2) related to the limit of addition, subtraction, multiplication or division of two functions when $x \longrightarrow a$ studied in the previous lesson is true when $x \longrightarrow \infty$

Example

1 Find:

a $\lim_{x \to \infty} (\frac{1}{x} + 3)$ **b** $\lim_{x \to \infty} (4 - \frac{3}{x^2})$

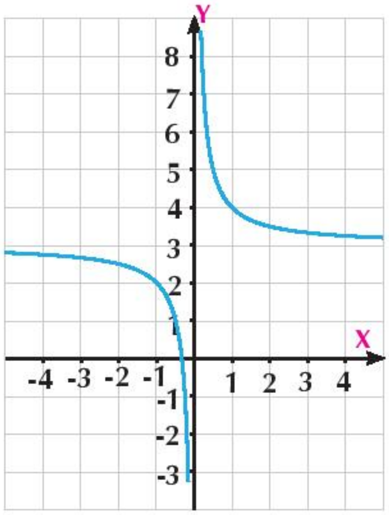
➢ Then check graphically using a graphical program.

Solution

a $\lim_{x \to \infty} (\frac{1}{x} + 3) = \lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} 3$

$= 0 + 3 = 3$

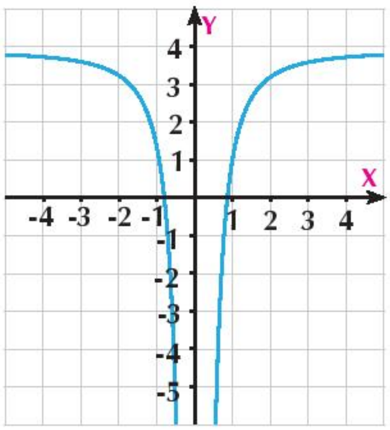
$\therefore \lim_{x \to \infty} (\frac{1}{x} + 3) = 3$



b $\lim_{x \to \infty} (4 - \frac{3}{x^2}) = \lim_{x \to \infty} 4 - \lim_{x \to \infty} \frac{3}{x^2}$

$= 4 - 3 \lim_{x \to \infty} \frac{3}{x^2} = 4 - 3 \times 0 = 4$

$\therefore \lim_{x \to \infty} (4 - \frac{3}{x^2}) = 4$



Try to solve

1 Find:

a $\lim_{x \to \infty} (\frac{5}{x} + 2)$ **b** $\lim_{x \to \infty} (\frac{2}{x^2} + 5)$

Example

2 Find: $\lim_{x \to \infty} (x^3 + 4x - 5)$

Solution

$$\lim_{x\to\infty}(x^3+4x-5)=\lim_{x\to\infty}x^3\left(1+\frac{4}{x^2}-\frac{5}{x^3}\right)$$, by taking off x^3 a common factor.

$$=\lim_{x\to\infty}x^3\times\lim_{x\to\infty}\left(1+\frac{4}{x^2}-\frac{5}{x^3}\right)=\infty\times1=\infty$$

Try to solve

2) Find each of the following limits:

a) $\lim_{x\to\infty}(x^3+7x^2+2)$ b) $\lim_{x\to\infty}(4-3x-x^3)$

Example

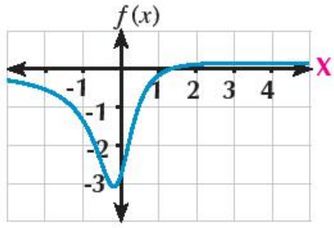
3) Find each of the following limits:

a) $\lim_{x\to\infty}\frac{2x-3}{3x^2+1}$ b) $\lim_{x\to\infty}\frac{2x^2-3}{3x^2+1}$ c) $\lim_{x\to\infty}\frac{2x^3-3}{3x^2+1}$

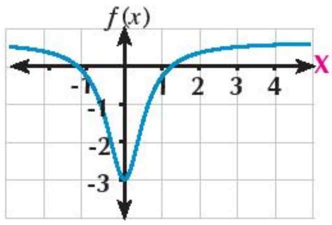
Solution

In all cases, divide both the numerator and the denominator by x^2 (biggest power of the variable x in the denominator).

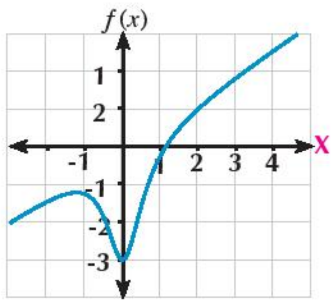
a) $$\lim_{x\to\infty}\frac{2x-3}{3x^2+1}=\frac{\lim_{x\to\infty}\left(\frac{2}{x}-\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(3+\frac{1}{x^2}\right)}=\frac{0-0}{3+0}=0$$



b) $$\lim_{x\to\infty}\frac{2x^2-3}{3x^2+1}=\frac{\lim_{x\to\infty}\left(\frac{2}{x}-\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(3+\frac{1}{x^2}\right)}=\frac{2-0}{3+0}=\frac{2}{3}$$



c) $$\lim_{x\to\infty}\frac{2x^3-3}{3x^2+1}=\frac{\lim_{x\to\infty}\left(2x-\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(3+\frac{1}{x^2}\right)}=\frac{\infty-0}{3+0}=\infty$$



We deduce from this example that: to find $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ where both $f(x)$ and $g(x)$ are polynomial functions, then:

- The limit gives a real number not equal zero if the degree of the numerator equals the degree of the denominator.
- The limit equals zero if the degree of the numerator is less than the degree of the denominator.
- The limit gives(∞ or $-\infty$) if the degree of the numerator is greater than the degree of the denominator.

Exercises (3 - 3)

Complete the following:

1) $\lim\limits_{x\to\infty} (1+\frac{3}{x}) =$ ....

2) $\lim\limits_{x\to\infty} (\frac{3}{x^2} - 2) =$ ....

3) $\lim\limits_{x\to\infty} (-7) =$ ....

4) $\lim\limits_{x\to\infty} (x^2 - 3) =$ ....

5) $\lim\limits_{x\to\infty} \frac{2x+1}{x}$ ....

6) $\lim\limits_{x\to\infty} \frac{x^3-5}{x^2+1} =$ ....

7) $\lim\limits_{x\to\infty} \frac{x^5+3}{x^3-5} =$ ....

8) $\lim\limits_{x\to\infty} \frac{3x}{\sqrt{x^2-1}} =$ ....

9) $\lim\limits_{x\to\infty} (3 - \frac{7}{x} + \frac{4}{x^2}) =$ ....

