Arts Section

General Mathematics

second secondary grade

Student book **second term**

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Introduction

بسم الله الرحمن الرحيم

We are pleased to introduce this book to show the philosophy on which the academic content has been prepared. This philosophy aims at:

1. Developing and integrating the knowledgeable unit in Math, combining the concepts and relating all the school mathematical curricula to each other.
2. Providing learners with the data, concepts, and plans to solve problems.
3. Consolidate the national criteria and the educational levels in Egypt through:
   - A) Determining what the learner should learn and why.
   - B) Determining the learning outcomes accurately. Outcomes have seriously focused on the following: learning Math remains an endless objective that the learners do their best to learn it all their lifetime. Learners should like to learn Math. Learners are to be able to work individually or in teamwork. Learners should be active, patient, assiduous and innovative. Learners should finally be able to communicate mathematically.
4. Suggesting new methodologies for teaching through (teacher guide).
5. Suggesting various activities that suit the content to help the learner choose the most proper activities for him/her.
6. Considering Math and the human contributions internationally and nationally and identifying the contributions of the achievements of Arab, Muslim and foreign scientists.

In the light of what previously mentioned, the following details have been considered:

★ This book contains three domains: algebra, relations and functions, calculus and trigonometry. The book has been divided into related and integrated units. Each unit has an introduction illustrating the learning outcomes, the unit planning guide, and the related key terms. In addition, the unit is divided into lessons where each lesson shows the objective of learning it through the title You will learn. Each lesson starts with the main idea of the lesson content. It is taken into consideration to introduce the content gradually from easy to hard. The lesson includes some activities, which relate Math to other school subjects and the practical life. These activities suit the students' different abilities, consider the individual differences throughout Discover the error to correct some common mistakes of the students, confirm the principle of working together and integrate with the topic. Furthermore, this book contains some issues related to the surrounding environment and how to deal with.

★ Each lesson contains examples starting gradually from easy to hard and containing various levels of thoughts accompanied with some exercises titled Try to solve. Each lesson ends in Exercises that contain various problems related to the concepts and skills that the students learned through the lesson.

★ Each unit ends in Unit summary containing the concepts and the instructions mentioned and General exams containing various problems related to the concepts and skills, which the student learned through the unit.

★ Each unit ends in an Accumulative test to measure some necessary skills to be gained to fulfill the learning outcome of the unit.

★ The book ends in General exams including some concepts and skills, which the student learned throughout the term.

Last but not least. We wish we had done our best to accomplish this work for the benefits of our dear youngsters and our dearest Egypt.

Contents

Unit 1 Sequences and series

Unit 2 Permutations, Combinations

Contents

Unit 3 Calculus and integration

Unit One

Sequences and series

Unit introduction

Undoubtedly, mathematics helps to discover and represent the patterns which might be finite or infinite where they can be existed in daily life situations or can be formed to serve different purposes. A lot of patterns are digital and have different uses in daily life. Patterns are found in forms of sequences and series. In the modern studies, the patterns have been developed from the theoretical domain into the practical one in the fields of science, geometry and statistics. Computers are the first technological appliances that can be improved and interacted uniquely and peerlessly to be used in analyzing the most complicated math and physics problems in all the knowledgeable branches.

Unit outcomes

By the end of this unit and doing all the activities involved , the student should be able to:

- Identify the concept of sequences and distinguish between sequences and series.
- Identify the arithmetic sequences and deduce its general term in different forms.
- Find the arithmetic mean of an arithmetic sequence and insert a finite number of arithemetic means between two numbers.
- Find the sum of a finite number of the terms of an arithmetic sequence in different forms.
- Identify the geometric sequence and deduce its general term in different forms.
- Find the geometric mean of a geometric sequence
- Insert a finite number of geometric means between two numbers.
- Deduce the relation between the arithmetic mean and geometric mean of two different positive numbers.
- Find the sum of a finite number of the terms of a geometric sequence in different forms.
- Find the sum of an infinite number of terms of a geometric sequence.
- Function the arithmetic and geometric sequences to interpret some life problems such as overpopulation.
- Use computers to do operations required to solve life and math problems on sequence and series.

Key terms

- Function
- Term
- Finite sequence
- Infinite sequence
- Increasing Seqence
- Decreasing Seqence
- Series
- Summation notation (Σ)
- Arithmetic sequence
- Common differnce
- Arithmetic mean
- Arithmetic series
- Geometric sequence
- Common ratio
- Geometric mean
- Geometric series
- Infinite geometric series
- Infinity

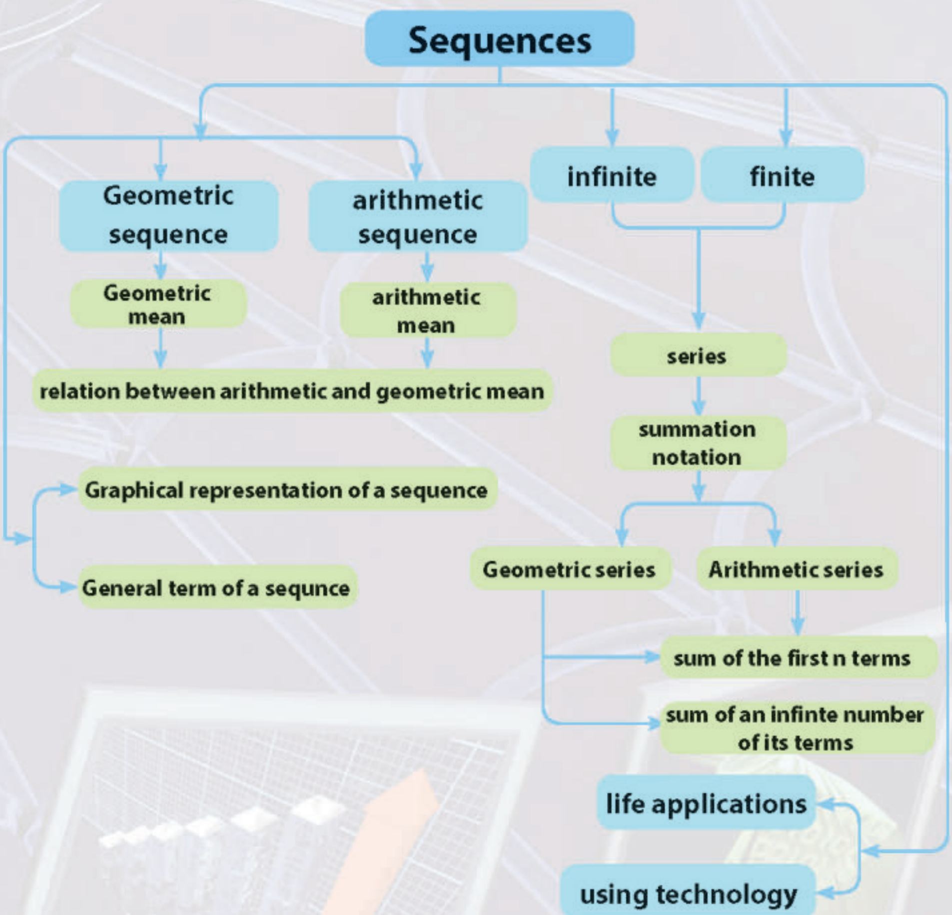
Unit Lessons

Lesson (1 - 1): Sequences and series.
Lesson (1 - 2): Arithmetic sequences.
Lesson (1 - 3): Arithmetic series.
Lesson (1 - 4): Geometric sequences.
Lesson (1 - 5): Geometric series.

Materials

- Scientifi calculator
- Graphics

Unit planning guide



Unit one

1 - 1 Sequences and Series

You will learn

- Definition of the sequence
- Finite and infinite sequence
- n^{th} term of the sequence
- Series and summation notation

Key terms

- Sequence
- Finite Sequence
- Infinite Sequence
- Set
- Term
- Series
- Summation notation

Materials

- Scientifi calculator
- Graphics

Think and discuss

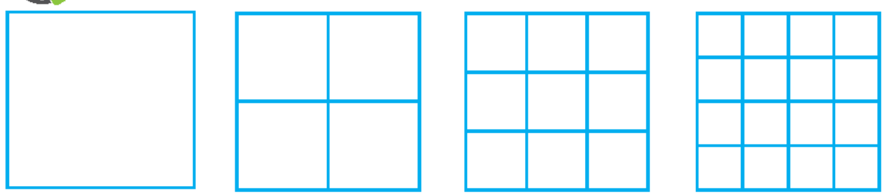


fig (1) fig (2) fig (3) fig (4)

The pattern above illustrates a square that can be divided into a number of smaller squares:

(1) find the number of small squares included in figure (5).
Draw figure (5).

(2) Can you find the number of small squares in figure 8?

(3) Can you find a relation between the number of small squares and the order of the figure?

Learn Sequence

The sequence is a function whose domain is a set of the positive integers Z^+ or a subset of it and its range is a set of the real numbers $\mathbb{R}$ where the first term is denoted by T_1, the second term is denoted by T_2 and the third term is denoted by T_3 and so on.... and the n^{th} term is denoted by Tn. The sequence can be expressed by writing down its terms between two brackets as follows: $(T_1, T_2, T_3, \dots, T_n)$ or denoted by the symbol (T_n).

Remember

The function is a relation between the two sets X and Y so that each element of X appears as a first projection only one time in a limited ordered pairs of the relation.

Example

1 Write down the first six terms for each of the following sequences:

a The sequence of the positive even numbers which starts from the number (2)

b The sequence of the number included between 10 and 30 in which each of them is divisible by 3.

Notice

(1) Terms of a sequence are the images of the elements of the sequence domain.
(2) The symbol (T_n) expresses the sequence while the symbol T_n expresses its n^{th} term.

Solution

a (2 , 4 , 6 , 8 , 10, 12) b (12 , 15 , 18 , 21 , 24 , 27)

Try to solve

1 Write down the first six terms for each of the following equences:

a The sequcne of the negative odd numbers which starts from the number (- 1).

b The sequence of the numbers included between 51 and 81 in which each of them is divisible by 5.

General term of a sequence

The general term of a sequcne (it is sometimes called the n^{th} term) is written as T_n where T_n is the image of the element whose order is **n** in the domain of the sequence and it can be deduced through the given terms of the sequence.

For example:

- The general term of the sequence of the even numbers : 2 , 4 , 6 , 8 , ... is $T_n = 2n$
- The general term of the sequence of the odd numbers : 1 , 3 , 5 , 7 , ... is $T_n = 2n - 1$
- The general term of the sequence: $-\frac{1}{3}$, $\frac{1}{4}$, $-\frac{1}{5}$, $\frac{1}{6}$, ... is $T_n = \frac{(-1)^n}{n+2}$

Critical thinking: is there a rule to find the general term of all sequences ? Explain .

Example

2 Write down the first five terms of the sequence (T_n) defined as follows:

$T_1 = -1$ and $T_{n+1} = 2T_n$ where $n \geqslant 1$

Solution

By substituting the value of n = 1 , 2 , 3 , 4 in the relation $T_{n+1} = 2T_n$:

let n = 1, then $T_2 = 2T_1$ **i.e.:** $T_2 = 2 \times -1 = -2$ (substituting by $T_1 = -1$)

let n = 2, then $T_3 = 2T_2$ **i.e.:** $T_3 = 2 \times -2 = -4$ (substituting by $T_2 = -2$)

let n = 3, then $T_4 = 2T_3$ **i.e.:** $T_4 = 2 \times -4 = -8$ (substituting by $T_3 = -4$)

let n = 4, then $T_5 = 2T_4$ **i.e.:** $T_5 = 2 \times -8 = -16$ (substituting by $T_4 = -8$)

The first five terms of the sequence are: (- 1 , - 2 , - 4 , - 8 , - 16)

Try to solve

2 Write down the first six terms of the sequence (T_n) defined as follows:

$T_1 = 3$, $T_n = 2T_{n-1}$ where $n \geqslant 2$

Finite seqence and infinite sequence

The sequence is finite if the number of its terms is finite (**i.e. can be counted can not be counted**).

The sequence is infinite if the number of its terms is infinite (**an infinite number of elements can't be counted**).

Example

(3) Write down each of the following sequences whose n^{th} term is given by the relation:

a $T_n = 2n + 1$ (to five terms starting from its first term).

b $T_n = n^2$ (to an infinite number of terms starting from its first term).

Solution

a Let n = 1 , 2, 3 , 4 , 5

$\therefore T_1 = 2(1) + 1 = 3$, $T_2 = 2(2) + 1 = 5$

$T_3 = 2(3) + 1 = 7$, $T_4 = 2(4) + 1 = 9$

$T_5 = 2(5) + 1 = 11$ $\therefore$ the sequence is: (3 , 5 , 7 , 9 , 11)**the sequence is finite**

b Let n = 1 , 2, 3 , 4 , 5 , ...

$\therefore T_1 = (1)^2 = 1$, $T_2 = (2)^2 = 4$

$T_3 = (3)^2 = 9$, $T_4 = (4)^2 = 16$

$T_5 = (5)^2 = 25$ $\therefore$ the sequence is: (1 , 4 , 9 , 16 , 25 , ...) **the sequence is infinite**

Try to solve

(3) Write down each of the following sequences whose n^{th} term is given by the relation:

a $T_n = 1 - 3n$ (to five terms starting from its first term).

b $T_n = n^3$ (to an infinite number of terms starting from its first term).

Series and summation notation

The series is the adding operation of the sequence terms .

for example: (2 , 5 , 8 , 11 , ...) is a sequcne while 2 + 5 + 8 + 11 + ... is the series related to the previous sequence and the summation notation " Σ " can be used. It is read as (sigma) to write the series in a short form.

Example

(4) Expand each of the following series , then find the sum of each expansion.

a $\sum_{r=1}^{4} r^2$ b $\sum_{r=3}^{7} (2r - 1)$

Solution

a let r = 1, then $T_1 = (1)^2 = 1$, let r = 2, then $T_2 = (2)^2 = 4$

let r = 3, then $T_3 = (3)^2 = 9$, let r = 4, then $T_4 = (4)^2 = 16$

i.e. the series is (1 + 4 + 9 + 16) and $\sum_{r=1}^{4} r^2 = 1 + 4 + 9 + 16 = 30$

b let r = 3, then $T_1 = 2 \times 3 - 1 = 5$, let r = 4, then $T_2 = 2 \times 4 - 1 = 7$

let r = 5, then $T_3 = 2 \times 5 - 1 = 9$, let r = 6, then $T_4 = 2 \times 6 - 1 = 11$

let r = 7, then $T_5 = 2 \times 7 - 1 = 13$

i.e. the series is (5 + 7 + 9 + 11 + 13) and $\sum_{r=3}^{7} (2r - 1) = 45$

Using the scientific calculator to find the sum of a series:

Scientific calculators help us do several complicated math problems quickly and accurately in a condition that the inputs must be correct . Finding the summation of a series is one of these essential operations . For example , we can check the summation of the series in the question (b) above as follows:

(1) Press the summation notation button Σ- (log) according to the specific color.

(2) Type the rule of the sequence (2r – 1) as follows:

start → 2 ALPHA) (x) – 1

(3) Use the button (Replay) to move as follows:

type the number of the terms of the sequence (7) in moving upwards,

type the order of the term by which we start with and its (3) in this example in moving downwards

(4) Press the button = to give the sum 45 and it is the same as the sum above.