Choose the correct answer:

10) $\lim\limits_{x\to\infty} \frac{6x}{2x+3}$ equal:

a) 0 b) 2 c) 3 d) ∞

11) $\lim\limits_{x\to\infty} \sqrt{\frac{4}{x}+1}$

a) 0 b) 1 c) 2 d) ∞

12) $\lim\limits_{x\to\infty} \frac{x+3}{2-x^2}$

a) 0 b) $\frac{1}{2}$ c) $\frac{3}{2}$ d) ∞

13) $\lim\limits_{x\to\infty} \frac{x^2+1}{2x-1}$

a) 0 b) $\frac{1}{2}$ c) 1 d) ∞

14) $\lim\limits_{x\to\infty} \sqrt{\frac{1+x}{4x-1}}$

a) -1 b) $\frac{1}{4}$ c) $\frac{1}{2}$ d) 1

Find the limit of the function at infinity

15) $\lim\limits_{x\to\infty} \frac{3}{x^2}$

16) $\lim\limits_{x\to\infty} (x^3 + 5x^2 + 1)$

17) $\lim\limits_{x\to\infty} \frac{2-7x}{2+3x}$

18) $\lim\limits_{x\to\infty} \frac{x^2}{x+3}$

19) $\lim\limits_{x\to\infty} \frac{4x^2}{x^2+3}$

20) $\lim\limits_{x\to\infty} \frac{5-6x-3x^2}{2x^2+x+4}$

21) $\lim\limits_{x\to\infty} \frac{2x-1}{x^2+4x+1}$

22) $\lim\limits_{x\to\infty} \frac{x^3-2}{3x^2+4x-1}$

23) $\lim\limits_{x\to\infty} \frac{2x^2-1}{4x^3-5x-1}$

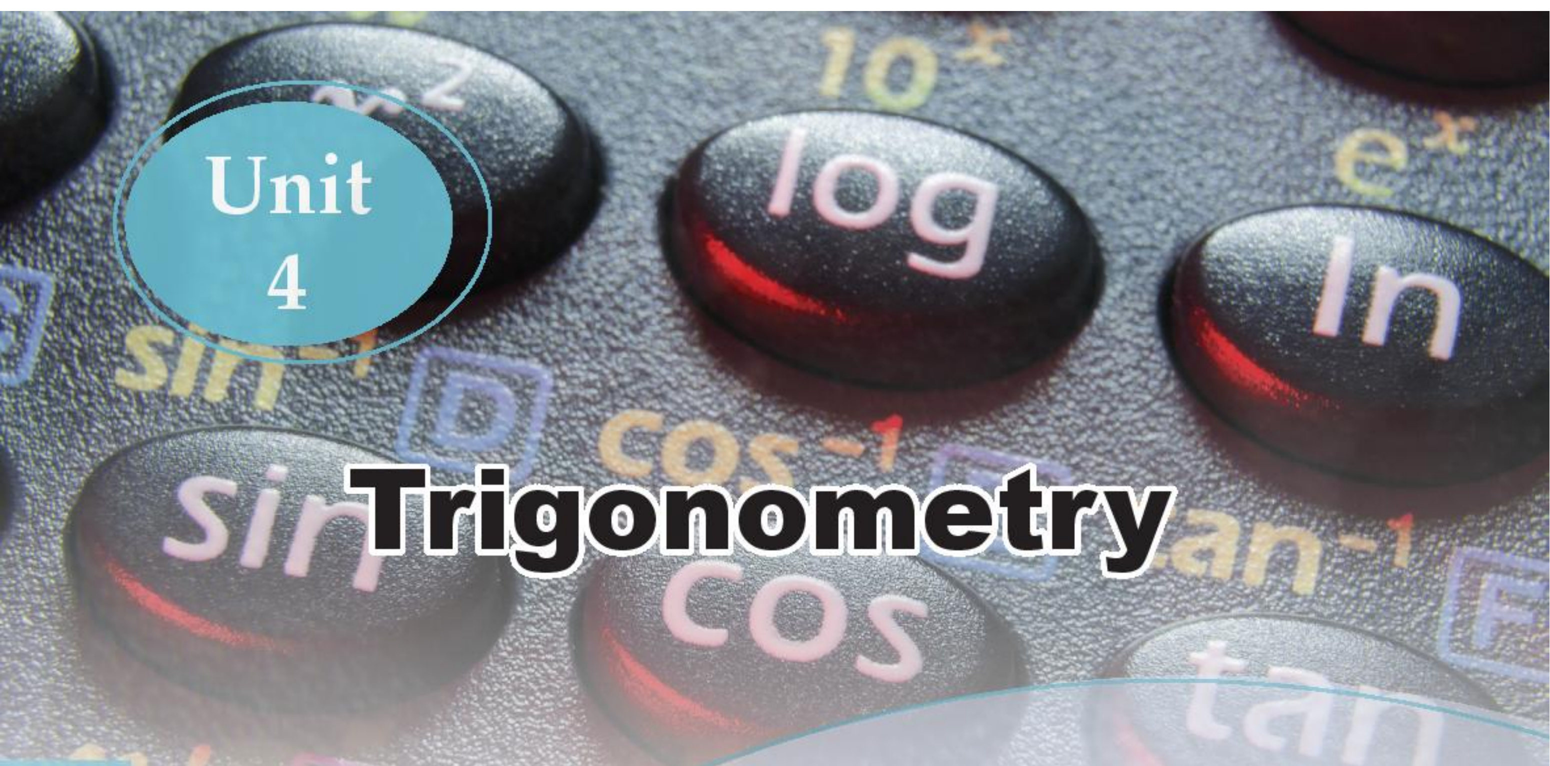
24) $\lim\limits_{x\to\infty} \frac{2x^2-6}{(x-1)^2}$

25) $\lim\limits_{x\to\infty} (7 + \frac{2x^2}{(x+3)^2})$

26) $\lim\limits_{x\to\infty} (\frac{1}{3x^2} - \frac{5x}{2+x})$

27) $\lim\limits_{x\to\infty} (\frac{x}{2x+1} + \frac{3x^2}{(x-3)^2})$

28) $\lim\limits_{x\to\infty} \frac{-x}{\sqrt{4+x^2}}$



Unit 4

Trigonometry

Unit introduction

Trigonometry (a latin term) is generally one of the mathematics branches and of geometry particulary where there is a relation among the side lengths of the triangle and the measurements of its angles in the form of trigonometrical functions(sine function , cosine function, tan function, ...) . Ancient Egyptians had been the first to practically use the rules of trigonometry. They had used these rules to build their pyramids and temples. Our knowledge about trigonometry is related to Greek who had originated the rules and theories. Beroni had produced a proof to the area of the triangle in terms of its side lengths. Western civilization had known Euclidean geometry via Arabs. Arab people had a long history in trigonometry. They had used the six trigonometric ratios where Altabani had discovered the special relation of the spherical right-angled triangle and the rule of finding the rule of sunrise.

Trigonometry has a lot of life applications in calculating distances and angles used in constructing the buildings, playgrounds, roads and industry of engines and electrical and mechanical appliances. Furthermore, Trigonometry is used to calculate geographical and astronomical distances and the exploratory systems of satellites.

Unit objectives

By the end of this unit, student should be able to:

- Identify the sine rule of any triangle that states in any triangle the side lengths are proportional to the sine of the opposite angles.
- Use the sine rule to find the side lengths of any triangle.
- Use the sine rule to find the measurements of the angles of any triangle.
- Deduce the relation between the sine rule of any triangle and radius length of its circumcircle.
- Deduce the cosine rule of any triangle.
- Use the cosine rule to find the unknown side length in the triangle.
- Use the cosine rule to find the unknown measurement of an angle in the triangle.
- Use sine and cosine rules of any triangle to solve this triangle:
- Use the calculator to solve various exercises and activities on sine and cosine rules of any triangle.

Key terms

- Trigonometry
- Sine rule
- Cosine rule
- Acute angle
- Obtuse angle
- Right angle
- Shortest side
- Longest side
- Missing length
- UnKnown angle
- Smallest angle
- Largest angle
- The area of the triangle
- The side lengthes of a Triangle
- Opposite angle

Unit lessons

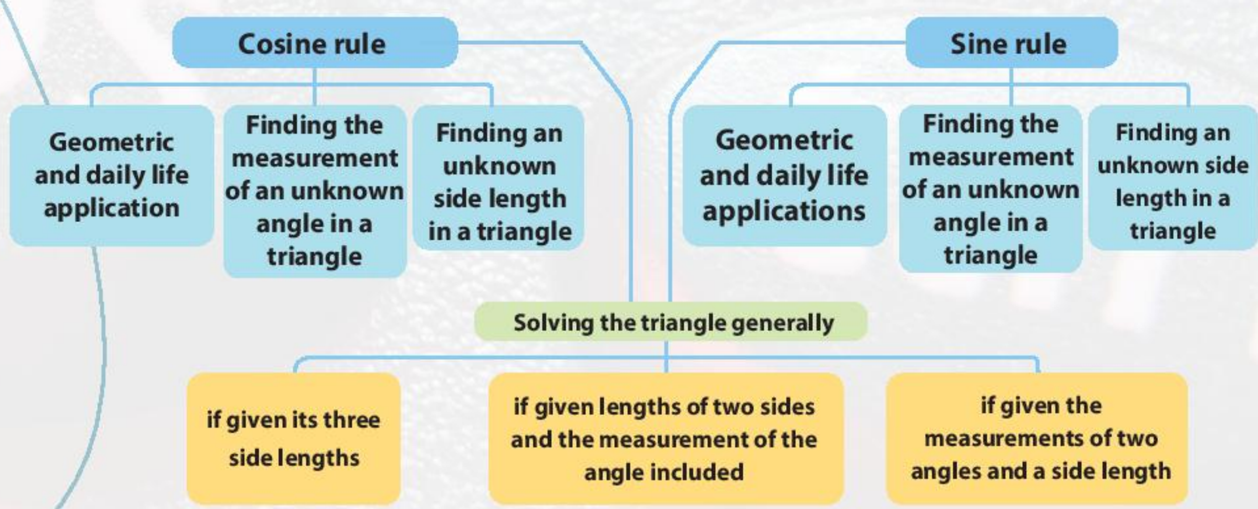
Lesson (4-1): Sine rule

Lesson (4-2): Cosine rule

Materials

Scientific calculator

Unit planning guide



Unit 4

4 - 1 Sine rule

You will learn

- The sine rule of any triangle.
- Using the sine rule to solve the triangle.
- Modeling and solving mathematical and life problems using the sine rule.
- The relation between the sine rule of any triangle, the radius length of the circumcircle of this triangle and solving problems on it.

Key terms

- Sine Rule
- Acute Angle
- Obtuse Angle
- Right Angle

materials

- Scientific calculator
- Graphical program

Remember

The inscribed angles which subtends the same arc in a circle are equal. The inscribed angle drawn in a semicircle is right angled.

Preface

You have learned how to solve the right-angled triangle. Now you are going to deal with triangles that don't have right angles to learn how to find the side lengths and the measurements of the angles of such triangles. You already know that each triangle is made up of sex elements- three sides and three angles. If any three elements are given- a side length is to be involved at least, you can find the other three elements by using the sine and cosine rules. Here , we can say that we can solve the triangle.

Learn

The sine rule

The following figures represents three type of triangles.

Acute-angled

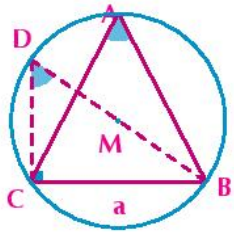


Figure (1)

$m(\angle A) = m(\angle D)$

Obtuse - angled

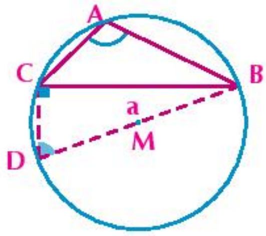


Figure (2)

$m(\angle A) = 180° - m(\angle D)$

Right- angled

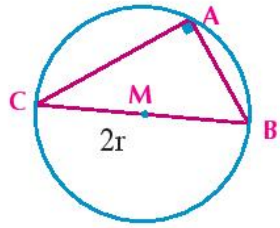


Figure (3)

$m(\angle A) = 90°$

In figure (1) where $\triangle ABC$ is an acute-angled triangle $\sin A = \sin D = \frac{a}{2r}$

Similarly, we can deduce that: $\sin B = \frac{b}{2r}$ and $\sin C = \frac{c}{2r}$

In figure (2) where $\triangle ABC$ is an obtuse-angled triangle at A

Notice: $\sin A = \sin (180° - D) = \sin D$

$\sin (180° - D) = \sin D]$

$\sin D = \frac{a}{2r}$ $\qquad \therefore \sin A = \frac{a}{2r}$

Similarly, we can deduce that :

$\sin B = \frac{b}{2r}$ and $\sin C = \frac{c}{2r}$

«Check it with your teacher»

Notice

a , b , c are symbols of the side lengths: $\overline{BC}, \overline{AC}, \overline{AB}$ in $\triangle ABC$ respectively.

Now: try to prove the same previous relation in $\triangle$ ABC which is right - angled at A.

In general The sine rule in $\triangle$ ABC states that:

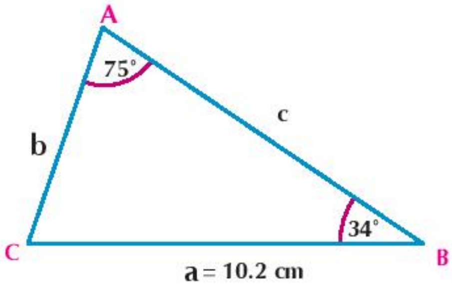
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$$ where r is the radius length of its circumcircle.

i.e. In any triangle , the side lengths are proportional to the sines of the opposite angles of these sides

Use the sine rule to find the side lengths of any triangle.:

Example

(1) In the triangle ABC, If m ($\angle$A) = 75°, m($\angle$B)= 34° and a = 10.2 cm, Find each of b and c to the nearest integer



Solution

$\because$ m($\angle$A) + m($\angle$B) + m($\angle$C) = 180°

$\therefore$ m($\angle$C) = 180° - (75° + 34°)

= 71°

Use the sine rule to find b and c

$$\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \therefore \frac{10.2}{\sin 75^\circ} = \frac{b}{\sin 34^\circ} = \frac{c}{\sin 71^\circ}$$

$$b = \frac{10.2 \times \sin 34^\circ}{\sin 75^\circ} \simeq 6\text{cm}$$ Use the calculator

1 0 . 2 × sin 3 4 °'") ÷ sin 7 5 °'") =

$$c = \frac{10.2 \times \sin 71^\circ}{\sin 75^\circ} \simeq 10\text{cm}$$ Use the calculator

1 0 . 2 × sin 7 1 °'") ÷ sin 7 5 °'") =

Try to solve

(1) In the triangle ABC, if m($\angle$C) = 61°, m($\angle$B) = 71° and b = 91cm, find each of a and c.

Finding the length of the longest side in the triangle

Remember

The longest side of a triangle is the side opposite to the largest angle and vice versa the smallest angle in a triangle is the angle opposite to the shortest side.

Example

(2) Find the length of the longest side in the triangle ABC in which m ($\angle$A) = 49° 11', m($\angle$B) = 76° 17' and c = 11.22cm to the nearest two decimals .

Solution

$\because$ m(∠C) = 180° - [m (∠A) + m(∠B)]

= 180 °- (49° 11' + 76° 17') = 54° 32'

The longest side is opposite to angle B. i.e, the required is to find b.