Try to solve

4 Expand each of the following series, then find the sum of expansion and check the sum using the calculator.

a $\sum_{r=1}^{5} (3r - 2)$ **b** $\sum_{r=1}^{4} (r + 1)^2$ **c** $\sum_{r=5}^{9} 3 \times 2^{r-1}$

Exercises 1 - 1

Complete:

1. The fifth term of the sequence (T_n) where $T_n = 2n - 1$ is ....

2. The fourth term of the sequence (T_n) where $T_n = n^2 + 3$ is ....

3. In the sequence (T_n) where $T_{n+1} = n\,T_n$. If $T_1 = 1$, then $T_2 =$ ....

Choose the correct answer:

4. The fifth term in the sequence of the natural numbers divisible by 5 is .... :
   a 5   b 25   c 20   d 10

5. The tenth term of the sequence whose n^{th} term is $T_n = \frac{2}{n} - 1$ where $n \in Z^+$ is:
   a $\frac{-4}{5}$ b $\frac{-1}{5}$ c $\frac{1}{5}$ d $\frac{4}{5}$

6. The rule of the sequence $((2 \times 3), (3 \times 4), (4 \times 5), (5 \times 6), ...)$ is:
   a $(n - 1)(n + 1)$ b $n(n + 1)$ c $2n(n + 1)$ d $(n + 1)(n + 2)$

Answer the following questions:

7. Show which of the following sequences is finite and which is infinite:
   a $(1, 4, 7, 11, ...)$
   b $(3, 5, 7, 9, ..., 21)$
   c The sequence (T_n) where $T_n = n^2 - 1$, $n \in Z^+$
   d The sequence (T_n) where $T_n = \frac{2}{n} + 3$, $n \in \{1, 2, 3, 4, 5\}$

8. Write down the first five terms for each of the sequences whose general term is given by the following rules:
   a $T_n = n + n^2$ b $T_n = \frac{1}{2n - 5}$ c $T_n = (\frac{1}{3})^n$ d $T_n = (-1)^n (n - 2)^2$

9. Discover the pattern , then write the next term
   a $65, 69, 73, 77, 81, ...$
   b $3, -6, 12, -24, 48, ...$
   c $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, ...$
   d $1, 3, 6, 10, 15, ...$

10. Expand each of the following series:
    a $\sum_{r=1}^{5} (3r - 2)$
    b $\sum_{r=1}^{8} \left((-1)^r + 4r\right)$
    c $\sum_{r=1}^{\infty} (\frac{1}{2})^{r-1}$
    d $\sum_{r=1}^{\infty} (\frac{1}{r} - \frac{1}{r+1})$

11 Expand each of the following series, then find the sum of the expansion and check the result using the calculator.

a $\sum_{r=3}^{7} (2r + 3)$

b $\sum_{r=2}^{6} (r^2 - 2)$

c $\sum_{r=1}^{5} 3 \times (\frac{1}{2})^{r+1}$

d $\sum_{r=3}^{6} (\frac{1}{r} + 2)$

Unit one

1 - 2 Arithmetic Sequence

You will learn

- Definition of the arithmetic sequence.
- The graphical representation of the arithmetic sequence.
- n^{th} term of the arithmetic sequence.
- Identify the arithmetic sequence
- Insert a finite number of arithmetic means between two numbers. defining the arithmetic mean

Key terms

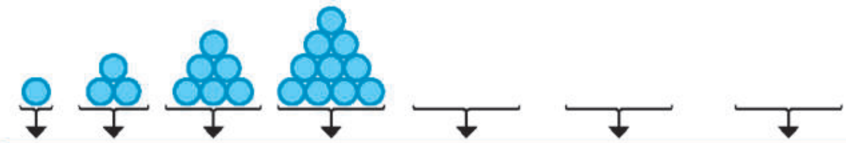
- Pattern
- Arithmetic sequence
- n^{th} Term
- Common difference
- Order of the term
- arithmetic mean

Materials

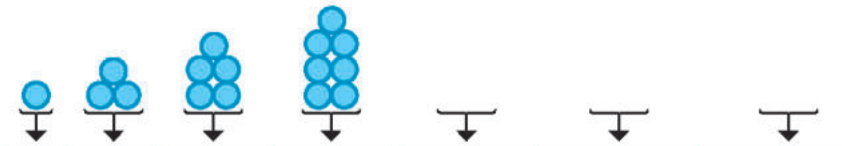
- Scientific Calculator
- Graphic programs

Activity

Study each of the following two patterns, then complete both of them till the seventh figure.



1	3	6	10	....	....	....



1	3	5	7	....	....	....

Answer the following questions:

(1) What are the aspects of similarity and difference between the first and second patterns?

(2) Write down the two sequences representing the two patterns above.

(3) What do you notice about the values of the sequence in the second pattern? Can you deduce a rule to relate the terms of this sequence? Write down this rule.

From the previous activity , we find that:

➢ The increase in the values of the elements of the first pattern varies, but the increase in the values of the elements of the second pattern is constant.

➢ The sequence representing the elements of the second pattern is: (1 , 3 , 5 , 7 , ...) where each element in it is greater than the directly previous term with a constant of 2. As a result, this sequence is called an arithmetic sequence.

Definition 1

Arithmetic Sequence

The arithmetic sequence is the sequence in which the difference between a term and the directly previous term equals a constant amount and it is called the common difference of the sequence. it is denoted by the symbol (d)

i.e.: $d = T_{n+1} - T_n$ for each $n \in Z^+$ and it can be formed in terms of its first term (a) and common difference (d).

Example

1 Which of the following sequences is an arithmetic sequence? Why?

a (7 , 10 , 13 , 16 , 19)

b $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6})$

c $T_n = 2n + 3$

Solution

a $\because T_2 - T_1 = 10 - 7 = 3$, $T_3 - T_2 = 13 - 10 = 3$

similarly $T_4 - T_3 = T_5 - T_4 = 3$

$\therefore T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = T_5 - T_4 = 3$ $\therefore$ the sequence is arithmetic and its common difference = 3

b $\because T_2 - T_1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$, $T_3 - T_2 = \frac{1}{4} - \frac{1}{3} = \frac{1}{12}$

$\therefore T_2 - T_1 \neq T_3 - T_2$ $\therefore$ the sequence is not arithmetic .

c $\because T_{n+1} - T_n = (2(n+1)+3) - (2n+3) = 2n + 2 + 3 - 2n - 3 = 2$

$\therefore$ the sequence is arithmetic and its common difference is 2

Try to solve

1 Which of the following sequences is an arithmetic sequence? Why?

a (38 , 33 , 28 , 23 , 18)

b (- 14 , - 8 , - 2 , 4 , 10)

c $T_n = 2n + 3$

d $T_n = 1 - \frac{3}{2} n$

Increasing sequence and decreasing sequence

The arithmetic sequence (T_n) is increasing if its common difference is positive (d > zero) such as: (1 , 5 , 9 , 13 , ...)

the arithmetic sequence (T_n) is decreasing if its common difference is negative (d < zero) such as: (4 , -1 , -6 , -11 , ...)

Try to solve

2 In the sequence (T_n) where $T_n = 3n - 5$

a Prove that (T_n) is an arithmetic sequence and find its common difference .

b Show that this sequence is increasing.

c Find the fifteenth term of the sequence.

d What's the value of n if $T_n = 85$?

Finding the n^{th} term of an arithmetic sequence:

From definition (1), the n^{th} term of the arithmetic sequence (T_n) whose first term is **a** and common difference is **d** can be deduced as follows :

$T_1 = a$, $T_2 = a + d$, $T_3 = a + 2d$

and by keeping this pattern, we find that the n^{th} term of this sequence is: $T_n = a + (n - 1) d$

Example

2 In the arithmetic sequence (13 , 16 , 19 , ..., 100)

a Find the tenth term b Find the number of the terms of the sequence

Solution

$\because$ The sequence is arithmetic $\therefore a = 13$, $d = 16 - 13 = 3$

a $\because T_n = a + (n - 1)\, d$ $\therefore T_{10} = 13 + (10 - 1) \times 3$

$= 13 + 9 \times 3 = 13 + 27 = 40$

b The required is to find the value of n when $T_n = 100$

$\because T_n = a + (n - 1)\, d$ $\therefore 100 = 13 + (n - 1) \times 3$ $\therefore 100 = 13 + 3n - 3$

i.e.: $3n = 100 - 10 = 90$ $\therefore n = 30$

Try to solve

3 Find the number of the terms of the arithmetic sequence (7 , 9 , 11 , ..., 65), then find the value of the tenth term from the end.

Identfing the arithmetic Sequence:

The arithmetic sequence can be identified when its first term and common difference are known.

Example

3 Find the arithmetic sequence (T_n) in which $T_7 = 18$ and $T_{15} = 34$

Solution

From the given data , we know that : $T_7 = 18$, $T_{15} = 34$

$\because T_n = a + (n - 1)\, d$ $\therefore 18 = a + (7 - 1)\, d$, then:

$\therefore a + 6d = 18$ **(1)**

similarly $34 = C + (15 - 1)\, d$

, $a + 14d = 34$ **(2)**

by solving the two equations **(1)**, **(2)** $d = 2$

and by substituting in the first equation

$\therefore a + 6 \times 2 = 18$ $\therefore a = 18 - 12 = 6$

$\therefore$ The arithmetic sequence is (6 , 8 , 10 , ...)

Notice

To get the value of d, then

$a + 14d = 34$

$-a - 6d = -18$

by multiplying the two sides of the first equation by (- 1)

by adding the two equations: $8d = 16$

by dividing the two sides of the equation by 2

$d = 2$

Using the calculator:

To check the solution of the two equations: $a + 6d = 18$ and $a + 14d = 34$ use the calculator and follow the next procedures:

Entering data

Press the operation button MODE and choose EQN from the list by typing the number typed in front of it or by pressing the button EXE in some calculators, then choose the linear equation anX + bnY = cn by pressing the button.

We enter the factors of (X), (Y), and the absolute term (cn) respectively for the first equation then the second equation directly as follows:

start ⟶ 1 = 6 = 18 = 1 = 1 4 = 3 4 =

Recalling the sums:

- Press the button = first time to get the value of the first variable, let it be (X) and the result is X = 6
- Press the button = another time to get the value of the second variable, let it be (Y) and the result is Y = 2

To exist the program : Press the buttons: start ⟶ MODE 1

Try to solve

4) Find the arithmetic sequence (T_n) in which $T_6 = 17$ and $T_3 + T_{10} = 37$

Arithmetic means:

You know that the arithmetic mean of the two numbers a and b is $\frac{a+b}{2}$

Considering that: (9 , 13 , 17 , 21 , 25) is an arithmetic sequence

- The arithmetic mean of the first and third terms $= \frac{9+17}{2} = 13$ **what do you notice?**
- The arithmetic mean of the second and fourth terms $= \frac{13+21}{2} = 17$ **what do you notice?**

Definition 2

If a , b and c are three consecutive terms in an arithmetic sequence, then b is called the arithmetic mean between the two numbers a and c where $b - a = c - b$,

i.e.: $2b = a + c$ then $b = \frac{a+c}{2}$ **So:** $\left(a, \frac{a+c}{2}, c\right)$ **is an arithmetic sequence**.

several arithmetic means: $x_1, x_2, x_3, \dots, x_n$ can be inserted between the two numbers a and b in a way that : $(a, x_1, x_2, x_3, \dots, x_n, b)$ **is an arithmetic sequence.**

Remember

The arithmetic mean of several quantities equals the sum of these numbers divided ny their numbers.

Example:

find the arithmetic mean of the numbers: 4 , 6 , 7 , 8 , 8 , 9

the arithmetic mean

$= \frac{4+6+7+8+8+9}{6}$

$= 7$

Verbal expression: Complate:

(1) If you form (3, 7, 11, ..., 43 , 47) to be an arithmetic sequence , then : 7, 11, 15, ..., 43 is called ....

(2) The number of arithmetic means = the number of the terms of the sequence ....

(3) The number of the terms of the arithmetic sequence = the number of means of this sequence ....

Insert an infinite number of arithmetic means between two number

Example

(5) Insert 5 arithmetic means between 6 and 48

Solution

First: Find the number of the sequence terms

Find five terms between the first and the last terms in the sequence. Thus, the number of the terms of the arithmetic sequence is

$n = 2 + 5 = 7$

Second: Find the value of d:

The n^{th} term of the arithmetic sequence is: $T_n = a + (n - 1)\, d$

by substituting: $a = 6$, $T_n = 48$, $n = 7$

$48 = 6 + (7 - 1)\, d$

i.e.: $6d = 42$ **by dividing the two sides by** 6 $\therefore d = 7$

Third: We use the value of d to find the required arithmetic means

$\boxed{6}$, 13 , 20 , 27 , 34 , 41 , $\boxed{48}$

+7 +7 +7 +7 +7 +7

The required means are: 13 , 20 , 27 , 34 , 41

Try to solve

(5) Insert four arithmetic means between the two numbers 13 , 48

Exercises 1 - 2

Determine which of the following sequences are arithmetic and which are non-arithmetic, then find the common difference in case the sequence is arithmetic:

(1) (12 , 15 , 18 , 21 , 24)

(2) (21 , 25 , 29 , 34 , 38)

(3) (x + 2y, 3x + 3y, 5x + 4y) where x and y are two positive quantities

Write down the first five terms of the arithmetic sequence in each of the following cases:

4) $a = 2$, $d = 5$

5) $a = 7$, $d = -3$

6) $a = -4$, $d = \frac{1}{4}$

Complete:

7) The seventh term of the arithmetic sequence (2 , 5 , 8 , ...) is ....

8) The eleventh term of the sequence (T_n) where $T_n = 3n - 5$ is ....

9) The n^{th} term of the arithmetic sequence (81 , 77 , 73 , ...) is ....

10) The n^{th} term of the arithmetic sequence $(\frac{1}{2}, \frac{1}{4}, 0, ...)$ is ....

Choose the correct answer from those given:

11) All the following sequences are arithmetic except:

a) (3 , 7 , 11 , 15 , ...)

b) (-11 , -15 , -19 , -23 , ...)

c) $(\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, ...)$

d) $(\frac{21}{5}, \frac{16}{5}, \frac{11}{5}, \frac{6}{5}, ...)$

12) If (T_n) is an arithmetic sequence where $T_n = 3n + 2$, then the arithmetic mean between T_5 and T_{11} equals:

a) 8 b) 16 c) 22 d) 26

Answer the following questions:

13) Find the twelfth and twentieth terms of the arithmetic sequence (4 , 7 , 10 , ...)

14) Find the number of the terms of the arithmetic sequence (63 , 59 , 55 , ... , - 133)

15) Write down the first three terms of the sequence $(2 + 5n)$, then find the order of the term whose value is 72 in this sequence .

16) (T_n) is an arithmetic sequence in which $T_1 = -51$ and $T_{22} = -156$, find the common difference of this sequence.

17) Find the arithmetic sequence whose fourth term = 18 and the seventh term = 27.

18) What is the value of n in the arithmetic sequence whose first term = 3 , $T_n = 39$ and $T_{2n} = 79$? find the sequence.

19) Find the arithmetic sequence whose fifth term = 21 and its tenth term = 3 times its second term.

20) (T_n) is an arithmetic sequence in which $T_1 + T_2 = 9$ and $T_5 = 22$. Find this sequence

21) Find the arithmetic sequence whose sixth term = 20 and the ratio between the fourth and tenth terms is 4: 7.

22) Find the arithmetic sequence whose fourth term = 11 and the sum of the fifth and ninth terms = 40, then find the order of the term whose value is 152 in this sequence .

23) If 36 , a , 24 , b are consecutive terms of an arithmetic sequence, find the value of a and b.

24) Insert 16 arithmetic means between 27 and - 24

1 - 3 Arithmetic Series

You will learn

- The concept of an arithmetic series
- Find the sum of n terms of an arithmetic sequence interms of its first and last terms .
- Find the sum of n terms of an arithmetic seuence in terms of its first term and the common difference.

Key terms

- Arithmetic series
- Summation notation (Σ)

Materials

- Scientific calculator

The sum of arithmetic series

The German scientist (Karl Gauss) had astonished his teacher when he was seven years old when he discovered the sum of adding the numbers from 1 to 100 mentally and in a fast way where he had noticed that the sum equals 50 pairs of the numbers which the sum of each is 101.

I.e.: $50 \times 101 = 5050$

Can you find the sum of the numbers from 1 to 20 mentally?

German scientist
Karl -Gauss
1777 - 1855

Definition

Arithmetic series

The arithmetic series is the operation of adding the terms of an arithmetic sequence.

For example: the sum of the first five terms of arithmetic sequence (3 , 5 , 7 , 9 , 11) is written as $S_5 = 3 + 5 + 7 + 9 + 11$

Sum of first n terms of an arithmetic series

First: finding the sum of first n terms of an arithmetic series in terms of its first and last terms.