$\because \frac{b}{\sin B} = \frac{c}{\sin C}$ $\therefore \frac{b}{\sin 17' 76^\circ} = \frac{11.22}{\sin 32' 54^\circ}$

$\therefore b = \frac{11.22 \times \sin 17' 76^\circ}{\sin 54^\circ 32'} \simeq 13.38$cm

Try to solve

(2) Find the length of the shortest side in the triangle ABC in which m(∠A) = 43°, m(∠B) = 65° and c = 8,4 cm, to the nearest decimal.

Solving the triangle using the sine rule

Solving the triangle means that we find the measurements of its unknown elements if three elements of the six elements are known in a condition that one of those known elements is to be a side length at least because the triangle can not be solved if the measurements of its three angles are known only. The sine rule enables us to solve the triangle if the measurements of two angles and a side length are known (given).

Solving the triangle if the measurements of two angles and a side length are given (known):

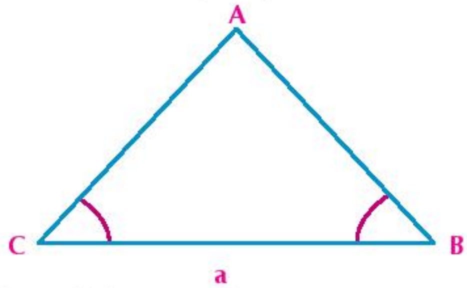
Note: To solve the triangle ABC if the mesurements of both angles B and C and the length of a are known, we follow the next steps:

1- We use the relation m(∠A) + m(∠B) + m(∠C) = 180° to find m(∠A)

2- We use the sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B}$ to find b

3- We use the sine rule: $\frac{a}{\sin A} = \frac{c}{\sin C}$ to find c

The following examples illustrate that:



Example

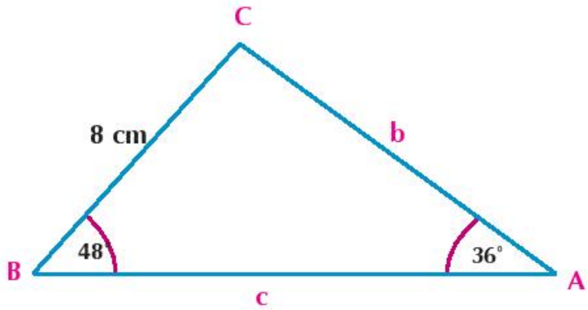
(3) Solve the triangle ABC in which m(∠A) = 36°, m(∠B) = 48° and a = 8 cm to the nearest three decimals.

Solution

Find m(∠C) from the relation :

m(∠C) = 180° - (36° + 48°) = 96°

We find b using the sine rule as follows:



$\because \frac{a}{\sin A} = \frac{b}{\sin B}$ $\therefore \frac{8}{\sin 36^\circ} = \frac{b}{\sin 48^\circ}$

$\because b = \frac{8 \times \sin 48^\circ}{\sin 36^\circ}$ $\therefore b \simeq 10.115$cm

Use the calculator as follows : 8 × sin 4 8 ÷ sin 3 6 =

$\because \frac{a}{\sin A} = \frac{c}{\sin C}$ $\therefore \frac{8}{\sin 36^\circ} = \frac{c}{\sin 96^\circ}$

$\therefore c = \frac{8 \times \sin 96^\circ}{\sin 36^\circ} \simeq 13.535$cm

Use the calculator as follows : 8 × sin 9 6 ÷ sin 3 6 =

Try to solve

3) Solve the triangle XYZ in which Y = 107.2cm, m(∠X) = 33° 16' and m(∠Z) = 44° 19'

Geometrical applications

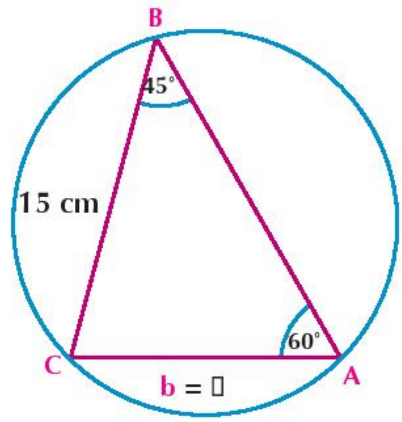
The relation between the sine rule and the radius length of the circumcircle of this triangle.

You have previously learned that: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$ where r is the radius length of the circumcircle of this triangle.

Example

4) ABC is a triangle in which a = 15cm, m(∠A) = 60° and m(∠B) = 45° . Find C and the radius length of circumcircle of the triangle ABC, to the nearest integer.

Solution



We find m(∠C) as follows:

m(∠C) = 180° - [60° + 45°]

= 180° - 105° = 75°

We use the sine rule to find c :

$\because \frac{c}{\sin C} = \frac{a}{\sin A}$ $\therefore \frac{c}{\sin 75^\circ} = \frac{15}{\sin 60^\circ}$

$c = \frac{15 \times \sin 75^\circ}{\sin 60^\circ} \simeq 17$ cm

To find the radius length of the circumcircle of triangle ABC, we use the relation:

$\frac{a}{\sin A} = 2r$ $\therefore \frac{15}{\sin 60^\circ} = 2r$ $\therefore 2r \times \sin 60^\circ = 15$

$\therefore r = \frac{15}{2 \sin 60^\circ} \simeq 9$cm

1 5 ÷) 2 sin 6 0 (=

Try to solve

4) ABC is a triangle in which m(∠A) = 64° 23' , m(∠B) = 72° 23' and c = 18 cm. Find each of a, b and the radius length of the circumcircle of triangle ABC.

Example

5) ABC is a triangle inscribed in circle M with radius length 100 cm. If AB = AC = 182 cm, find:

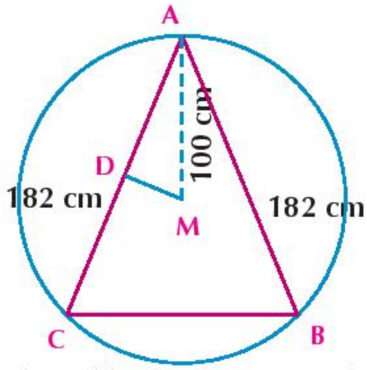
a) Length of $\overline{BC}$ to the nearest decimal.

b) The area of the triangle ABC to the nearest square centimetre.

Rappel

The area of a triangle = $\frac{1}{2}$ porduct of the lengths of any two sides × sine the subtnded angle.

Solution



We find m(∠B) as follows:

In △ ABC :

$\frac{AC}{\sin B} = 2r$ (sine rule)

$\frac{182}{\sin B} = 200$ $\sin B = \frac{182}{200} = 0.91$

$\therefore m(\angle B) = 65°\ 30'\ 19''$

(m(∠B) = m(∠C) because △ ABC is an isosceles triangle and both are acute angles)

We find m(∠A)

$m(\angle A) = 180° - 2 \times (65°\ 30'\ 19'') \simeq 48°\ 59'\ 22''$

We find the length of $\overline{BC}$ using the sine rule as follows:

$\therefore \frac{BC}{\sin 48°\ 59'\ 22''} = \frac{182}{\sin 65°\ 30'\ 19''}$ $\therefore BC = \frac{182 \times \sin 48°\ 59'\ 22''}{\sin 65°\ 30'\ 19''} \simeq 150.9\text{cm}$

1 8 2 × sin 4 8 °′″ 5 9 °′″ 2 2 °′″) ÷

sin 6 5 °′″ 3 0 °′″ 1 9 °′″) =

The area of the triangle ABC = $\frac{1}{2}$ AB × AC sin A

$= \frac{1}{2} \times 182 \times 182 \sin 48°\ 59'\ 22'' \simeq 12497\ cm^2$

Try to solve

5) ABC is a triangle inscribed in a circle with radius length 8.4 cm. If AB= AC = 10.3 cm, Find:

a) The length of the base $\overline{BC}$

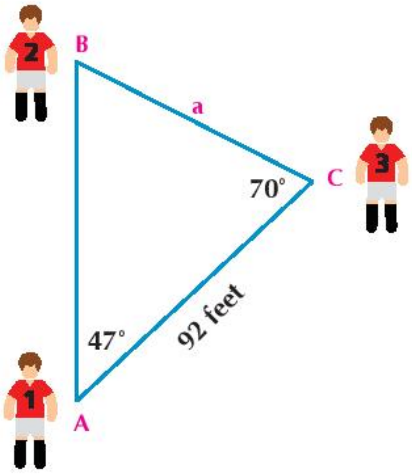
b) The area of the triangle ABC

Daily life applications on the sine rule

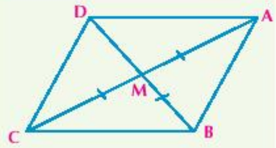
The sine rule can be used to solve many applications. It could be done by drawing a triangle , then solving that triangle to find the required data .

Example

6 **Sports:** the opposite figure represents three players from a football team during a match. Find the distance between the second player and the third player to the nearest feet.



Hint



The area of parallelogram ABCD = 2A(ΔABC) and the area of ΔABC = $\frac{1}{2}$ AB □ AC sin BAC

Solution

$m(\angle B) = 180° - (70° + 47°) = 63°$

The distance between the second player and the third player is a.

Then $\frac{a}{\sin 47°} = \frac{92}{\sin 63°}$ $\therefore a = \frac{92 \times \sin 47°}{\sin 63°} \simeq 76$ feet

Use the calculator: 9 2 × sin 4 7 (÷ sin 6 3 (=

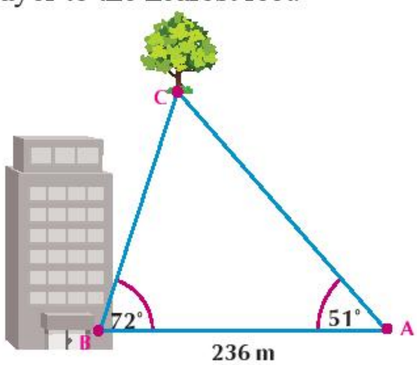
The distance between the second player and the third player is approximately 76 feet

Try to solve

6 Find the distance between the first player and the second player to the nearest feet.

Example

7 **Geography:** In the following figure, there are three geographical positions forming a triangle. If the distance between position A and position B is 236 km, the measurement of the angle at position B is 72° and the measurement of the angle at position A is 51°, Find:



a The distance between position C and position B. to the nearest integer

b The area of land which positions A , B and C represent its vertices to the nearest square meter.

Solution

a We find $m(\angle C)$ in Δ ABC : $m(\angle C) = 180° - (51° + 72°) = 57°$

We use the sine rule to find the length of $\overline{BC}$:

$\because \frac{BC}{\sin A} = \frac{AB}{\sin C}$ (sine rule) $\therefore \frac{BC}{\sin 51°} = \frac{236}{\sin 57°}$

then $BC = \frac{236 \times \sin 51°}{\sin 57°} = 218.6871 \simeq 219$ meters

b We find the surface area of the triangle ABC in terms of a, b and $m(\angle B)$.

$\because$ The area of the triangle ABC $= \frac{1}{2}$ a c sin B

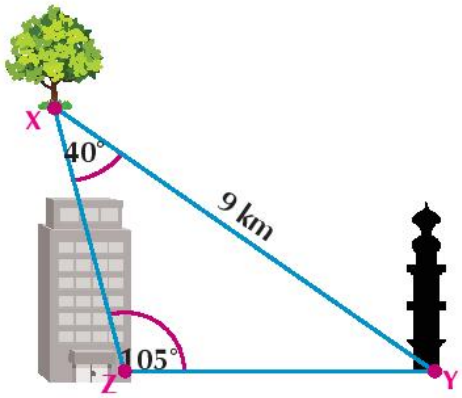
$= \frac{1}{2} \times 218.6871 \times 236 \times \sin 72° \simeq 24542\ m^2$.

Try to solve

7) In the opposite figure, there are three geographical positions forming a triangle. If the distance between position X and position Y is 9 km, the measurement of the angle at position X is 40° and the measurement of the angle at position Z is 105°, Find :

a) The distance between position X and position Z.

b) The area of the triangle whose vertices are X, Y, Z.



Exercises (4 - 1)

Complete :

1) In any triangle , the side lengths are proportional to ....

2) ABC is an equilateral triangle whose side length is $10\sqrt{3}$ cm the diameter length of the circumcircle of this triangle is ....

3) ABC is a triangle in which $m(\angle A) = 60°$, $m(\angle C) = 40°$ and $c = 8.4$ cm, then $a =$ .... cm

4) In the triangle ABC, $\frac{2\,b}{\sin B} =$ .... r

5) The diameter length of the circumcircle of the acute triangle ABC is 20 cm and BC = 10cm then $m(\angle A) =$ ....°

6) If the area of the equilateral triangle whose side length is 6 cm is ....

Choose the correct answer .

7) The radius length of the circumcircle of the triangle ABC in which $m(\angle A) = 30°$ and $a = 10$ cm is..........

a) 10cm b) 20cm c) 5cm d) 40cm

8) If the radius length of the circumcircle of the triangle ABC is 4 cm and $m(\angle A) = 30°$, then the length of a is..........

a) 4cm b) 2cm c) $4\sqrt{3}$ d) $\frac{1}{16}$

9) In the triangle ABC, the expression $2r \sin A$ is equal to..........

a) a b) b c) c d) $A(\triangle ABC)$

10) If r is the radius length of the circumcircle of the triangle XYZ , then $\frac{y}{\sin Y}$ is equal to ....

a) r b) 2 r c) $\frac{1}{2}$r d) 4 r

(11) The triangle LMN in which m(∠L) = 30° and MN = 7 cm, then the diameter length of its circumcircle is ....:

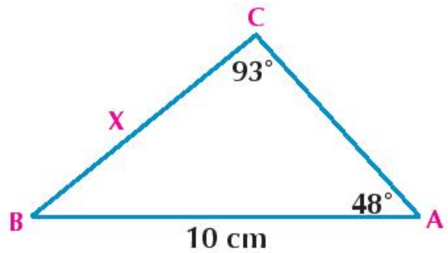
(a) 7cm (b) 3.5cm (c) 14cm (d) $\frac{14}{\sqrt{3}}$

(12) In the triangle XYZ, if 3 sin X = 4 sin Y = 2 sin Z, then x : y : z equal:

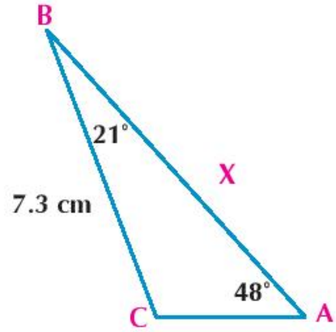
(a) 2 : 3 : 4 (b) 6 : 4 : 3 (c) 3 : 4 : 6 (d) 4 : 3 : 6

(13) Use the sine rule to find the value of x to the nearest tenth.