If we have an arithmetic series of first term a, common difference d, last term ℓ and number of terms n, then the sum of n terms of this series is denoted by the symbol S_n where:

$$s_n = a + (a + d) + (a + 2d) + ... + (\ell - d) + \ell \qquad (1)$$

This sum can be expressed in another form as follows:

$$s_n = \ell + (\ell - d) + (\ell - 2d) + ... + (a + d) + a \qquad (2)$$

By adding the two equations (1) , (2) we deduce:

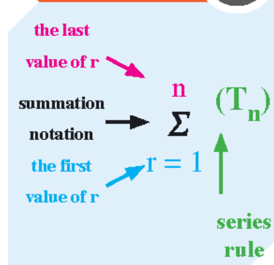
$$2s_n = (a + \ell) + (a + \ell) + (a + \ell) + ... + (a + \ell) \quad \text{to n times}$$

i.e. $2s_n = n\,(a + \ell)$ **by dividing the two sides by 2** $s_n = \frac{n}{2}(a + \ell)$

Example

1) **Using the summation notation Σ :** find $\sum_{r=5}^{24} (4r - 3)$

Remember



Solution

Because the expression inside the summation notation is of first degree, it represents an arithmetic sequence

$n = 24 - 5 + 1 = 20$ **Find the number of the sequence terms**

$T_n = 4n - 3$ **n^{th} term of the sequence**

$T_5 = 4 \times 5 - 3 = 17$, $T_{24} = 4 \times 24 - 3 = 93$

$s_n = \frac{n}{2}(a + \ell)$ **summation formula**

$s_{20} = \frac{20}{2}(17 + 93) = 1100$ **by substituting: $a = 17$, $\ell = 93$, $n = 20$**

Try to solve

1) Find:

a) $\sum_{k=1}^{20} (6k + 5)$

b) $\sum_{m=7}^{32} (12 - 5m)$

Example

2) Find the sum of the arithmetic series $2 + 5 + 8 + ... + 62$

Solution

$\ell = a + (n - 1)d$ **n^{th} term of the sequence**

$62 = 2 + (n - 1) \times 3$ **by substituting $a = 2$, $d = 3$, $\ell = 62$**

i.e.: $3n - 3 + 2 = 62$

$3n - 1 = 62$ **then $n = 21$**

$s_n = \frac{n}{2}(a + \ell)$ **summation formula**

$s_{21} = \frac{21}{2}(2 + 62) = 672$ **by substituting $a = 2$, $n = 21$, $T_n = 62$**

Second: Finding the sum of n terms of an arithmetic series in terms of its first term and the common difference

You know that $\ell = a + (n - 1)\, d$, $s_n = \frac{n}{2}(a + \ell)$

by substituting from the first relation in the second relation, then:

$s_n = \frac{n}{2}[\, a + a + (n - 1)\, d\,]$ **i.e.:** $s_n = \frac{n}{2}[2a + (n - 1)\, d]$

Example

(3) In the arithmetic series 5 + 8 + 11 + ... find :

- **a** The sum of the first twenty terms of the series.
- **b** The sum of ten terms starting from the seventh term.
- **c** The sum of the series terms starting from T_{10} to T_{20}

Solution

$a = 5$, $d = 8 - 5 = 3$

a $s_n = \frac{n}{2}[2a + (n - 1)\, d]$ **summation formula**

$s_{20} = \frac{20}{2} \times [2 \times 5 + (20 - 1) \times 3]$ **by substituting a = 5 and d = 8 - 5 = 3**

$s_n = 10\, (10 + 19 \times 3)$

$= 10 \times 67 = 670$ **by simplifying**

b $T_n = a + (n - 1)\, d$ **n^{th} term of the sequence**

$T_7 = a + 6d$

$= 5 + 6 \times 3 = 23$ **by substituting a = 5 , d = 3 and n = 7**

$s_{10} = \frac{10}{2} \times [2\, T_7 + (10 - 1) \times 3]$ **by substituting in summation formula**

$s_{10} = 5 \times [2 \times 23 + 27]$

$= 5 \times 73 = 365$ **by simplifying**

c The sum of the sequence terms starting from T_{10} to T_{20}

$T_n = a + (n - 1)\, d$ **n^{th} term of the sequence**

$T_{10} = a + 9\, d$

$= 5 + 9 \times 3 = 32$ **by substituting a = 5 , d = 3**

$T_{20} = a + 19\, d = 5 + 19 \times 3 = 62$

$s_n = \frac{n}{2}(a + \ell)$ **summation formula (n = 20 - 10 + 1 = 11)**

$s_{11} = \frac{11}{2}(T_{10} + T_{20})$

$= \frac{11}{2}(32 + 62) = 517$ **by substituting T_{10} = 32 , T_{20} = 62, n = 11**

Constructing the arithmetic sequence

Example

4 Find the arithmetic sequence in which: $T_1 = 11$, $T_n = 87$ and $s_n = 980$

Solution

a Finding the value of n

$s_n = \frac{n}{2}(a + \ell)$ — summation formula

$980 = \frac{n}{2}(11 + 87)$ — by substituting $T_1 = 11$, $T_n = 87$ and $S_n = 980$

$98 \times \frac{n}{2} = 980$ then: $n = 20$ terms — by simplifying $T_{20} = 87$

b Finding the value of d

$T_n = a + (n - 1)d$ — n^{th} term

$87 = 11 + 19d$ — by substituting $T_1 = 11$, $n = 20$ and $T_n = 87$

$19d = 87 - 11 = 76$ — by simplifying and by dividing by 19

$\therefore d = 4$

c Constructing the sequence: $T_2 = 11 + 4 = 15$, $T_3 = 15 + 4 = 19$

The arithmetic sequence is (11 , 15 , 19 , ……, 87)

Try to solve

2 Find the arithmetic sequence in which

a $T_1 = 23$, $T_n = 86$, $s_n = 545$

b $T_1 = 17$, $T_n = -95$, $s_n = -585$

Exercises 1 - 3

Complete:

1. The sum of the consecutive integers which starts from number 1 and ends in the number 20 equals ....

2. The sum of the first ten even numbers in the set of the natural numbers equals ....

3. The set of the odd natural numbers which is greater than 10 and less than 30 equals ....

4. The set of the natural numbers divisible by three and included between 30 and 50 equals ....

5. The sum of the first nine terms of the arithmetic sequence whose first term is 2 and last term is 18 is..........

6. $\sum_{k=1}^{5} (2k + 1) =$ ....

Choose the correct answer

7. The value of the arithmetic series $\sum_{r=1}^{4} (2r + 1)$ equals:
   a) 25 b) 30 c) 35 d) 24

8. The formula of the series: $4 + 9 + 14 + ... + 5n - 1$ by using the summation notation equals:
   a) $\sum_{r=4}^{n} (5r - 1)$ b) $\sum_{r=1}^{n} (5r - 1)$ c) $\sum_{r=1}^{n} (5r + 1)$ d) $\sum_{r=1}^{5n-1} (3r + 1)$

9. The formula of the series: $7 + 12 + 17 + 22$ by using the summation notation equals:
   a) $\sum_{r=1}^{4} (5r + 2)$ b) $\sum_{r=1}^{4} (4r + 3)$ c) $\sum_{r=1}^{4} (7r + 1)$ d) $\sum_{r=1}^{4} (3r + 4)$

Answer the following questions

10. Find the sum of first ten terms of the arithmetic sequence (14 , 18 , 22 , ...)

11. Find the sum of first fifteen terms of the arithmetic sequence whose first term is 4 and the fifteenth term is 26 .

12. Find the sum of the even numbers from 2 up to 40 .

13. Find the sum of the first 20 terms of the arithmetic series $(6 + 4 + 2 + ...)$.

14. Find the sum of first thirty terms of the sequence (T_n) where $T_n = (2n + 3)$

15. Find the sun of the terms of the arithmetic sequence (2 , 5 , 8 , ... , 80).

16. Find the number of the terms necessary to be taken off from the sequence (16 , 20 , 24 , ...) starting from the first term to get a sum equal to 456.

17. How many terms are needed to be taken off from the sequence (- 16 , - 14 , -12 , ...) starting from its first term to get a sum equal to zero.

18. Find the number of the terms necessary to be taken off from the sequence (27 , 24, 21 , ...) starting from the first term to vanish the sum.

19. **Saves:** Zyad saves 15 L.E from his daily work. If he saves an amount increases for 2 L.E every day more than the day before directly, find how much he can save within 15 days.

Unit one

1 - 4 Geometric sequence

You will learn

- Definition of the geometric sequence
- Graphical representation of the geometric sequence
- n^{th} term of the geometric sequence
- Identifying the geometric sequence
- Geometric means
- The relation between an arithmetic mean and a geometric mean of two numbers

Key terms

- Geometric Sequence
- n^{th} Term
- Increasing Sequence
- Decreasing Seqence
- Alternating signal Sequence
- Geometrical Mean

Materials

- Scientific calculator
- Graphic program

Activity

(1) On cardboard, draw an isosceles right - angled triangle.

(2) Cut the constructed triangle into two isosceles right- angled triangles.

(3) Repeat the work as shown below and find the number of the triangles resulted each time .

(4) Answer the following question :

a Does the number of the resulted small triangles construct an arithmetic sequence? Explain.

b Is there a relation relating the numbers of the resulted sequence? What is this relation?

c Can you find the number of the triangles resulted in fifth and sixth figures by repeating the pattern above?

From the previous activity, we deduce that :

The sequence resulted from the previous figures is(1 , 2 ,4 , 8 , ...). It is not an arithmetic sequence because $T_{r+1} - T_r \neq$ a constant, but we notice that if any term is divided by the directly previous term, it gives a constant (it is the number 2) . This sequence is called a geometric sequence.

Definition 1

➢ The sequence (T_n) where $T_n \neq 0$ is called a geometric sequence If $\frac{T_{n+1}}{T_n}$ = a constant for each $n \in Z_+$ and the constant is called the common ratio of the sequence and is denoted by the symbol(r).

Example

1 Show which of the following sequences (T_n) is geometric , then find the common ratio of each:

a $T_n = 2 \times 3^n$ **b** $T_n = 4\,n^2$

c The sequence (T_n) where : $T_1 = 12$, $T_n = \frac{1}{4} \times T_{n-1}$ (where $n > 1$)

Solution

a $\because \frac{T_{n+1}}{T_n} = \frac{2 \times 3^{n+1}}{2 \times 3^n} = 3^{n+1-n} = 3$ **(constant)**

$\therefore$ the sequence is geometric and its common ratio r = 3

b $\because \frac{T_{n+1}}{T_n} = \frac{4(n+1)^2}{4n^2}$ **(is not constant)**

$\therefore$ The sequence is not geometric

c $\because (T_n) = (\frac{1}{4} \times T_{n-1})$ (where n > 1)

$\therefore \frac{T_n}{T_{n-1}} = \frac{1}{4}$ **(constant)**

$\therefore$ The sequence is geometric and its common ratio $r = \frac{1}{4}$

Try to solve

1 Tell which of the following sequences is geometric , then find its common ratio in case it is a geometric sequence :

a $(T_n) = (96, 48, 24, 12, 6, 3)$ **b** $(T_n) = (\frac{1}{243}, \frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \frac{1}{3})$

c $(T_n) = (5 \times 2^n)$ **d** $(T_n) = (3(n+1)^2)$

Finding the n^{th} term of a geometrical sequence

From definition (1), the n^{th} term of the geometric sequence (T_n) whose first term is a and common ratio is r can be deduced as follows:

$T_1 = a$, $T_2 = ar$, $T_3 = ar^2$ and $T_4 = ar^3$

By continuing this pattern, we find that the n^{th} term of this sequence is: : $T_n = ar^{n-1}$

Example

2 In the geometric sequence (2 , 4 , 8 , ...), find:

first : The fifth term. **second:** the order of the term whose value is 512.

Solution

$\because$ a = 2 , $r = \frac{4}{2} = 2$, $T_n = a \times (2)^{n-1}$

$\therefore T_5 = ar^4 = 2 \times 2^4 = 2 \times 16 = 32$ **i.e. the value of the fifth term is 32**

$\because T_n = a \times r^{n-1}$ $\therefore 2 \times 2^{n-1} = 512$ **By dividing the two sides by 2**

$\therefore 2^{n-1} = 2^8$ $\therefore n - 1 = 8$ $\therefore n = 9$

i.e. the term whose value is 512 is the ninth term.

Try to solve

2 Prove that the sequence (T_n) where $T_n = 2 \times 3^{n-5}$ is a geometric sequence, then find its seventh term.

Identifying the geometrical sequence

The geometric sequence can be identified whenever its first term and common ratio are known (given).

Example

3 (T_n) is a geometric sequence, if $T_4 = 40$ and $T_7 = 320$, find this sequence.

Solution

$\because T_4 = a r^3$ $\therefore a r^3 = 40$ .... (1)

$\because T_7 = a r^6$ $\therefore a r^6 = 320$ .... (2)

By dividing the two sides of the equations (1) and (2)

$\therefore \frac{a r^6}{a r^3} = \frac{320}{40}$ (where $a r \neq 0$) $\therefore r^3 = 8$

$\therefore r = 2$

By substituting in equation (1)

$\therefore a (2)^3 = 40$ i.e. : $8 a = 40$

By dividing the two sides by 8, then $a = 5$ $\therefore$ the sequence is (5 , 10 , 20 , ...)

Verbal expression :

What do you expect if the power of the common ratio r is an even number? Explain .

Using the scientific calculator to write the geometric sequence:

To write down the geometric sequence in which $a = 5$ and $r = 2$, we do the following:
We write the value of a (number 5), then press [=] then press [×] and put the value of r (number 2), then press [=] then we get the second term of the sequence. By pressing [=] repeatedly, we get the next terms and so on....

Try to solve

3 (T_n) is a geometric sequence in which $T_3 = 12$ and $T_8 = 384$. find this sequence

4 Find the geometric sequence whose terms are positive, its second term equals 6 and its tenth term equals 1536 .

Example

Education:

4 The increasing rate of second secondary grade in an educational administration is 4 % yearly more than the directly previous year. How many students will be there after 6 years if the number of the students is 2400 students right now?

Solution

$\because$ Number of students now = 2400

$\therefore$ Number of students in the second year $= 2400 + 2400 \times 4\ \%$

$= 2400\,(1 + 0.04)$

$= 2400\,(1.04)$

Number of students in the third year $= 2400\,(1.04) + 2400(1.04) \times 0.04$

$= 2400\,(1.04)\,(1 + 0.04) = 2400(1.04)^2$ and so on ...

Remember

$4\% = \frac{4}{100} = 0.04$

i.e. the numbers of students construct a geometric sequence

$(2400\ ,\ 2400(1.04)\ ,\ 2400(1.04)^2, \ldots)$

$a = 2400 \quad , \quad r = 1.04 \quad , \quad n = 6$

by substituting in the rule of n^{th} term of the geometric sequence $T_n = a \times r^{n-1}$

$T_n = (2400) \times (1.04)^5 = 2919.966966$

i.e. the number of students after 6 years equals 2920 students approximately .

Geometric Means

The geometric means are similar to the arithmetic means. They are the terms located between two non- successive terms in the geometric sequence . the common ratio of the geometric sequence is used to find these means.

Definition 2

- **If a, b and c are three successive terms of a geometric sequence, then b is known as the geometric mean between the two numbers a and c where: $\frac{b}{a} = \frac{c}{b}$ i.e. $b^2 = a\,c$, then $b = \pm\sqrt{a\,c}$**

Tip

the geometric mean of a set of positive real values T_1, T_2, T_3 ,

Finding the geometric means :

Example

5 Insert 5 geometric means between 4 and 2916

Solution

First : Find the number of the sequence terms

There are five means between the first and last terms in the geometric sequence

Thus , the number of the terms of the sequence is $n = 2 + 5 = 7$

Second : Find the value of d

The n^{th} term of the arithmetic sequence : $T_n = a\,r^{n-1}$

By substituting : $a = 4$, $T_n = 2916$, $n = 7$

$2916 = 4 \times r^{7-1}$ i.e. : $4 \times r^6 = 2916$

by dividing the two sides by 4 : $r^6 = 729$ i.e. : $r^6 = (\pm 3)^6$ then $r = \pm 3$

Third : Use the value of r to find the geometric means required :

(4) , 12 , 36 , 108 , 324 , 972 , (2916) or

×3 , ×3 , ×3 , ×3 , ×3 , ×3

(4) , -12 , 36 , -108 , 324 , -972 , (2916)

× -3 , × -3 , × -3 , × -3 , × -3 , × -3

the required means are: 12 , 36 , 108 , 324 , 972 or - 12 , 36 , - 108 , 324 , - 972

Critical thinking: What do you expect about the relation between the arithmetic and geometric means of two equal positive real numbers?

Example

6 If you insert four geometric means between two numbers and the sum of the first and fourth means equals 90 and the sum of the second and third means equals 60, find the two numbers.

Solution

$\because$ Number of means =4 $\therefore$ number of sequence terms = 4 + 2 = 6

$\therefore$ let the first number = a, then the first and fourth terms are T_2 and T_5

$\therefore T_2 + T_5 = 90$ $\therefore a\,r + a\,r^4 = 90$

$\therefore a\,r\,(1 + r^3) = 90$ **(1)**

$\because$ the second and third terms are T_2 and T_4

$\therefore a\,r^2 + a\,r^3 = 60$ $\therefore a\,r^2\,(1 + r) = 60$ **(2)**

By dividing **(1)** , **(2)** $\dfrac{ar(1+r)(a - r + r^2)}{ar^2(1+r)} = \dfrac{3}{2}$

$\therefore 2r^2 - 2r + 2 = 3r$ $\therefore 2r^2 - 5r + 2 = 0$ $\therefore (r - 2)(2r - 1) = 0$

$\therefore r = 2$ or $r = \frac{1}{2}$

By substituting $r = 2$ $\therefore a = 5$

By substituting $r = \frac{1}{2}$ $\therefore a = 160$ the two numbers are: 160 , 5

Exercises 1 - 4

Determine the geometric sequences in the following , then find the common ratio in case the sequence is geometric:

1. (1 , 4 , 9 , 16 , ...)
2. (243 , 81 , 27 , 9 , ...)
3. $(\frac{1}{128}, \frac{1}{64}, \frac{1}{32}, \frac{1}{16}, ...)$
4. (- 1 , 3 , - 9 , - 27 , ...)