(a)



(b)



Solve each triangle ABC by using the sine rule, if you know that:

(14) m(∠A) = 75°, m(∠B)= 34°, a = 10.2cm

(15) m(∠A) = 19°, m(∠C)= 105°, c = 11.1cm

(16) m(∠A) = 116°, m(∠C) = 18°, a = 17cm

(17) m(∠A) = 36°, m(∠B)= 77°, b = 2.5cm

(18) X(∠A) = 49° 11' , m(∠B) = 67° 17' , c = 11.22cm

(19) X(∠B) = 115° 4' , m(∠C) = 11° 17' , c = 516.2cm

Find the diameter length of the circumcircle of the triangle ABC in each of the following case :

(20) m(∠A) = 75°, a = 21cm

(21) m(∠B)= 50°, b = 90cm

(22) m(∠C)= 102°, c = 11cm

(23) m(∠A) = 70°, a = 8.5cm

(24) In the triangle ABC, m(∠A) = 67° 22' , m(∠C) = 44° 33' and b = 100 cm, find the perimeter of the triangle ABC and its surface area.

(25) In the triangle XYZ, If y = 68.4cm, m(∠Y) = 100° and X(∠Z) = 40° , find x and the radius length of the circumcircle of the triangle XYZ, then find the surface area of the triangle.

(26) ABC is a triangle in which m(∠A) = 22° 37' , m(∠B) = 67° 23', and its perimeter is 30 cm. Find each of a and b to the nearest centimeter.

Unit 4

4 - 2 Cosine rule

You will learn

- The cosine rule for any triangle.
- Using the cosine rule to solve the triangle.
- Modeling and solving daily life and mathematical problems using cosine rule.

Key terms

- Cosine rule
- Acute angle
- Obtuse angle
- Right angle

Materials

- Scientific calculator

Think and discuss

Each of the following triangles is given the lengths of two sides ; 3cm and 4cm.

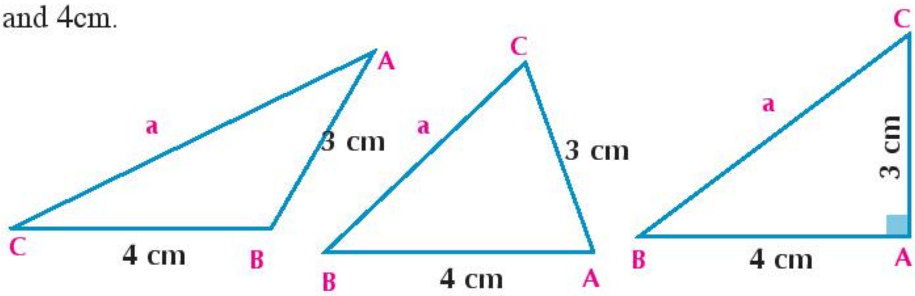


figure (1) figure (2) figure (3)

a) From figure (1), m(∠A) is a right angle. Find a.

b) What are the possible values of "a" if ∠A is an acute angle (figure 2)?

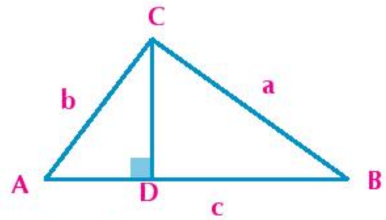
c) What are the possible values of "a" if ∠A is an obtuse angle (figure 3)?

d) Can you solve the two triangles in figures (2) and (3) if (∠A) is given using the sine rule? Explain.

Cosine rule helps us solve such triangles .

Learn

The cosine rule



In the figure opposite: $\overline{CD} \perp \overline{AB}$

In △ BCD:

$(BC)^2 = (CD)^2 + (BD)^2$ **(Pythagoras theory)**

$(BC)^2 = (CD)^2 + (AB - AD)^2$ Expanding the brackets

$= (CD)^2 + (AD)^2 + (AB)^2 - 2AB.AD$

$= (AC)^2 + (AB)^2 - 2AB.AD$

$a^2 = b^2 + c^2 - 2\,bc\cos A$

Think: Find the value of b^2 and c^2 in terms of a, b , c and the measurements of the angles of △ ABC.

Notice

$(AC)^2 = (AD)^2 + (CD)^2$

$AD = AC \cos A$

The cosine rule states that : in any triangle ABC :

$a^2 = b^2 + c^2 - 2\,bc\cos A$, $b^2 = c^2 + a^2 - 2\,c\,a\cos B$

, $c^2 = a^2 + b^2 - 2\,a\,b\cos C$

Finding the length of an unknown side in a triangle

Example

1 XYZ is a triangle in which $x = 24.3$cm , $y = 22.8$ cm and $m(\angle Z) = 42^\circ$, find z to the nearest decimal.

Solution

$z'^2 = x^2 + y^2 - 2x'\,y\cos Z$

$= (24.3)^2 + (22.8)^2 - 2 \times 24.3 \times 22.8\cos 42^\circ \simeq 286.87$

$Z \simeq 16.9$ cm

Use the calculator as follows:

2 4 . 3 x^2 + 2 2 . 8 x^2 − 2 × 2 4 . 3 × 2 2 . 8 cos 4 2 = √ ANS =

Try to solve

1 ABC is a triangle in which $a = 72.8$ cm, $b = 58.4$ cm and $m(\angle C) = 64.8^\circ$, find c to the nearest decimal

Finding the measurement of an angle if its three side lengths are given

You have previously learned that :

$a^2 = b^2 + c^2 - 2\,bc\cos A$ (Cosine rule)

i.e. $2b\,c\cos A = b^2 + c^2 - a^2$

then $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ (dividing by 2bc)

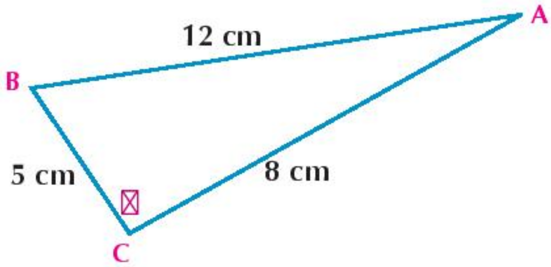
We can deduce that:

$\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ and $\cos C = \dfrac{a^2 + b^2 - c^2}{2\,ab}$

Using the cosine rule of any triangle to find the measurement of an unknown angle in this triangle.

Example

(2) From the opposite figure , find m(∠C)



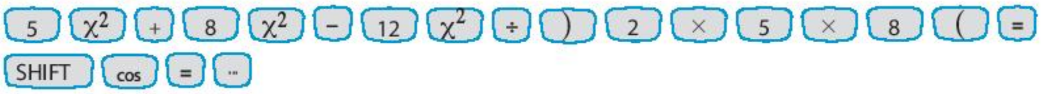
Solution

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab} \quad \text{(cosine rule)}$$

$$= \frac{(5)^2 + (8)^2 - (12)^2}{2 \times 5 \times 8} \quad \text{(by substituting)}$$

$$= \frac{-55}{80}$$

The calculate can be used as follows:

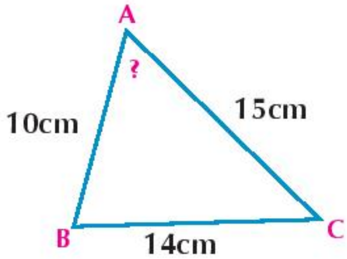


We notice that the cosine of the angle is negative, so ∠C is obtuse and

m(∠C) ≃ 133° 25' 57''

Try to solve

(2) From the opposite figure, find m(∠A)



Example

(3) Find the measurement of the largest angle in the triangle LMN if it is known that $l = 7.5$ cm, m = 12,5 cm and n = 17,5 cm, then prove that:

$\cos N - 3\sqrt{3}\,\sin N + 5 = 0$

Remember

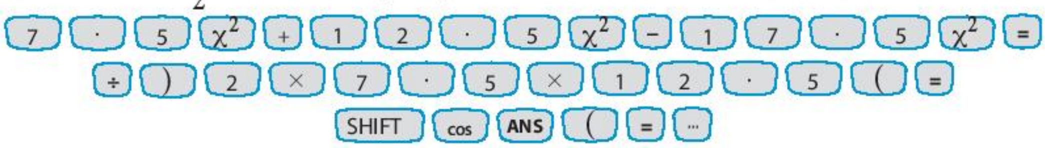
$\cos 120^\circ = \cos(180^\circ - 60^\circ)$
$= -\cos 60^\circ = -\frac{1}{2}$
$\sin 120 = \sin(180^\circ - 60^\circ)$
$= \sin 60^\circ = \frac{\sqrt{3}}{2}$

Solution

The largest angle is the angle opposite to the longest side, so ∠N is the largest angle in the triangle.

Then : $\cos N = \frac{l^2 + m^2 - n^2}{2\,l\,m} = \frac{(7.5)^2 + (12.5)^2 - (17.5)^2}{2 \times 7.5 \times 12.5} = -\frac{1}{2}$

$\therefore \cos N = -\frac{1}{2}$ $\qquad \therefore m(\angle N) = 120^\circ$



The left side $= \cos N - 3\sqrt{3}\,\sin N + 5 = \cos 120^\circ - 3\sqrt{3}\sin 120^\circ + 5$

$= -\frac{1}{2} - 3\sqrt{3} \times \frac{\sqrt{3}}{2} + 5 = 0 =$ The right side.

Try to solve

(3) In the triangle ABC , if a = 12 cm , b = 15cm and c = 18 cm, prove that m(∠C) = 2 m(∠A).

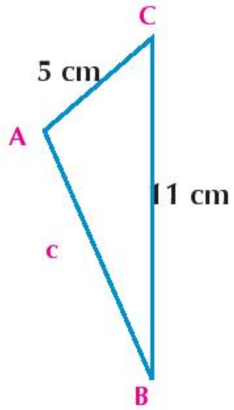
Using the cosine rule in solving the triangle

The cosine rule allows us to solve the triangle in terms of the lengths of two sides and the measurement of the angle included. In this case, there is only one triangle.

Solving the triangle in terms of the lengths of two sides and the measure of the included angle :

Example

(4) Solve the triangle ABC in which a = 11cm, b = 5cm and m(∠C) = 20°



Solution

$\because$ $c^2 = a^2 + b^2 - 2ab \cos C$

$\therefore$ $c^2 = (11)^2 + (5)^2 - 2 \times 11 \times 5 \cos 20°$

$\therefore$ $c = \sqrt{(11)^2+(5)^2 - 2 \times 11 \times 5 \text{Cos } 20}$

$\simeq 6.529$cm

√ 1 1 x² + 5 x² − 2 × 1 1 × 5 cos 2 0 =

$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$= \frac{(5)^2 + (6.529)^2 - (11)^2}{2 \times 5 \times 6.529} \simeq -0.817$

$\therefore m(\angle A) \simeq 144.786°$

$m(\angle B) = 180° - [m(\angle A) + m(\angle C)]$

$= 180° - [144.786° + 20°]$

$= 15.214°$

$\therefore c = 6.529$cm , $m(\angle A) = 144° 47' 96''$

$m(\angle B) \simeq 15° 12' 50''$

Remember

Solving the triangle means that we find the unknown elements. In this case, the required is to find c , m(∠A) and m(∠B)

Tip

To find the measurement of an angle in a triangle in terms of the lengths of two sides and the measurement of the angle included, it is better using the cosine rule instead of the sine rule. In case of using the sine rule, the sine of the acute angle or the obtuse angle is always positive because the sine is positive in the first and second quadrants. In case of using the cosine rule, if the angle is obtuse, its cosine is negative. If the angle is acute, its cosine is positive.

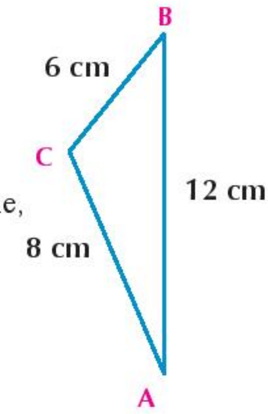
Try to solve

(4) Solve the triangle ABC in which a = 24.6cm , c = 14.2cm and m(∠B) = 42° 18'

Solving the triangle in terms of the lengths of its three sides

Example

(5) Solve the triangle ABC in which a = 6 cm, b = 8 cm and c = 12 cm.

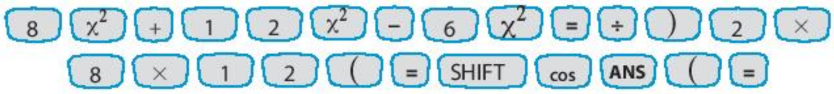


Solution

Required is to find the measurements of the three angles of the triangle, then

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c} = \frac{(8)^2 + (12)^2 - (6)^2}{2 \times 8 \times 12} = \frac{43}{48}$$

$\therefore$ m $(\angle A) \simeq 26^\circ\ 23'\ 4''$



$$\cos B = \frac{c^2 + a^2 - b^2}{2\,ca} = \frac{(12)^2 + (6)^2 - (8)^2}{2 \times 12 \times 6} = \frac{29}{36}$$

$\therefore$ m $(\angle B) \simeq 36^\circ\ 20'\ 10''$

$\therefore$ m $(\angle C) = 180^\circ - [26^\circ\ 23'\ 4'' + 36^\circ\ 20'\ 10'']$

$= 117^\circ\ 16'\ 46''$

Try to solve

(5) Solve the triangle ABC in which a = 12,2 cm, b = 18,4cm and c = 21.1 cm

Example

(6) **Geometry:** ABCD is a quadrilateral in which AB = 9 cm, BC = 5 cm, CD = 8 cm, DA = 9 cm and AC = 11 cm. Prove that the figure ABCD is a cyclic quadrilateral.

Solution

In the triangle ABC

$$\cos B = \frac{(9)^2 + (5)^2 - (11)^2}{2 \times 9 \times 5} = -\frac{1}{6}$$

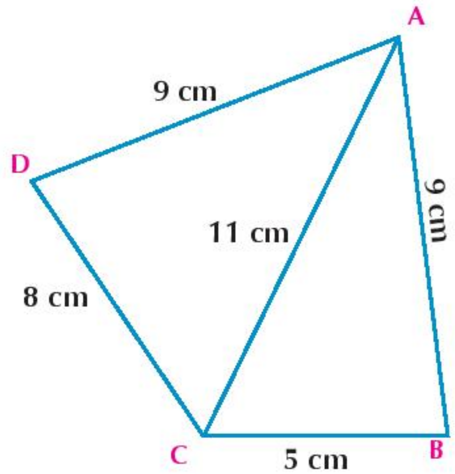
In the triangle ADC

$$\cos E = \frac{(9)^2 + (8)^2 - (11)^2}{2 \times 9 \times 8} = \frac{1}{6}$$

Then $\cos D = -\cos B$

i.e. $m(\angle D) + m(\angle B) = 180^\circ$

Where $\angle D$ and $\angle B$ are two opposite supplementary angles in the figure ABCD.