Write down the first five terms of the geometric sequence, if known that :

5. a = 2 , r = 4
6. a = -4 , r = 2
7. a = 1 , $r = \frac{-1}{2}$
8. a = -128 , $r = \frac{-1}{2}$

Complete

9. The seventh term of the geometric sequence (64 , 32 , 16 , ...) equals ....
10. The sixth term of the geometric sequence $(\frac{1}{243}, \frac{1}{81}, \frac{1}{27}, ...)$ is ....
11. The fifth term of the sequence (T_n) where $T_n = 2 \times (3)^{n-1}$ equals ....
12. The n^{th} term of the geometric sequence (3 , - 6 , 12 , ...) is ....
13. The geometric mean of the two numbers 4 and 16 is ....
14. If the geometric mean of the two numbers 9 and y is 15 , then y equals ....
15. If a , b and c are three consecutive and positive terms of a geometric sequence, then b = ...

Choose the correct answer:

16. The fifth term of the geometric sequence $(8, 6, \frac{9}{2}, ...)$ is :
    a) $\frac{27}{8}$ b) $\frac{27}{16}$ c) $\frac{9}{4}$ d) $\frac{81}{32}$
17. All the following sequence are geometric except :
    a) (3 , -6 , 12 , -24 , ...)
    b) (-2, 4, 10, 16, ...)
    c) $(\frac{3}{2}, 1, \frac{2}{3}, \frac{4}{9})$
    d) $(\frac{3b}{a}, 6, \frac{12a}{b}, \frac{24a^2}{b^2}, ...)$ where a > 0 , b > 0

Answer the following questions:

18. If (T_n) is a sequence where $T_n = 5 \times 2^n$, prove that it is a geometric sequence, then write its first three terms.
19. In the geometric sequence $\left(\frac{1}{8}, -\frac{1}{4}, \frac{1}{2}, -1, ...\right)$, find :
    a) Its tenth term b) The order of the term whose value = - 1024 .
20. Show that the sequence (T_n) where $T_n = \frac{3}{8}(2)^n$ is a geometric sequence , then find its eighth term and the order of the term whose value is 768.
21. Find the geometric sequence whose common ratio = $\frac{1}{2}$ and its third term = 24 .

22. Find the geometric sequence whose first term = 9 and its sixth term = 288 .

23. Find the geometric sequence (T_n) in which $T_3 = 12$ and $T_8 = 384$.

24. Find the geometric sequence whose third term = 18 , and its sixth term = 486

25. Find the geometric sequence (T_n) in which $T_2 = 10$ and $T_6 = 160$.

26. Find the geometric mean between 16 and 49

Unit one

Geometric Series 1 - 5

You have previously learned that the series is the sum of a sequence terms and you also learned how to find the sum of an arithmetic series. Can you find the sum of the following geometric series?

95 + 285 + 855 + ... 1869885. Notice that it is quite difficult to find the sum using the traditional way. As a result, we are in need to a rule to find this sum easily and quickly. This is what we are going to know now .

The sum of geometric series

The geometric series is the sum of the terms of the geometric sequence between which the addition sign (+) is placed and the sum of its terms is denoted by the symbol S.

Sum of first n terms of a geometric sequence

First : Finding the sum of n terms of a geometric series in terms of its first term and the common ratio

If $a + ar + ar^2 + ... + ar^{n-1}$ is a geometric series whose first term is a and common ratio is r, then the sum S_n of this series can be found as follows :

$s_n = a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1}$ .... (1)

By multiplying the two sides by r, then :

$rs_n = ar + ar^2 + ar^3 + ...+ ar^{n-1} + ar^n$ .... (2)

By subtracting the two equations :

$s_n - rs_n = a - ar^n$ i.e. :

$s_n (1 - r) = a(1 - r^n)$

By dividing the two sides by (1- r) in a condition $(1 - r) \neq 0$

$$s_n = \frac{a(1 - r^n)}{1 - r}, \quad r \neq 1$$

Example

1 Find the sum of the geometric series in which : a = 3 , r = 2 and n = 8

You will learn

- Sum of the geometric series.
- Using summation notation
- Infinite geometric series
- Sum of infinite geometric series.
- Converting the recurring decimal into an irrational number .

Key terms

- Geometric Series
- Infinite Geometric Series

Materials

- Scientific calculator
- Graphical programs

Solution

the sum of the geometric sequence : $s_n = \frac{a(1 - r^n)}{1 - r}$

by substituting : a = 3 , r = 2 , n = 8

$s_8 = \frac{3(1 - 2^8)}{1 - 2}$ by simplifying $s_8 = 3 \times 255 = 765$

Second: finding the sum of n terms of a geometric series in terms of its first and last terms

We know that : $s_n = \frac{a - ar^n}{1 - r}$ .... (1)

and : $\ell = a r^{n-1}$ by multiplying the two sides by r then $\ell r = a r^n$ ...... (2)

by substituting from (2) in (1), then : $s_n = \frac{a - \ell r}{1 - r}$, $r \neq 1$

Example

(2) Find the sum of the geometric series: 1 + 3 + 9 + ... + 6561

Solution

The sum formula of the geometric sequence : $s_n = \frac{a - \ell r}{1 - r}$

By substituting : a = 1 , r = 3 , ℓ = 6561

$s = \frac{1 - 6561 \times 3}{1 - 3}$ by simplifying $s = \frac{19682}{2} = 9841$

Using the Summation Notation.

Example

(3) Find : $\sum_{r=5}^{12} 3(2)^{r-1}$

Solution

As the expression inside the summation notation is in the exponential form, it represents a geometric sequence

$T_5 = a = 3(2)^{5-1} = 48$, r = 2 , n = 12 – 5 + 1 = 8

the sum of the geometric series : $s_n = \frac{a(1 - r^n)}{1 - r}$

by substituting : a = 48 , r = 2 , n = 8

$s_8 = \frac{48(1 - 2^8)}{1 - 2}$ **by simplifying** $s_8 = 48 \times 255 = 12240$

Example

Forming the geometric sequence

(4) Find the geometric sequence whose first term = 243, last term = 1 , and the sum of its terms is 364.

Solution

$\because$ a = 243 , ℓ = 1 , s_n = 364 , $s_n = \frac{a - \ell r}{1 - r}$

$\therefore\ 364 = \dfrac{243 - r}{1 - r}$ $\therefore\ 364(1 - r) = 243 - r$

$\therefore\ 364 - 364r = 243 - r$ $\therefore 364r - r = 364 - 243$

$\therefore\ 363r = 121$ by dividing the two sides by 363 $\therefore\ r = \frac{1}{3}$

The geometric sequence is (243 , 81 , 27 , ...)

Try to solve

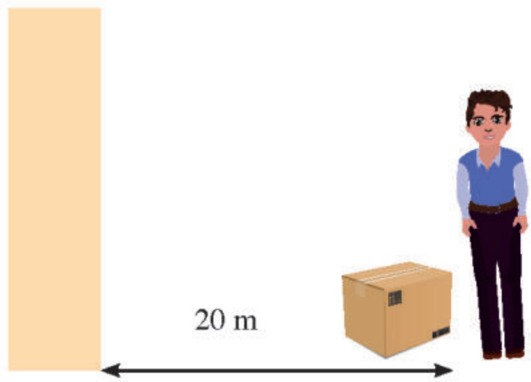
1 Find the geometric sequence whose first term is 243, last term is 1, and the sum of its terms is 364.

Infinite geometric series

Think and discuss

Zyad has wanted to move a box in the direction of a wall distant 20 m from him over some stages so that the distance travelled by the box equals half the remaining distance after each stage. Can Zyad reach the wall?

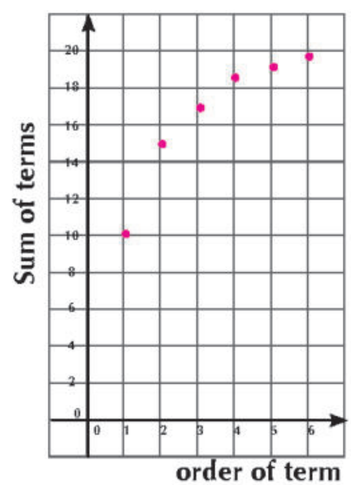
You can answer this question through learning the infinite geometric series .



Definition 2

The infinite geometric series has an infinite number of terms. If their sum is a real number, the series is convergent because its sum gets near to a real number . If the series doesnot have a sum , it is divergent.

In Think and discuss, the sum of the distances traveled by Zyad is given by the series 10 + 5 + 2.5 +1.25 + ... the more its terms increase, the nearer its sum gets to 20 m. It is the real sum of the series. As a result , we can assume that Zyad will reach the wall when the terms of the sequence increase infinitely. The figure opposite illustrates the graphical representation of the sum of S_n , so the sum of the convergent series gets near to a real number where $|r| < 1$ and the series is divergent if the sum does not get near to a real number where $|r| \geqslant 1$



Example **Convergent and divergent series**

5 Which of the following geometric series can an infinite number of its terms be summed? Explain.

a 75 + 45 + 27 + ... b 24 + 36 + 54 + ...

Solution

a Find the common ratio of the geometric series $r = \frac{45}{75} = \frac{3}{5}$, then an infinite numbers of terms of the series can be summed because $-1 < \frac{3}{5} < 1$

b Find the common ratio of the geometric series $r = \frac{36}{24} = \frac{3}{2}$, then the infinite number of terms of the series can not be summed because: $\frac{3}{2} > 1$

The sum of the infinite geometric series

We knew that the sum of n terms of the geometric series is given by the relation $S_n = \frac{a(1 - r^n)}{1 - r}$

and when we sum an infinite number of its terms, then r^n gets near to zero when $-1 < r < 1$

and the sum became : $s_\infty = \frac{a}{1 - r}$

Example

6 Find the sum for each of the following two geometric series if found :

a $\frac{81}{8} + \frac{27}{4} + \frac{9}{2} + ...$

b $\frac{2}{3} + \frac{5}{6} + \frac{25}{24} + ...$

Solution

a Find the common ratio of the geometric sequence : $r = \frac{27}{4} \div \frac{81}{8} = \frac{27}{4} \times \frac{8}{81} = \frac{2}{3}$

$\because -1 < \frac{2}{3} < 1$ $\therefore$ The series has a sum

$\because a = \frac{81}{8}$, $r = \frac{2}{3}$ **by substituting in the sum formula** $s_\infty = \frac{a}{1 - r}$

$\therefore s_\infty = \frac{\frac{81}{8}}{1 - \frac{2}{3}} = \frac{\frac{81}{8}}{\frac{1}{3}} = \frac{243}{8}$

b Find the common ratio of the geometric sequence : $r = \frac{5}{6} \div \frac{2}{3} = \frac{5}{6} \times \frac{3}{2} = \frac{5}{4}$

$\because \frac{5}{4} > 1$ $\therefore$ The series is divergent and has no sum

Try to solve

7 Find the sum for each of the following two geometric series if found :

a $12 - 24 + 48 - 96$ ....

b $\frac{7}{5} + \frac{21}{10} + \frac{63}{20} + ...$

Example **Using the summation notation**

7 Find $\sum_{r=1}^{\infty} 42(\frac{6}{7})^{r-1}$

Solution

Sum of the geometric sequence : $s_\infty = \frac{a}{1 - r}$

by substituting : $a = 42$ and $r = \frac{6}{7}$: $s_\infty = \frac{42}{1 - \frac{6}{7}} = 294$

Exercises 1 - 5

Choose the correct answer from those given:

1. The sum of the first five terms of the geometric sequence in which $a = 1$ and $r = 2$ equals :
   - a) 32
   - b) 31
   - c) 30
   - d) 29

2. The sum of an infinite number of terms of the sequence (4 , 2 , 1 , ...) is :
   - a) 8
   - b) 12
   - c) 16
   - d) 20

3. If the sum of an infinite number of terms of a geometric sequence whose common ratio is $\frac{1}{4}$ equals 4 , then its first term equals :
   - a) 1
   - b) 2
   - c) 3
   - d) 4

4. If the sum of an infinite number of terms of the geometric sequence whose first term is 12 equals 96, then its common ratio equals :
   - a) $\frac{1}{3}$
   - b) $\frac{1}{2}$
   - c) $\frac{7}{8}$
   - d) $\frac{3}{4}$

5. If the first term of a geometric sequence equals the sum of the next terms to infinity, then the common ratio of this sequence equals :
   - a) 0.5
   - b) 0.333
   - c) 0.25
   - d) 0.666

Answer the following questions:

6. Find the sum for each of the following geometric sequences :
   - a) (6 , 12 , 24 , ... to 6 terms)
   - b) (125 , 25 , 5 , ... to 6 terms)
   - c) (3 , - 6 , 12 , ... , 768)

7. Which of the following geometric sequences can be summed up to ∞, then find the sum if possible :
   - a) (24 , 12 , 6 , ...)
   - b) (3 , - 6 , 12 , ...)
   - c) $(\frac{1}{32}, \frac{1}{16}, \frac{1}{8}$ ...)
   - d) $(2 \times 5^{1-n})$

8. Find the sum of an infinite number of terms for each of the following geometric sequence :
   - a) (32 , 16 , 8 , ...)
   - b) $(\frac{81}{16}, \frac{27}{8}, \frac{9}{4}$ .....)
   - c) $(2, \sqrt{2}, 1,$ .....)
   - d) $(\frac{27}{4}, \frac{27}{16}, \frac{27}{64}$ .....)
   - d) $(T_n) = (3^{3-n})$
   - e) $(T_n) = (\frac{2}{3})^{n-1}$

9. Find the geometric sequence whose first term = 243 , last term = 1 , and the sum of its terms = 364

10. Find the geometric sequence whose sum is 1093, last term is 729 and its common ratio is 3

(11) Find the sum of the terms of the geometric sequence $(T_n) = (3^{n-1})$ starting from its fourth term up to tenth term.

(12) In the geometric sequence (1 , 3 , 9 , ...), find the least number of terms that should be taken off starting from its first term to make the sum of these terms greater than 1000

(13) Find the geometric sequence whose sum of an infinite number of its terms equals 48 and its second term equals 12 .

(14) How many terms need to be taken off from the geometric sequence (3 , 6 , 12 , ...) starting from its first term to make the sum of these terms = 381.

(15) Prove that the sequence $(T_n) = (10 \times 2^{n-2})$ is a geometric sequence , then find the number of the terms starting from the first term to make the sum 2555 .

(16) (T_n) is a geometric sequence whose terms are positive $T_2 = 6$, $T_3 - T_1 = 9$. find the sequence and sum of the first twelve terms .

(17) Find the geometric sequence whose terms are positive , the sum of its first four terms = 45 , and its sixth term is greater than its second term by 90 .

(18) If the first term of a geometric sequence of an infinite number of terms = 18 , and its fourth term = $\frac{16}{3}$, find its sum.

(19) Find the geometric sequence whose sum of its first and second terms = 16 and the sum of an infinite number of its terms = 25

(20) An infinite geometric sequence whose first term = the sum of next terms to infinity and the sum of the first and second terms = 9. Find the sequence .

(21) An infinite geometric sequence in which any term of it = twice the sum of the next terms to infinity. If its fourth term = 3 , find this sequence.

(22) **Income:** A worker has started to work at a factory for a yearly income of 7200 L.E so that he got a yearly raise of 0.6 % of the directly previous year. Calculate his income in the seventh year and the sum of what he got within the first seven years. .

(23) **Income :** An employee has started to work for a yearly income of 3600 and this income increases each year at a rate of $\frac{1}{20}$ more than the directly previous year. How much will he have after 11 years? What is the sum of money which this employee receives during this period of time?

(24) **Saving :** At the age of 6, Kreem could save 150 L.E. in the year and in each directly next year, he could save twice of what he saved in the directly previous year. He is 10 years-old now and wants to buy a computer for 5000 L.E. Is the sum of what he saved during this period enough to buy this computer? Explain .

1. The arithmetic series: $3 + 7 + 11 ...... + 35$ is written using the summation notation as follows:

   a. $\sum_{r=1}^{8} (4r - 1)$ b. $\sum_{r=1}^{9} (4r - 1)$ c. $\sum_{r=1}^{9} (3r - 4)$ d. $\sum_{r=1}^{8} (3r - 4)$

2. All the following sequences are arithmetic except :

   a. (11 , 15 , 19 , ...) b. (- 23 , - 28 , - 33 , ...)

   c. $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...)$ d. $(\frac{32}{5}, \frac{26}{5}, \frac{20}{5}, ...)$

3. If $2a + 2$, $6a - 2$ and $7a$ are three consecutive terms of an arithmetic sequence , then a equals :

   a. 1 b. 2 c. 3 d. 4

4. The sum of the arithmetic series $\sum_{r=3}^{8} (2r + 1)$ equals :

   a. 64 b. 72 c. 76 d. 80

5. The sum of the first n terms of a geometric sequence is given by the relation $S_n = (3^n - 1)$, then its fifth term equals::

   a. 162 b. 243 c. 81 d. 729

6. If the sum of the second and fourth terms of an arithmetic sequence equals 2 and the sum of the sixth, seventh and eighth terms equals- 45, find this sequence .

7. Three numbers are arithmetically sequent whose sum equals 12 and their product = 48, find these numbers .

8. Three numbers form an arithmetic sequence whose sum equals 27 and the square of its second term is greater than the product of the first and third terms by 16, find these numbers.

9. If several arithmetic means are inserted between 3 and 53 and the ratio between the sum of the first two means to the sum of the last two means is 3 : 13 , find these means .

10. In the arithmetic sequence (9 , 12 , 15 , ...). Find:

    a. Sum of first 15 terms.

    b. Sum of terms of the sequence starting from the fifth term up to the fifteenth term

    c. Number of terms whose sum = 750 starting from the first term.

11. In the geometric sequence (2 , 6 , 18 , ...), find the least number of the terms that must be taken off starting from the first term to make the sum of these terms greater than 6500.

12. (T_n) is a geometric sequence of positive terms in which $T_6 = \frac{1}{9} T_4$, $T_3 + T_6 = 28$ find the sequence , then find the sum of its first ten terms.

13. A geometric sequence of positive terms whose sum of the first four terms = 60 , and the sum of the next four terms = 960 . Find this sequence , then find the sum of seven terms starting from its third term.

Unit Two

Permutations, Combinations

Unit introduction

Counting is an important part of the basic skills in mathematics. We regularly face a lot of problems that need to be solved. We need to do counting operations in deferent ways to solve them. In this unit, we are going to identify different strategies for counting such as the fundamental counting principle and the most important applications of it:

Permutations. They are used to know the number of methods used to order the elements of a set with all possible methods.

Combinations. They mean to choose disregarding the order.

Scientists such as Omar Alkhaiam, Isac Newton and pascal had played a great role in this field which is still ongoing up to day.

Unit objectives

By the end of the unit and doing the activities involved , the student should be able to:

- Identify the counting principle and simple applications on it.
- Identify an introduction about the permutation and combination and the relation between them .
- Use the computer to calculate each of permutations and combinations.

Key - terms

- Fundamental Counting Principle
- Conditional Counting Principle
- Operation
- Tree Diagram
- Factorial
- Permutations
- Combinations
- Elements
- Order
- Committee
- Subset

Unit lessons

Lesson (2 - 1): Counting Principle .

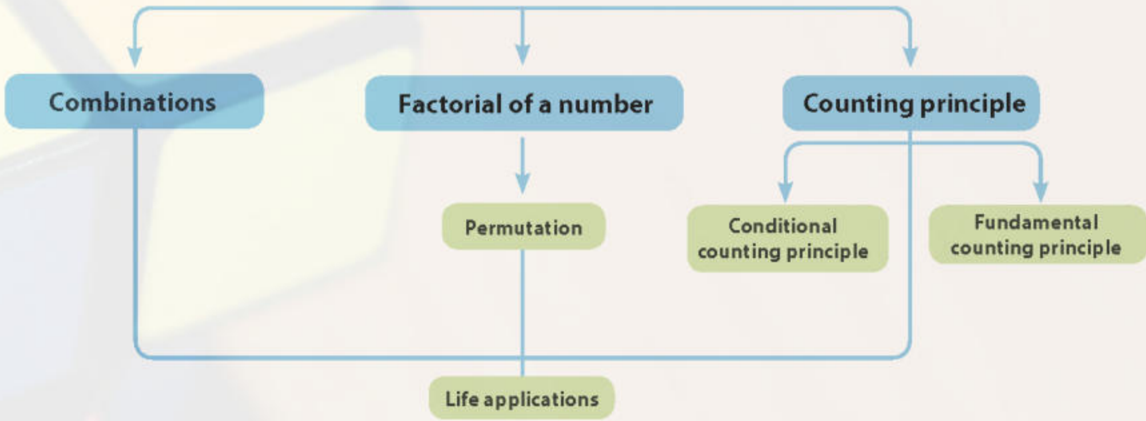
Lesson (2 - 2): Factorial of a number - permutations.

Lesson (2 - 3): Combinations

Materials

Scientific calculator - computer - graphic programs

Unit planning guide



Unit Two

2 - 1 Counting Principle

We will learn

- The concept of counting principle and simple applications on it.
- Fundamental counting principle.
- Conditional counting principle.

Key - term

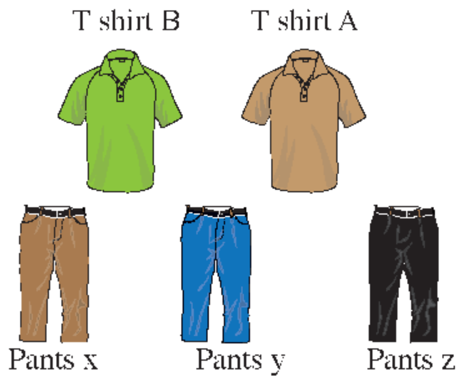
- Fundamental Counting Principle
- Operation
- Tree Diagram

Matrials

- Scientific Calculator
- Graphic program

Think and discuss

How many ways are there to choose, if you are asked to wear a t-shirt and a pair of pants out of 2 t-shirts and 3 pairs of pants?



Example

1. How many ways are there to choose a male student out of three students (Ashraf - Mohamed - Hassan) and a female student out of two students (Samar - Mona)?

Solution

In this example, we find that it is easy to know the number of ways of choosing. For example, we can choose Ashraf, Samar or Ashraf, Mona or Mohamed, Mona or Hassan, Samar .. etc. We are going to express that by the following graphical diagram called tree diagram:

Male students	Female students	Choice
Ashraf	Samar	Ashraf، Samar
	Mona	Ashraf، Mona
Mohamed	Samar	Mohamed، Samar
	Mona	Mohamed، Mona
Hassan	Samar	Hassan، Samar
	Mona	Hassan، Mona

The number of ways to choose a male student out of three students = 3 ways.

The number of ways to choose a female student out of two students = 2 ways

∴ Number of ways of choosing = $3 \times 2 = 6$ ways

Try to solve

1. In **think and discuss** , How many possible ways of choosing are there?

Example

2. How many three- digit numbers can be formed so that the unit digit is from the elements of {3, 7} the tens digit is from the elements of {2, 4, 9} and the hundreds digit is from the elements of {1, 5} ?

Solution

Unit digit	tens digit	hundreds digit	Number
3	2	1	123
		5	523
	4	1	143
		5	543
	9	1	193
		5	593
7	2	1	127
		5	527
	4	1	147
		5	547
	9	1	197
		5	597

From the tree diagram, we find that:

The number of ways to choose the unit digit × number of ways to choose tens digit × number of ways to choose hundreds digit $= 2 \times 3 \times 2 = 12$ ways

The previous examples support the following definition:

Learn

Fundamental Counting Principle

Definition: If the number of ways to perform a certain task is m_1 way , the number of ways to perform another certain task is m_2 way and the number of ways to perform a third certain task is m_3 way and so on ..., then the number of ways to perform these tasks together $= m_1 \times m_2 \times m_3 \times ... \times m_n$

Example

3. How many ways can khaled have a meal out of three meals (Liver - chicken - fish) and a beverage out of three beve rages (orange - lemon - Mango)?

Solution

Number of ways to choose a meal $= 3$ ways and the number of ways to choose a drink $= 3$ ways.

The total number of ways to choose $= 3 \times 3 = 9$ ways.

Try to solve

2. A restaurant presents 6 types of pies, 4 types of salads and 3 types of beverages. How many meals can the restaurant present daily so that a meal includes a type from each of pies, salads and beverages?

Example **Conditional Counting Principle**

4. How many different three -digit numbers can be formed from the numbers $\{0 , 1 , 2 , 3 , 4\}$?

Solution

Start with the conditional hundreds digit (Zero can't be used left side)

Number of ways of choosing the digit in hundreds digit = 4

Number of ways of choosing the digit in tens digit = 4

Number of ways of choosing the digit in unit digit = 3

place	ones	tens	hundreds
No of ways	3	4	4

∴ The total number of ways of choosing = $4 \times 4 \times 3 = 48$ ways

Try to solve

3. How many ways can a different four -digit number be formed from the numbers {2, 3, 4, 7}, so that the tens digit is even.

Exercises (2 - 1)

Choose the correct answer:

1. The number of ways of sitting 4 students on 4 seats in a row equals :
   - a) 1
   - b) $4 + 4$
   - c) 4×4
   - d) $4 \times 3 \times 2 \times 1$

2. The number of the different two - digit numbers taken off from the numbers { 5, 3, 0, 2} equals:
   - a) 3×2
   - b) 4×2
   - c) 3×3
   - d) 3×4

3. The number of the different odd three-digit numbers taken off from the numbers {2 , 3, 6, 8} equals:
   - a) $8 \times 6 \times 3$
   - b) $4 \times 3 \times 3$
   - c) $4 \times 3 \times 2$
   - d) $2 \times 3 \times 1$

4. How many three -digit numbers can be formed from the elements {2, 3, 5}?

5. How many different four - digit numbers can be formed taken off from the elements {2, 3, 6, 8} so that the unit digit is 6?

6. The licence plates of cars in a governorate start with three letters followed by three digits except Zero. How many plates can be got assuming that there is no repetition for any letter or digit in the licence plates?

7. How many different three-digit numbers can be taken off from the numbers {2, 5, 8, 9} so that these numbers are less than 900?

8. If you know that the set of the numbers of mobile networks in a country is made up of an eleven-digit number. If the number (025) is constant on left side, find the greatest number of phone lines which the mobile network can stand .

Factorial of a Number - Permutations

Think and discuss

Use what you learned in the previous lesson to answer the following questions:

1) How many ways can 4 students sit on three seats in a row?

2) How many ways can 5 racers stand on the edge of a swimming pool to jump?

Learn

Definition 1

Factorial: The factorial of a positive integer n is written as $\lfloor n$ and equals the product of all the positive integers which are lesser than or equal n where:

$$\lfloor n = n\,(n-1)\,(n-2)\ldots 3 \times 2 \times 1$$

Notice that

- When n = 0 **then** $\lfloor 0 = 1$
- When n = 1 **then** $\lfloor 1 = 1$
- $\lfloor 4 = 4 \times 3 \times 2 \times 1 = 4\lfloor 3$,
  $\lfloor 6 = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6\lfloor 5$

In general: $\lfloor n = n\lfloor n-1$ where $n \in Z^+$

Example

1) a) Find $\dfrac{\lfloor 10}{\lfloor 8}$ b) If $\lfloor n = 120$ find the value of n

Solution

a) $\dfrac{\lfloor 10}{\lfloor 8} = \dfrac{10 \times 9 \lfloor 8}{\lfloor 8} = 10 \times 9 = 90$

b) $\lfloor n = 5 \times 4 \times 3 \times 2 \times 1$ $\therefore \lfloor n = \lfloor 5$ then n = 5

Try to solve

1) Find: a) $\dfrac{\lfloor 15}{\lfloor 12}$ b) $\dfrac{\lfloor 7}{\lfloor 5} + \dfrac{\lfloor 9}{\lfloor 7}$

Example

2) Find the solution set of the equation:- $\dfrac{\lfloor n}{\lfloor n-2} = 30$

We will learn
- Factorial of a number
- Permutations

Key - term
- Factorial of a Number
- Permutatuins
- Sub-Permutation

Matrials
- Scientific calculator
- Graphical programs

Solution

$\because \dfrac{\underline{|n}}{\underline{|n-2}} = \dfrac{n(n-1)\ \underline{|n-2}}{\underline{|n-2}} = 30$ $\qquad \therefore\ n(n-1) = 6 \times 5$ $\qquad \therefore\ n = 6$

Try to solve

(2) If $\dfrac{1}{\underline{|n}} + \dfrac{2}{\underline{|n+1}} = \dfrac{56}{\underline{|n+2}}$ find the value of n

Critical thinking: if: $\underline{|a} = \underline{|0}$ find the value of a.

Permutations

Introductory example: How many ways can different three-digit numbers be formed from the set of numbers {2, 3, 5} ?

The numbers are: 532 , 352 , 523 , 253 , 325, 235. Each number of these numbers is called a permutation of the numbers

and its number $= 3 \times 2 \times 1$ **and is written as** $^3p_3$ **and is read as** (3 p 3).

The following table illustrates that:

Unit digit	Tens	Hundreds
1	2	3

So, the permutation for a number of objects is to put them in a certain arrangement

Definition 1

The number of permutations of n different objects taking r at a time is denoted by the symbol ^np_r where:

$^np_r = n(n-1)(n-2)\ ...\ (n-r+1)$ where $r \leqslant n$, $r \in N$, $n \in Z^+$

$^np_0 = 1$

For example:

- $^6p_3 = 6 \times 5 \times 4 \times \dfrac{3 \times 2 \times 1}{3 \times 2 \times 1} = \dfrac{\underline{|6}}{\underline{|6-3}}$

  $^6p_3 = \dfrac{\underline{|6}}{\underline{|6-3}}$

- $^7p_5 = 7 \times 6 \times 5 \times 4 \times 3 \times \dfrac{2 \times 1}{2 \times 1} = \dfrac{\underline{|7}}{\underline{|7-5}}$

of the previous , we deduce that:

$^np_r = \dfrac{\underline{|n}}{\underline{|n-r}}$ where $r \in N$, $n \in Z^+$, $r \leqslant n$

using calculator

Permutations are denoted by the symbol ^np_r in the calculator and we use the buttons Shift × **To calculate the value of $^5p_2$ by the calculator, press the following buttons consecutively:**

5 Shift × 2 =

The answer = 20

Example

(3) Find the value for reach of the following:

a $^7p_4$ b $^4p_4$ c $^4p_3$

Solution

a $^7p_4 = 7 \times 6 \times 5 \times 4 = 840$

b $^4p_4 = 4 \times 3 \times 2 \times 1 = 24$

c $^4p_3 = 4 \times 3 \times 2 = 24$. What do you notice in the two phrases b and c?

Try to solve

3) Calculate the value of the following: **a** $^{5}p_{2} + {}^{6}p_{3}$ **b** $\frac{^{5}p_{5}}{^{5}p_{4}}$

Example

4) Find the number of the different ways, for 5 students to sit on 7 seats in one row.

Solution

We have 7 seats. Among the 7 seats, 5 should be selected each time

$\therefore$ Number of ways $= {}^{7}p_{5} = 7 \times 6 \times 5 \times 4 \times 3 = 2520$

Use the calculator: 7 SHIFT × (npr) 5 =

Try to solve

4) How many words can be formed from five different Alphabetic letters?

Example

6) If $^{7}p_{r} = 840$, find the value of $\lfloor\underline{r - 4}$

Solution

Start by dividing the number 840 by 7, then divide the quotient by 6, then divide the resulted quotient by 5 and so on till you reach number 1

840	7
120	6
20	5
4	4
1	

$\therefore$ Number $840 = 7 \times 6 \times 5 \times 4 = {}^{7}p_{4}$

$\therefore\ {}^{7}p_{r} = {}^{7}p_{4}$ $\therefore\ r = 4$ $\therefore\ \lfloor\underline{r - 4} = \lfloor\underline{0} = 1$

Try to solve

6) If $^{9}p_{r-1} = 504$, find the value of $\lfloor\underline{r + 1}$

Critical thinking: 1) Find the value of: $^{7}p_{7}$, $\lfloor\underline{7}$. **What do you notice**?

Exercises (2 - 2)

Choose the correct answer:

1. How many ways can a president and vice president be selected from a 12 - member committee?
   - a 2
   - b 23
   - c 66
   - d 132

2. If $^{5}p_{r} = 60$, then r equals:
   - a 4
   - b 3
   - c 2
   - d 5

3. If $^{n}p_{3} = 120$, then the value of n equals:
   - a 6
   - b 5
   - c 4
   - d 3

4. The number of ways to arrange the letters of the word HELP equals:
   - a 4
   - b 9
   - c 10
   - d 24

5. The number of ways to select a different two - digit number of the set of numbers {3, 4, 5, 6} equals:
   - a 48
   - b 30
   - c 12
   - d 4

6. How many ways can Hossam have a meal out of three meals (Kofta - chicken - fish) and a beverage out of two beverages (Juice - soft drink)? Represent it using the tree diagram.

7. How many ways can a two -digit number be formed from the numbers 1 , 2 , 3 , 4?

8. How many ways can a different two -digit number be formed from the numbers 1 , 2 , 3 , 4?

9. How many ways can a different two -digit even number be formed from the numbers 1 , 2 , 3, 4?

10. How many ways can a committee of a man and a woman be formed from 3 men and 4 woman?

11. An Ice-cream shop offers three sizes (small - medium - large) and five flavors (strawberry , mango, lemon , milk, chocolate)
    How many ways are available to buy an ice -cream cone?

12. From the set of the letters {a , b, c , d, e , f}, find
    - a Number of ways to select a letter.
    - b Number of ways to select two different letters.