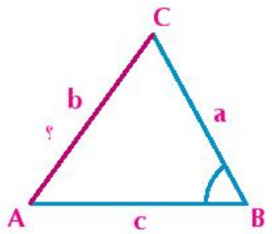


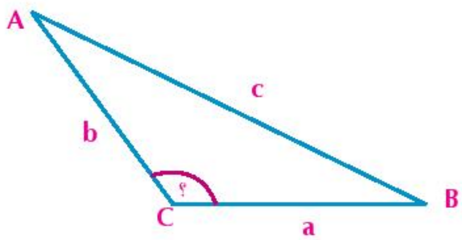
$\therefore$ The figure ABCD is a cyclic quadrilateral . **(Q.E.D)**

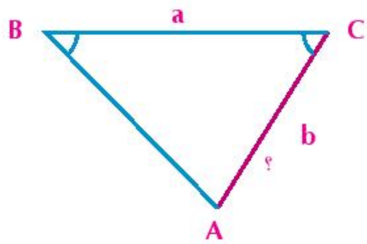
Try to solve

6) ABCD is a quadrilateral in which AB = 2.7 cm, AC = 7.2 cm, BC = 6.3 cm, CD = 4.5 cm and BD = 7.2 cm. Prove that the figure ABCD is a cyclic quadrilateral.

Discussion : For each the following triangle, write the correct formula to the sine rule or cosine rule in order to find what is required (referred to in red). Use the given data referred to in blue only.







Remember

The cyclic quadrilateral is a figure whose four vertices belong to one circle

- the figure is cyclic quadrilateral if:
- There are two opposite supplementary angles.
- The measurement of the exterior angle at any vertex of its vertices equals the measurement of the interior angle opposite to the adjacent angle of this angle.
- There are two equal angles in measurement and drawn on one base and on one side of it.
- Its vertices are equidistant from a fixed point

Exercises (4 - 2)

Complete the following:

1. In any triangle XYZ, $x^2 = y^2 + \square^2$ .... , $\cos X = \dfrac{y^2 + z^2 \text{.......}}{\text{.......}}$

2. A triangle whose side lengths are 13, 17 and 15 cm, then the measurement of its largest angle is ....

3. A triangle whose side lengths are 5,7 cm, 7,5 cm and 4,2 cm, then the measurement of its smallest angle is ....

4. A triangle ABC in which a = 10 cm, b = 6 cm and $m(\angle C) = 60°$, then c = ...

5. In triangle LMN, $m^2 + n^2 - \ell^2 =$ ....

Choose the correct answer:

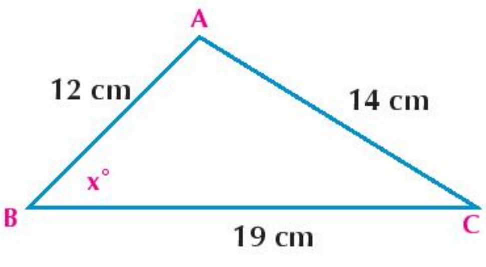
6. The measurement of the largest angle in the triangle whose side lengths are 3, 5 and 7 is ....
   - a) 150°
   - b) 120°
   - c) 60°
   - d) 30°

7. In which triangle LMN , the expression $\dfrac{\ell^2 + m^2 - n^2}{2\,\ell m}$ is equal to:
   - a) sin L
   - b) cos M
   - c) cos N
   - d) sin N

8. In triangle XYZ, $y^2 + z^2 - x^2 = 2yz$ ....
   - a) cos X
   - b) sin Z
   - c) cos Z
   - d) sin X

9. In triangle ABC, if a : b : c = 3 :2 : 2, then cos a equals ....
   - a) $\frac{1}{2}$
   - b) $-\frac{1}{8}$
   - c) $\frac{1}{2}$
   - d) $\frac{3}{4}$

Use the cosine rule to find the value of x to the nearest tenth.

10.


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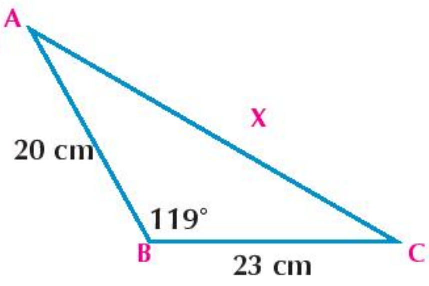

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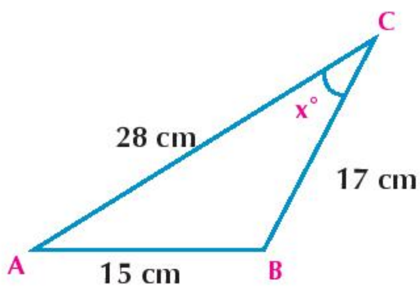

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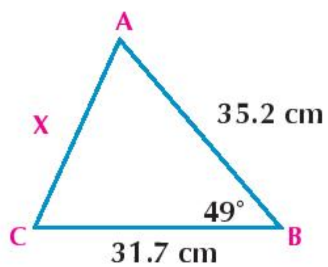

In triangle ABC if:

14. $a = 5$, $b = 7$ and $c = 8$, prove that $m(\angle B) = 60°$
15. $a = 3$, $b = 5$ and $c = 7$, prove that $m(\angle C) = 120°$
16. $a = 13$, $b = 7$ and $c = 13$, find $m(\angle C)$
17. $a = 13$, $b = 8$ and $c = 7$, find $m(\angle A)$
18. $a = 10$, $b = 17$ and $c = 21$, find the measurement of the smallest angle.
19. $a = 5$, $b = 6$ and $c = 7$, find the measurement of the largest angle.
20. $a = 17$, $b = 11$ and $m(\angle C) = 42°$, find c to the nearest two decimals.
21. $b = 16$ cm, $c = 14$ cm , $m(\angle A) = 72°$, find a to the nearest two decimals.
22. The triangle ABC in which $a = 3$cm , $b = 5$cm and $c = \sqrt{19}$ cm, find :
    - a) $m(\angle C)$
    - b) area of triangle ABC
23. ABC is a triangle in which $a = 9$ cm , $b = 15$ cm and $c = 21$ cm, find the measurement of the largest angle of the triangle and prove that it satisfies the relation $\cos C - 5\sqrt{3} \sin C + 8 = 0$
24. ABCD is a quadrilateral in which AB = 3 cm, AC = 8 cm, BC = 7 cm, CD = 5 cm and BD = 8 cm, prove that the figure is a cyclic quadrilateral.
25. ABCD is a quadrilateral in which AB = 15 cm, BC = 20 cm, CD = 16 cm, $\overline{AC}$ = 25 cm and $m(\angle ACD) = 36° 52'$, find length of $\overline{AD}$ to the nearest centimeter, then find the area of the quadrilateral ABCD.
26. ABCD is a parallelogram in which AB = 12 cm, BC = 10 cm and the diagonal length $\overline{BD}$ = 14 cm, find the diagonal length $\overline{AC}$ to the nearest centimeter.
27. ABCD is quadrilateral in which BD = 78 cm, CD = 96 cm, $m(\angle BCD) = 97°$, $m(\angle ABD) = 72°$, and $m(\angle ADB) = 43°$, find $\overline{AB}$.