13 **Find the value of the following:**

a $\lfloor 7 \div \lfloor 5$ b $3\lfloor 2 - \lfloor 3$ c ${}^{5}p_{3} \times \lfloor 2$

d ${}^{3}p_{3} \times {}^{2}p_{2}$ e ${}^{8}p_{1} + {}^{8}p_{2}$ f ${}^{7}p_{0} + {}^{7}p_{7}$

14 **Find the value of n which satisfies the following:**

a $\lfloor n = 24$ b $\dfrac{\lfloor n+1}{\lfloor n-1} = 42$

c ${}^{15}p_{n} = 2730$ d ${}^{n}p_{0} + {}^{n}p_{1} + {}^{n}p_{2} = 50$

15 Find the value of n if:

a $7p_{n} = 210$ b $n\lfloor 2n-1 = 12$

16 If ${}^{n}p_{4} = 14 \times {}^{n-2}p_{3}$ find the value of n.

17 Find the number of ways to select a president, vice president and a secretary out of a ten-person committee.

18 How many ways can the physical education instructor choose three students (one after another) to participate in the teams of soccer , basketball and volleyball respectively from eight students?

19 Prove that: $\dfrac{\lfloor n+2}{\lfloor n} = {}^{n+2}p_{2}$

Unit Two

2 - 3 Combinations

We will learn

- The concept of combinations and simple applications on them.
- Pascal's triangle.

Key - term

- Combinations
- Elements
- Order
- Committee
- Subset

Matrials

- Scientific calculator
- Computer

Tip

Combination can be written in the form of $^nC_r = \binom{n}{r}$

Introduction

Two clubs of a four - club set { a , b, c , d} are to be selected for a soccer match, then the possible permutations are:

(a, b), (a, c), (a, d), (b, a) , (b, c), (b, d), (c, a), (c, b), (c, d), (d, a), (d, b), (d, c).

From the previous data, we notice that selecting (a, b) is different from selecting (b, a) and so on...

If we want to select from the previous disregarding the order, then all the possible choices are: {a, b}, {a, c}, {a, c}, {b, c}, {b, d}, {c, d}. and each choice of these choices is called" Combination"

Combinations

Definition 1

The number of combinations formed from r of objects chosen from n elements at the same time is ^nC_r where, $r \leqslant n$, $r \in N$, $n \in Z+$

In the previous introduction , we find that:

the number of combinations of two elements taken from four elements is denoted by the symbol $^4C_2$ and is read as (4 c 2) or by the symbol $\binom{4}{2}$ and is read as 4 c 2. In the introduction above, we notice that the number of ways of choosing = 6 ways

i.e.: $^4C_2 = \frac{^4P_2}{\lfloor 2} = \frac{4 \times 3}{2 \times 1} = 6$, $\quad ^nC_r = \frac{^nP_r}{\lfloor r}$

Example

1. Find the value for each of the following

a $^7C_5$ **b** $^7C_2$ (what do you notice)?

Solution

a $^7C_5 = \frac{^7P_5}{\lfloor 5} = \frac{7 \times 6 \times 5 \times 4 \times 3}{5 \times 4 \times 3 \times 2 \times 1} = 21$ **b** $^7C_2 = \frac{7 \times 6}{2 \times 1} = 21$

We notice that: $^{7}C_5 = {}^{7}C_2$ $(5 + 2 = 7)$

Important corollaries: $^{n}c_r = \dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$ $\qquad$ $^{n}C_r = {}^{n}C_{n-r}$

Try to solve

(1) Find the value of $^{12}C_9$: $^{17}C_{14}$ without using the calculator.

Activity

Using the calculator

The buttons [SHIFT] [÷] can be used from left to right to write the symbol of combinations($^{n}C_r$)

1) Use the calculator to find the value of $^{5}C_4 + {}^{7}C_2$

Solution

Press the following buttons consecutively

start ⟶ [5] [SHIFT] [÷] [4] [+] [7] [SHIFT] [÷] [2] [=]

The result = 26

Example

(2) If : $^{28}C_r = {}^{28}C_{2r-47}$, find the values of r.

Solution

$\because$ $^{28}C_r = {}^{28}C_{2r-47}$

either : $r = 2r - 47$ i.e.: $r = 47$

it is greater than the value of n, so it is refused.

or : $r + 2r - 47 = 28$ $\qquad \therefore 3r = 75$ $\qquad \therefore r = 25$

Try to solve

(2) If $^{28}C_r = {}^{28}C_{2r-5}$, then find the value of r.

Example

(3) How many ways can a four - person team be chosen from a 9 - person set?

Solution

So that the choice disregards the order, then each choice is called a combination.

Number of choices = $^{9}C_4 = \dfrac{^{9}P_4}{\underline{|4}} = 126$

Try to solve

(3) 7 people have participated in a chess game so that a game is held between each two players. How many matches are there?

Example Counting principle

(4) How many ways can a committee of two men and a woman be selected out of 7 men and 5 women?

Solution

Number of ways to select 2 men out of 7 men = ${}^{7}C_{2} = 21$

Number of ways to select a women out of 5 women = ${}^{5}C_{1} = 5$

According to the counting principle, the number of ways to form the committee = $21 \times 5 = 105$ ways

Critical thinking: How many ways can a committee of 4 men and 3 women be selected out of 6 men and 5 women ?

Try to solve

(4) How many ways can a five-member committee formed from 3 male students and 2 female students be selected out of a class contains 10 male students and 8 female students?

Activity Pascal's triangle

Blaise pascal (1623 - 1662):

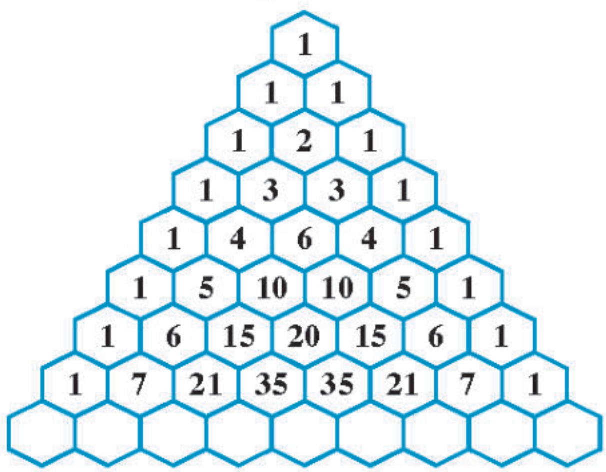
Blaise pascal is a french philosopher, mathematician and physicist. He had stated the theory of probabilities and designed a triangular array of numbers called pascal's triangle in calculating the probabilities. Furthermore, he had invented a calculator to perform the addition and multiplication operations

Check the opposite number triangle , then answer the following questions:

1- What do you notice about how numbers are written in this triangle?

2- Is there a relation among the number of elements of each row and the row directly next to it?

3- Is there a symmetry among the numbers existed on the two sides of the triangle?



After performing the activity, we can notice that:

- **First row:** represents (n = 1) of the elements taken off from r = 0 or r = 1

  So: ${}^{1}C_{0} = 1$, ${}^{1}C_{1} = 1$

- **Second row:** represents (n = 2) of the elements taken off from r = 0 or r = 1 or r = 2 in each time.

  then: ${}^{2}C_{0} = 1$, ${}^{2}C_{1} = 2$, ${}^{2}C_{2} = 1$ and so on.

As we notice that:

- Each row starts with one because ${}^{n}C_0 = 1$, and ends in one because ${}^{n}C_n = 1$
- Each number in any row except for the first row equals the sum of the two numbers located above it in the row directly above it.
- **In the third row , we find:** 1 , 1 + 2 , 2 + 1, 1
- **In the fourth row , we find:** 1 , 1 + 3 , 3 + 3 , 3 + 1 , 1 and so on.
- There is a symmetry about the number located at the middle of the row (if n is even)
- There is a symmetry about the two numbers located at the middle of the row (if n is odd)
- This coincides the previous relation ${}^{n}C_r = {}^{n}C_{n-r}$

Application on the activity:

Prove that: ${}^{5}C_0 + {}^{5}C_1 + {}^{5}C_2 + {}^{5}C_3 + {}^{5}C_4 + {}^{5}C_5 = 2^5$

Exercises (2 - 3)

Choose the correct answer from those given:

1. The number of ways to choose 3 people out of five people equals.........................
   - a 15
   - b 10
   - c 20
   - d 35
2. The number of ways to answer 4 questions only out of a 6 - question exam equals.........................
   - a 30
   - b 15
   - c 24
   - d 10
3. The number of ways to choose a red ball and a white ball out of 5 red balls and 3 white balls equals.........................
   - a 15
   - b 8
   - c 60
   - d 2

Answer the following questions:

4. Calculate the value of ${}^{6}C_3$, ${}^{9}C_1$, ${}^{12}C_{11}$ and ${}^{100}C_0$
5. If ${}^{n}C_3 = 120$, find the value of ${}^{n}C_{n-9}$
6. If ${}^{n+1}C_4 = \frac{5}{2}\,{}^{n}C_3$, find the value of n.
7. If ${}^{n}C_3 = 30\frac{1}{3}$ n, find the value of n.
8. How many ways can a five-people committee take the majority of a decision?
9. How many ways can a five - student activity committee formed from three male students and two female students out of a class contains 10 male students and 8 female students?
10. Write in terms of permutations each of:
    - a ${}^{8}C_3$
    - b ${}^{19}C_2$
    - c ${}^{5}C_0$
    - d ${}^{x}C_{x-y}$
11. Use the form ${}^{n}C_r$ to write each of:
    - a $\frac{{}^{8}P_2}{\lfloor 2}$
    - b $\frac{{}^{9}P_3}{\lfloor 3}$
    - c $\frac{{}^{10}P_4}{\lfloor 4}$
    - d $\frac{{}^{8}P_0}{\lfloor 0}$

General Exercises

Complete:

1. The number of ways to form a different two -digit number of the set of the numbers 1, 2, 3, 4 equal.........................

2. If you are asked to make a three-digit code doesnot include zero for an iron treasury, then the number of ways equals.........................

3. If $\lfloor n = \lfloor 1$ then n = ...

Choose the correct answer:

4. A man has wanted to buy a car of three models {a, b, c} and he wanted to select one of colors:{white - red - silver - black}. How many ways can this man choose a car?

   a 7   b 12   c 14   d 24

5. How many ways can a different three- digit number be formed from the numbers 1, 3, 6, 7

   a 9   b   c 24   d 64

6. if X = {x:x ∈N and 1 ⩽ x⩽ 5} and Y = {(a, b): a ,b∈ x and a ≠b}. How many elements are there in Y?

   a 7   b 10   c 20   d 25

7. $^{12}C_4 + {}^{12}C_3$ equals

   a 715   b 710   c 716   d 720

8. If $^{n}p_r = 336$, $^{n}C_r = 56$ then n , r are

   a (3, 2)   b (8 , 3)   c (7, 4)   d (7, 3)

9. If $^{n}C_{10} = {}^{n}C_{14}$, then $^{25}C_n$ equals.

   a 24   b 25   c 1   d 49

10. Calculate the value of each:

    a $^{5}C_5$ b $\lfloor 6$ c $^{23}p_1$

    d $^{50}C_{49}$ e $\frac{\lfloor 13}{\lfloor 14}$ f $^{4}p_4$

(11) What is the value of $\lfloor n-5$, if ${}^{2n-1}C_3 = 84$?

(12) What is the value of ${}^{2r}C_4$, if ${}^{8}P_{r+1} = 336$?

(13) How many ways can 7 students be selected from 10 students to take a historical trip?

(14) If 3 students are selected from (n) number of students to attend a seminar so that the number of ways of choosing is 10 ways, find the number of students.

(15) How many ways can a committee of a man and two women be selected from 7 men and 5 women?

(16) It is wanted to choose a teacher out of 4 teachers to train the Olympics students in mathematics and another teacher to prepare the test, find the number of ways of choosing.

(17) Find the solution set of the following equations:

a $\dfrac{\lfloor n+3}{\lfloor n+1} = 42$

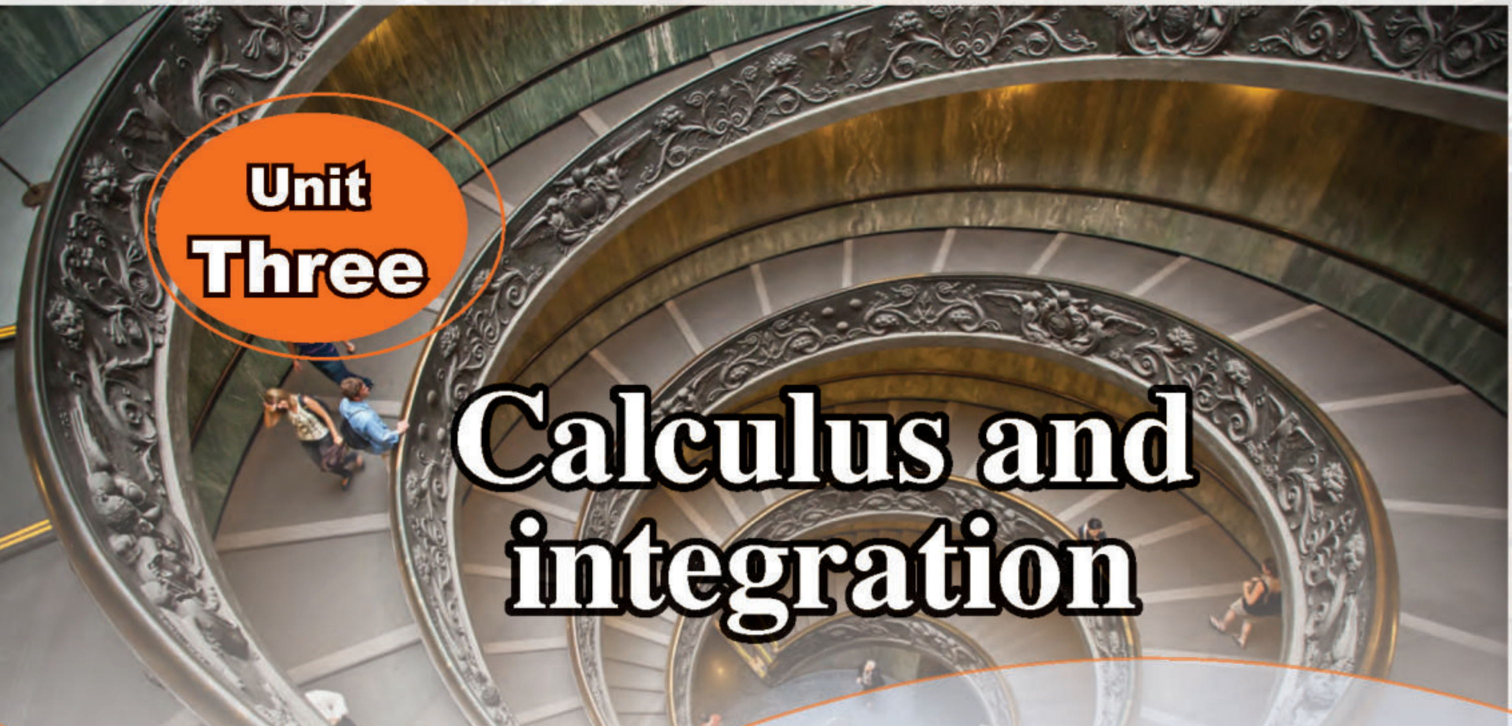
b $12 \times \lfloor n = \lfloor n+2$

c $\lfloor n-4 = 120$

d $\lfloor n-5 = 1$

e ${}^{n}C_3 = 84$

f ${}^{12}C_r = {}^{12}C_{2r+3}$



Unit Three

Calculus and integration

Unit introduction

There is no doubt aware that the calculus one of the modern science, which are used in many branches of knowledge, and in various scientific contexts such as engineering Physics, medicine, economics and geography, calculus is interested in studying the differentiation changes and differences)variable quantities(through accounts include averages, observes and rates of change compared to the rate of change is attributed to the change in average temperatures - price - speed - production ... as well as the study of the behavior Functions such as the functions of costs and income in the economy; to maximize profit or study the effect of taking a particular drug in reducing the duration of treatment or study schemes belt Earthquakes when planning for the construction, urban stress and expense rates, etc., The calculus is looking to find a quantity Bmalomah rate Change, and is used to calculate the area under the curve, and the amount of labor resulting from the impact of a changing force, and includes calculus operations With infinitely small quantities in contrast to algebra. Finally, we sought to understand the minutes of this article will help solve a lot of problems with overlapping many of the minute, which are built by Renaissance Science Scientific and cultural that we wish for our country..

Unit objectives

By the end of this unit, the student should be able to:

- Recognize the concept of variation function, average rate of change function and the rate of change function.
- Conclude the first derivation function.
- Recognize the geometrical meaning of the first derivative (the slope of the tangent).
- Identify some derivative rules .
  - the derivative of the constant function.
  - the derivative of the function $f: f(x) = x^n$
  - the derivative of the function $f: f(x) = x$
  - the derivative of the function $f: f(x) = a\,x^n$
  - the derivative of the sum and difference between two function
  - the derivative of product of two functions.
  - the derivative of the quotient of functions.
  - the derivative of the composite function - the chain rule.
  - the derivative of the function $y = (f(x))^n$
- Use the derivatives in geometrical applications like finding the equations of the tangent to a curve at a point on it.
- Recognize the concept of integral -antiderivative.
- Recognize the following integral rules:
  - $\int x^n\,dx = \frac{1}{n+1}\,x^{n+1} + C,\ n \neq -1$
  - $\int a\,f(x)\,dx = a\int f(x)dx$, where a is a constant
  - $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
  - $\int (a\,x + b)^n\,dx = \frac{1}{a\,(n+1)}\,(a\,x + b)^{n+1} + C,\ n \neq -1$

Key terms

- Variation
- Average Rate of Change
- Rate of Change
- Differentiation
- Differintiable Function
- First Derivative
- Product
- Quotient
- Antiderivative
- Integration

Unit Lessons

Lesson (3 - 1): The rate of change.

Lesson (3 - 2): Differentiation.

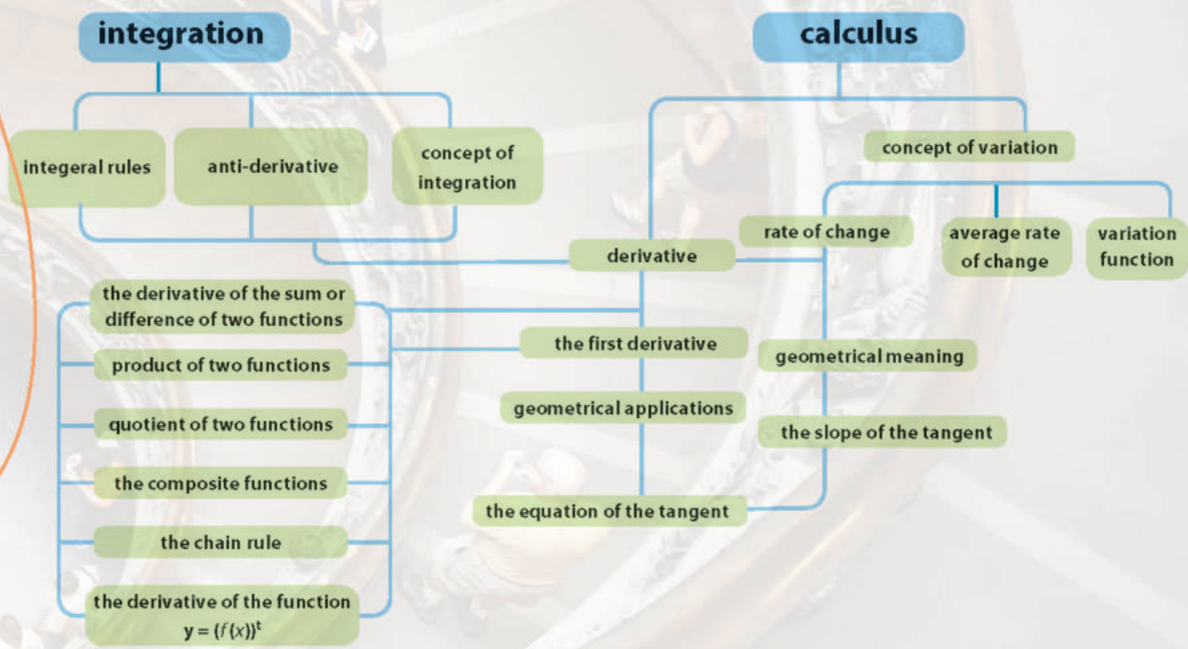
Lesson (3 - 3): Derivative rules.

Lesson (3 - 4): Integration

Materials

Scientific calculator - Computer - Graphing program

Unit planning guide



Unit Three

3-1 Rate of Change

We will learn

- The concept of the variation function.
- The concept of the average rate of change.
- The concept of the rate of change.

Key - term

- Function of Variation
- Average rate of Change
- Rate of Change

matrials

- Scientific calculator

Think and discuss

Why leave spaces between the railway bridges or metal rods breaks? If the temperature increases from 30° to 42° in a time period or decreases from 48° to 22° in another time period Calculate the change in temperature for each period ... What do you notice?

If the length of one of the rails is L meters and changed its temperature x from x_1 to x_2 then we say that a change has occurred in the temperature and be: The change in the value of x = the value of x at the end of the change - the value of x at the beginning of change

Is the length of the rails varies depending on the temperature change?؟

Learn

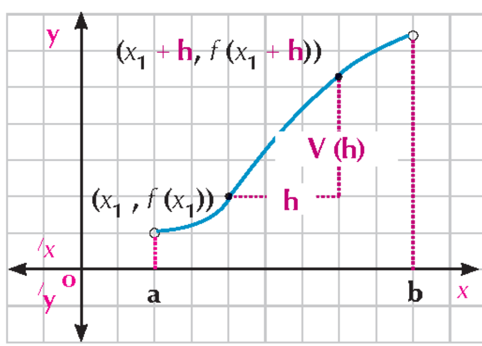
The Variation Function

If $f:]a, b[\longrightarrow \mathbb{R}$ where $y = f(x)$ then any change in the value of x from x_1 to x_2 in the domain of f corresponds change in the value of y from $f(x_1)$ to $f(x_2)$ then :

The value of the change of $x = \triangle x$ (read as delta x) $= x_2 - x_1$,

The value of the change of $y = \Delta y = f(x_2) - f(x_1)$

and by considering $(x_1, f(x_1))$ a point on the curve of the function f, then for each change in the coordinate of x from x_1 to $x_2 = x_1 + h$ such that $x_1 + h \in]a, b[$, $h \neq 0$ then there is a corresponding change in the y-coordinate given by the relation :



$V(h) = f(x_1 + h) - f(x_1)$ the function V is called the variation function of the function f at $x = x_1$

Notice:

Each of symbols Δx , h represents the variation of x

Example

1) If $f(x) = 3x^2 + x - 2$ and x changes from 2 to 2 + h find the variation function V, then calculate the variation of f when

a) $h = 0.3$ b) $h = -0.1$

Solution

$\because f(x) = 3x^2 + x - 2$, x changed from 2 to 2 + h

$\therefore x_1 = 2$, $f(2) = 3 \times 4 + 2 - 2 = 12$, and :

$f(2 + h) = 3(2 + h)^2 + (2 + h) - 2 = 12 + 12h + 3h^2 + 2 + h - 2$

$= 3h^2 + 13h + 12$

$V(h) = f(2 + h) - f(2)$

$= (3h^2 + 13h + 12) - 12 = 3h^2 + 13h$

a) when $h = 0.3$

$V(0.3) = 3(0.3)^2 + 13 \times 0.3$

$= 4.17$

b) when $h = -0.1$

$V(-0.1) = 3(-0.1)^2 + 13(-0.1)$

$= -1.27$

Try to solve

1) If $f(x) = x^2 - x + 1$ find the variation function V when $x = 3$ then calculate:

a) $V(0.2)$ b) $V(-0.3)$

Learn **The Average rate of change function**

when we divide the variation function V by h where $h \neq 0$ we get a new function called the function of the average rate of change of f at $x = x_1$ where :

$$A(h) = \frac{V(h)}{h} = \frac{f(x_1 + h) - f(x_1)}{h} \quad \text{or} \quad \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example

2) If $f : [0, \infty[\longrightarrow \mathbb{R}$ where $f(x) = x^2 + 1$ find :

a) The function of the average rate of change of f when $x = 2$ then calculate $A(0.3)$

b) The average rate of change of f when x changes from 3 to 4

Solution

a) $f(x_1) = f(2) = (2)^2 + 1 = 5$, $f(x_1 + h) = f(2 + h)$

$\therefore\ f(2+h) = (2+h)^2 + 1 = h^2 + 4h + 5$

$\because A(h) = \dfrac{f(x_1+h) - f(x_1)}{h}$

$\therefore A(h) = \dfrac{h^2 + 4h + 5 - 5}{h} = h + 4$ then $A(0.3) = 4.3$

b When x changes from 3 to 4 then $x_1 = 3$, $x_2 = 4$

then $f(3) = 9 + 1 = 10$, $f(4) = 16 + 1 = 17$

The average rate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{17 - 10}{4 - 3} = 7$

Try to solve

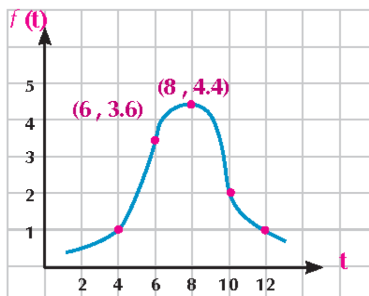
2 If $f(x) = x^2 + 3x - 1$ find :

a The average rate of change function at $x = 2$ then find A (0.2)

b The average rate of change when x changes from 4.5 to 3

Example

3 The opposite graph shows the curve $g = f(t)$ where g is the total sales of one of computer stores estimated in millions of pounds, t the time in months. From the graph find , the average rate of change in the total sales when the time changes from.



a t = 4 to t = 8 **b** t = 8 to t = 10

Solution

a From the graph : $f(8) = 4.4$, $f(4) = 1$

the average rate of change $f = \dfrac{f(8) - f(4)}{8 - 4} = \dfrac{4.4 - 1}{4} = 0.85$ million pound / month

i.e. the average of the total sales increases by 0.85 million pound monthly during this period .

b From the graph : $f(10) = 2$, $f(8) = 4.4$

the average rate of change of $f = \dfrac{f(10) - f(8)}{10 - 8} = \dfrac{2 - 4.4}{2} = -1.2$ million pound / month

i.e. the average of the total sales decreases by 1.2 million pound monthly during this period.

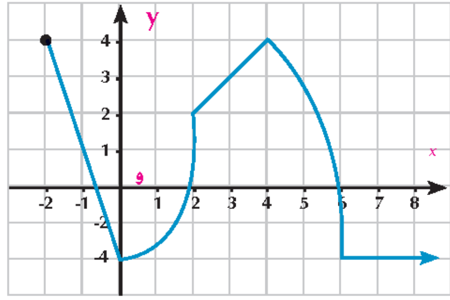
Try to solve

3 Using the graph shown in example 3 . Find the average rate of change in the total sales when the time changes from :

a t = 4 to t = 6 **b** t = 6 to t = 10 **c** t = 4 to t = 12

Critical thinking :

The opposite graph shows the curve of the function f where $y = f(x)$. Determine the intervals over the average rate of change of f is constant. Explain your answer.



Learn

The rate of change function

If $f:]a, b[\longrightarrow \mathbb{R}$ where $y = f(x)$, x_1, $x_1 + h \in]a, b[$ then:

the function of the rate of change of f at $x_1 = \lim\limits_{h \to 0} \dfrac{f(x_1 + h) - f(x_1)}{h} = \lim\limits_{h \to 0} A(h)$ on condition that the limit exist.

Example

(4) Find the function of the rate of change of f when $x = x_1$ then find this rate at the indicated values of x

a $f(x) = 3x^2 + 2$, $x = 2$

Solution

a $\because f(x) = 3x^2 + 2$ $\therefore$ then $x = x_1$ then $f(x_1) = 3x^2_1 + 2$,

$f(x_1 + h)^2 = 3(x_1 + h)^2 + 2 = 3x_1^2 + 6x_1 h + 3h^2 + 2$

the function of the rate of change of $f = \lim\limits_{h \to 0} \dfrac{f(x_1 + h) - f(x_1)}{h}$

$= \lim\limits_{h \to 0} \dfrac{6x_1 h + 3h^2}{h} = \lim\limits_{h \to 0} (6x_1 + 3h) = 6x_1$

when $x = 2$ $\therefore x_1 = 2$ then the rate of change of $f = 6 \times 2 = 12$

Exercises 3 - 1

Choose the correct answer

1. If the average rate of change of $f = 2.4$ when x changes from 3 to 3.2 then the variation of f equals ....

   a) 0.32 b) 0.48 c) 3.6 d) 7.2

2. If the average rate of change of $f = 5$ when x changes from 2 to 4 , $f(2) = 6$ then $f(4)$ equals........

   a) - 4 b) 7 c) 8 d) 16

3. The average rate of change of the volume of a cube when its edge length changes from 5cm to 7cm equals ....

   a) 125 b) 343 c) 218 d) 109

4. The average rate of change of the function f where $f(x) = x^2 + 3x + 5$ when x changes from 1 to 3 equals.......

   a) 1 b) 3 c) 7 d) 9

Answer the following:

5. If $f(x) = x^2 + 2x - 1$ find the variation of f when

   a) x changes from 2 to 2.1 b) $x = -2$, $h = 1$

6. Find the function of the rate of change of f when $x = x_1$ then deduce the rate of change of f at the indicated values of x_1 in each of the following :

   a) $f(x) = 2x^3, \quad x_1 = 2$

7. **Areas:** Asquared lamina shrinking by cooling preserves its squared shape , calculate the rate of change of the area of the lamina with respect to its side length when the side length is 8cm.

Unit Three

3 - 2 Differentiation

We will learn

- The first derivative function
- The geometrical meaning of the first derivative (the slope of the tengant)

Key - term

- First Derivative
- Slope
- Tangent

matrials

- Scientific calculator
- Computer graphing program

Think and discuss

1. Figure (1) shows the curve $f :]a, b[\longrightarrow \mathbb{R}$ where $y = f(x)$, $\overleftrightarrow{CD}$ intersect the curve at the two points $C(x_1, f(x_1))$, $D(x_1 + h, f(x_1 + h))$. find the slope of the transversal $\overleftrightarrow{CD}$.

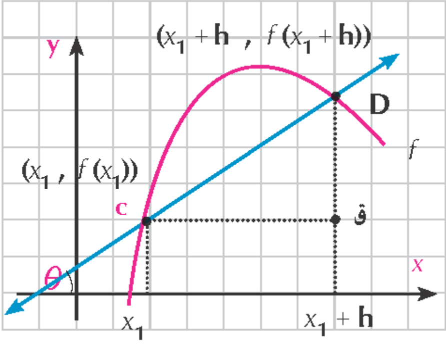


figure (1)

2. Consider x changes from x_1 to $x_1 + h$ compare the average rate of change and the slope of the transversal $\overleftrightarrow{CD}$. Is the following relation true? the slope of the secant $\overleftrightarrow{CD} = \tan\theta = \dfrac{f(x_1 + h) - f(x_1)}{h} = A(h)$

3. If the point $C(x_1, f(x_1))$ is fixed on the curve of the function f, and the point D is moving on the curve approaching the point C so $\overleftrightarrow{CD}$ take the situation on $\overleftrightarrow{CN}$ and become a tangent to the curve at the point C.

   i.e. $h \longrightarrow 0$

find the slope of the tangent to the curve of f at c

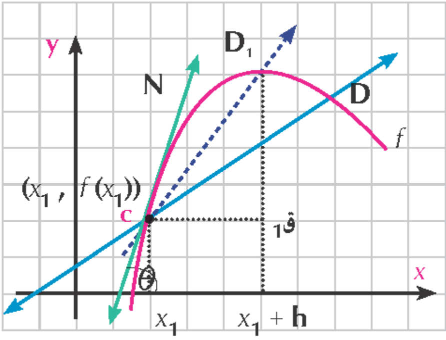


figure (2)

Notice :

the slope of the tangent at $C = \tan\theta = \lim\limits_{h \to 0} \dfrac{f(x_1 + h) - f(x_1)}{h}$ if exist

i.e. :

the slope of the tangent to the curve of the function f where $y = f(x)$ at the point $(x_1, f(x_1))$ equals the rate of change of f at $x = x_1$

Example

(1) Find the slope of the tangent to the curve of the function f where $f(x) = 3x^2 - 5$ at the point A(2, 7) then find the measure of the positive angle which the tangent makes with the positive direction of the x-axis at the point A to the nearest minute .

Solution

$\because f(2) = 3(2)^2 - 5 = 7$ $\therefore$ the point A (2, 7) lies on the curve of f

the slope of the tangent at $(x = 2)$ = the rate of change of f at $(x = 2)$

$$= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$\therefore \text{ the slope of the tangent} = \lim_{h \to 0} \frac{3(2+h)^2 - 5 - 7}{h} = \lim_{h \to 0} \frac{12h + 3h^2}{h}$$

$$= \lim_{h \to 0} (12 + 3h) = 12$$

then $\tan \theta = 12$ $\therefore \theta = \tan^{-1}(12) \simeq 85° 14'$

Try to solve

(1) Find the slope of the tangent to the curve of the function f where $f(x) = x^3 - 4$ at the point A (1, - 3) then find the measure of the positive angle which the tangent makes with the positive direction of the x-axis at the point A to the nearest minute .

Learn — The Derivative Function

Each value of the variable x in the domain of f corresponds a unique value of the rate change of f so the rate of change is also a function in the variable x called "the derivative function" or : " the first derivative of the function" or " the first differential coefficient "

Definition

If $f :]a, b[\longrightarrow \mathbb{R}$, $x \in]a, b[$ then the derivative function f' :

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ provided this limit exists.

The derivative function symbols :

If $y = f(x)$ then the first derivative of the function f by the one of the symbols

y' or f' and read as " y derivative" or " f derivative "

$\frac{dy}{dx}$ and read as " dy by d x " or " the derivative of y with respect to x"

Notice that the slope of the tangent to the curve of $y = f(x)$ at the point $(x_1, f(x_1))$ is $f'(x_1)$

Example

(2) Using the definition of the derivative. Find the derivative of the function f where $f(x) = x^2 - x + 1$ then find the slope of the tangent at the point $(-2, 7)$

Solution

$\because f(x) = x^2 - x + 1$

$\therefore f(x + h) = (x + h)^2 - (x + h) + 1 \quad = x^2 + 2x\,h + h^2 - x - h + 1$,

$f(x + h) - f(x) = (2x + h - 1)\,h$

$\because f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $\qquad \therefore f'(x) = \lim_{h \to 0} \frac{(2x + h - 1)\,h}{h}$

$f'(x) = \lim_{h \to 0} (2x + h - 1)$ $\qquad f'(x) = 2x - 1$

$\because f(-2) = (-2)^2 - (-2) + 1 = 7$ $\qquad \therefore$ the point $(-2, 7)$ lies on the curve of f

The slope of the tangent at the point $(-2, 7) = f'(-2) = 2(-2) - 1 = -5$

Try to solve

(2) Using the definition of the derivative to find the derivative of the function f when $f(x) = 3x^2 + 4x + 7$, then find the slope of the tangent to the curve at the point $(-1, 6)$.

Learn

Differintiability of a Function

The function f is said to be differentiable at $x = a$ (a belong to the domain of the function) if and only if $f'(a)$ exist where $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Example

(3) By using the definition, find the derivative of the function f where $f(x) = \sqrt{x - 1}$ then find $f'(5)$

Solution

$\because f(x) = \sqrt{x - 1}$ $\qquad \therefore$ the domain $f = [1, \infty[$

$\therefore f(x + h) = \sqrt{x + h - 1}$

$f(x + h) - f(x) = \sqrt{x + h - 1} - \sqrt{x - 1}$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\sqrt{x + h - 1} - \sqrt{x - 1}}{h}$ multiplying by the conjugate of the numerator

$$f'(x) = \lim_{h \to 0} \frac{(x+h-1)-(x-1)}{h(\sqrt{x+h-1}+\sqrt{x-1})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h-1}+\sqrt{x-1})} = \frac{1}{2\sqrt{x-1}}, \quad x > 1$$

notice that f is not differentiable at $x = 1$ because of the non-existance of the limit

$$f'(5) = \frac{1}{2\sqrt{5-1}} = \frac{1}{2 \times 2} = \frac{1}{4}$$

Try to solve

3. By using the definition , find the derivative of the function f where $f(x) = \sqrt{x+5}$

Exercises 3 - 2

1. Find the derivative function of the function f in each of the following :
   - a) $f(x) = 5x + 2$
   - b) $f(x) = 3x^2$
   - c) $f(x) = x^3 - 1$
   - d) $f(x) = x^2 + 2x$.

2. Find the first derivative of the function f in each of the following and identify the values of x at which the function is not differentiable:
   - a) $f(x) = \frac{1}{x}$
   - b) $f(x) = \frac{1}{x+3}$
   - c) $f(x) = \frac{3}{2x-5}$
   - d) $\sqrt{x-4}$

3. Find the derivative of the function f where $f(x) = x^3 + 4$ then find the slope of the tangent to the curve at the point (-1, 3) which lies on the curve .

4. Find the derivative of the function f where $f(x) = a\,x + b$ at any point (x, y) where $a, b \in R$.

5. Find the slope of the tangent to the curve of the function f where $f(x) = 3x^2 - 8$ at the point A (2, 4) then find the measure of the positive angle which the tangent makes with the positive direction of x-axis .

Unit Three

3 - 3 Rules of Differentiation

We will learn

- The derivative of the constant function
- The derivative of $f(x) = x^n$
- The derivative of $f(x) = x$
- The derivative of $f(x) = ax^n$
- The derivative of the sum of two functions an the difference between them
- The derivative of product of two functions
- The derivative of quotient of two function
- The derivative of the composite function (the chain rule)
- The derivative of $y = (f(x))^n$.

Key - term

- First Derivative
- Product
- Quotient
- Chain Rule

matrials

- Scientific calculator.
- Computer graphing program.

Explore

1 - Using the definition, find the derivative of each of the following :

$f(x) = x^3$ $\qquad f(x) = x^5$

2 - Can you discover the derivative of $f(x) = x^7$ without using the definition?

3 - Can you conclude a rule to derivative f where $f(x) = x^n$?

Learn

Derivative of a Function

1 - Derivative of the constant function

If $y = c$ **where:** $c \in \mathbb{R}$ **then:** $\frac{dy}{dx} = 0$

Notice that :

$y = f(x) = c, \quad f(x+h) = c$

$\because \frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\therefore \frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{c - c}{h} = 0 \ (h \neq 0)$

2 - the derivative of the function $f(x) = x^n$

If $y = x^n$ **where:** $n \in \mathbb{R}$ **then:** $\frac{dy}{dx} = n x^{n-1}$

If $y = x$ **then:** $\frac{dy}{dx} = 1$

If $y = a x^n$ **where:** $a, n \in \mathbb{R}$ **then:** $\frac{dy}{dx} = a n x^{n-1}$

Example

(1) find $\frac{dy}{dx}$ for each of the following:

a) $y = -3$ b) $y = x^4$ c) $y = 5x$

d) $y = \frac{3}{x^2}$ e) $y = \sqrt{x^3}$

Solution

a) $\because y = -3 \quad \therefore \frac{dy}{dx} = 0$ b) $\because y = x^4 \quad \therefore \frac{dy}{dx} = 4x^3$

c $\because\ y = 5x$ $\quad\therefore \frac{dy}{dx} = 5$

d $\because y = \frac{3}{x^2} = 3x^{-2}$ $\quad\therefore \frac{dy}{dx} = -6x^{-3}$

e $\because y = \sqrt{x^3} = x^{\frac{3}{2}}$ $\quad\therefore \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$ where $x \geqslant 0$

Try to solve

1 Find $\frac{dy}{dx}$ for each of the following:

a $y = x^7$ b $y = x^{10}$ c $y = x^{\frac{3}{2}}$ d $y = x^{-4}$

e $y = x^{-\frac{5}{3}}$ f $y = \frac{1}{x^5}$ g $y = \sqrt{x^9}$ h $y = \sqrt[3]{x^7}$

i $y = 3x^6$ j $y = -2x^{-5}$ k $y = \pi x^{-4}$ l $y = -4\sqrt[3]{x}$

The derivative of the sum or difference of two functions

If g and h are two differentiable functions with respect to the variable x , then $g \pm h$ is also differentiable with respect to x and $\frac{d}{dx}(g \pm h) = \frac{dg}{dx} \pm \frac{dh}{dx}$ and in general:

If $f_1, f_2, ..., f_n$ **are differentiable functions with respect to the variable x then:**

$$\frac{d}{dx}(f_1 \pm f_2 \pm f_3 \pm ... \pm f_n)(x) = f_1'(x) \pm f'_2(x) \pm f'_3(x) \pm ... \pm f'_n(x)$$

Example

2 Find $\frac{dy}{dx}$ in each of the following:

a $y = 2x^6 + x^{-9}$ b $y = \frac{\sqrt{x} - 2x}{\sqrt{x}}$

Solution

a $\because\ y = 2x^6 + x^{-9}$

$\therefore \frac{dy}{dx} = 12x^5 - 9x^{-10}$

b $\because\ y = \frac{\sqrt{x} - 2x}{\sqrt{x}}$

$= 1 - 2\sqrt{x} = 1 - 2x^{\frac{1}{2}}$

$\therefore \frac{dy}{dx} = 0 - 2 \times \frac{1}{2}x^{\frac{1}{2}-1} = -x^{-\frac{1}{2}}$

Try to solve

2 Find $\frac{dy}{dx}$ If:

a $y = 3x^2 + 5$ b $y = 2x^3 - 4x + 7$ c $y = 3x^{-4} + 6x^{\frac{2}{3}}$

d $y = 7x^4 - \frac{1}{4x}$ e $y = 3x^8 - 2x^{-5} + 8$ f $y = \frac{5}{x} + x\sqrt{x} - 4$

The derivative of the product of two functions:

If g and h are two differentiable functions with respect to the variable x then the function $(g \,.\, h)$ is also differentiable with respect to the variable x and $\frac{d}{dx}(g \,.\, h) = g \frac{dh}{dx} + h \,.\, \frac{dg}{dx}$

Example

3 Find $\frac{dy}{dx}$ if $y = (x^2 + 1)(x^3 + 3)$ then find $\frac{dy}{dx}$ when $x = -1$

Solution

$\because y = (x^2 + 1)(x^3 + 3)$

$$\therefore \frac{dy}{dx} = (x^2 + 1) \times 3x^2 + (x^3 + 3) \times 2x$$
$$= 3x^4 + 3x^2 + 2x^4 + 6x$$
$$= 5x^4 + 3x^2 + 6x$$

When $x = -1$ $\quad \therefore \frac{dy}{dx} = 5(-1)^4 + 3(-1)^2 + 6(-1) = 2$

Try to solve

3 Find $\frac{dy}{dx}$ if :

a $y = (2x + 3)(3x - 1)$

b $y = (2x + 5)^2$

c $y = \sqrt{x}\,(\sqrt{x} + 4)$

d $y = (\sqrt{x} - 1)(\sqrt{x} + 1)$

e $y = (4x^2 - 1)(7x^3 + x)$ then find $\frac{dy}{dx}$ at $x = 1$

The derivative of the quotient of two functions:

If g and h are two differentiable functions with respect to the variable x and $h(x) \neq 0$ then the function $(\frac{g}{h})$ is also differentiable with respect to the variable x and $\frac{d}{dx}(\frac{g}{h}) = \frac{h\frac{dg}{dx} - g\frac{dh}{dx}}{h^2}$

i.e. $(\frac{g}{h})' = \frac{hg' - gh'}{h^2}$

Example

4 Find $\frac{dy}{dx}$ if $y = \frac{x^2 - 1}{x^3 + 1}$

Solution

$\because y = \frac{x^2 - 1}{x^3 + 1}$

$$\therefore \frac{dy}{dx} = \frac{(x^3 + 1) \times 2x - (x^2 - 1) \times 3x^2}{(x^3 + 1)^2}$$
$$\therefore \frac{dy}{dx} = \frac{2x^4 + 2x - 3x^4 + 3x^2}{(x^3 + 1)^2}$$
$$= \frac{-x^4 + 3x^2 + 2x}{(x^3 + 1)^2}$$

Try to solve

4) Find $\frac{dy}{dx}$ if:

a) $y = \frac{2x - 1}{x + 1}$ b) $y = \frac{2x + 5}{1 - 4x}$ c) $y = \frac{2x^2 - 1}{x + 5}$

The derivative of the Composike unction (Chain rule)

Theorem

If $y = f(z)$ is differentiable with respect to the variable z, and $z = g(x)$ is differentiable with respect to the variable x then $y = f(g(x))$ is differentiable with respect to the variable x and: $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$

notice that y is a function of function of x

this theorem is known as the chain rule

Example

5) If $y = (x^2 - 3x + 1)^5$ **find** $\frac{dy}{dx}$

Solution

let $z = x^2 - 3x + 1$ $\therefore y = z^5$

y is differentiable with respect to z (polynomial of z) then $\frac{dy}{dz} = 5z^4$

also z is differentiable with respect to x (polynomial of x) then $\frac{dz}{dx} = 2x - 3$,

Applying the chain rule $\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = 5z^2 \times (2x - 3)$,

substituting by z $\therefore \frac{dy}{dx} = 5(x^2 - 3x + 1)^4 \times (2x - 3)$

Try to solve

5) If $y = (2x + 3)^5$ **find** $\frac{dy}{dx}$

the derivative of the funciton $[f(x)]^n$

If $z = [f(x)]^n$ where f is differentiable with respect to the variable x, n is a real number,

then: $\frac{dz}{dx} = n[f(x)]^{n-1} \times f'(x)$

Example

6 Find $\frac{dy}{dx}$ if

a $y = (6x^3 + 3x + 1)^{10}$

b $y = (\frac{x-1}{x+1})^5$

Solution

a $y = (6x^3 + 3x + 1)^{10}$

$\therefore \frac{dy}{dx} = 10(6x^3 + 3x + 1)^9 \times \frac{d}{dx}(6x^3 + 3x + 1) = 10(18x^2 + 3)(6x^3 + 3x + 1)^9$

$= 30(6x^2 + 1)(6x^3 + 3x + 1)^9$

b $y = (\frac{x-1}{x+1})^5$

$\therefore \frac{dy}{dx} = 5(\frac{x-1}{x+1})^4 \times \frac{(x+1)\times 1 - (x-1)\times 1}{(x+1)^2}$

$= 5(\frac{x-1}{x+1})^4 \times \frac{x+1-x+1}{(x+1)^2}$

$= \frac{10}{(x+1)^2} \times (\frac{x-1}{x+1})^4 = \frac{10(x-1)^4}{(x+1)^6}$

Try to solve

6 Find $\frac{dy}{dx}$ if

a $y = (2x^3 - 4x + 1)^9$

b $y = (\frac{5x^2}{3x^2+2})^3$

Example

7 If $f(x) = \frac{1}{3}x^3 - 2x^2 + 5x - 4$ find the values of x which make $f'(x) = 2$

Solution

$f'(x) = \frac{1}{3} \times 3x^2 - 2 \times 2x + 5 \times 1$

$= x^2 - 4x + 5$

When $f'(x) = 2$ $\therefore x^2 - 4x + 5 = 2$ $\therefore x^2 - 4x + 3 = 0$

$\because (x-1)(x-3) = 0$ $\therefore x = 1$ or $x = 3$

Try to solve

7 Find the values of x which make $f'(x) = 7$ in each of the following:

a $f(x) = x^3 - 5x + 2$

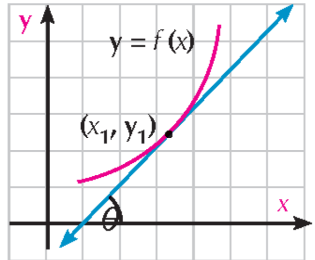
b $f(x) = (x-5)^7$

Critical thinking: Find $\frac{dy}{dx}$ if :

$y = (x-1)(x+1)(x^2+1)(x^4+1)(x^8+1)(x^{16}+1)$

Geometric Applications on the Derivative

In the geometrical explaining of the first derivative of the function f where $y = f(x)$ we found that the slope of the tangent (m) at the point (x_1, y_1) equals the first derivative of the function at that point.



i.e. $m = \frac{dy}{dx}$ at the point (x_1, y_1) and it can be written at the form

$$m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)}$$

and if θ is the measure of the positive angle which the tangent makes with the positive direction of x-axis, **then:**

$$m = \tan\theta = \left[\frac{dy}{dx}\right]_{(x_1, y_1)}$$

Notice that:

1 If m_1, m_2 are the slopes of two straight lines L_1, L_2 then :

$L_1 // L_2$ if and only if $m_1 = m_2$ (the condition of parallelism)

, $L_1 \perp L_2$ if and only if $m_1 m_2 = -1$ (the condition of perpendicularity)

2 The slope of the tangent to a curve at any point on it is known as the slope of the curve at this point , also perpendicular to the tangent to a curve at the point of tangency is known as the normal to the curve at this point .

$\therefore$ the slope of the normal at the point $(x_1, y_1) = \frac{-1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}}$

Example

8 Find the slope of each of the tangent and the normal to the curve $y = 2x^3 - 4x + 5$ at the point (-2, -3) which lies on it.

Solution

$\because y = 2x^3 - 4x + 5$ $\therefore$ the slope of the tangent at any point $= \frac{dy}{dx} = 6x^2 - 4$

at the point (-2 , -3)

the slope of the tangent $= \left[\frac{dy}{dx}\right]_{(-2, -3)} = 6(-2)^2 - 4 = 20$

the slope of the normal $= \frac{-1}{\text{the slope of the tangent}} = \frac{-1}{20}$

Try to solve

8 Find the slope of each of the tangent an the normal to the following curves at the indicated points:

a $y = x - 7$ when $x = -1$

d $y = (x^3 - 2)(x + 1)$ when $x = 1$

Example

9 Find the points on the curve $y = x^2 - 6x + 5$ at which :

a the slope of the tangent = 2 **b** the tangent is parallel to x axis

c the tangent is perpendicular to the straight line $4y + x - 1 = 0$

Solution

$\because y = x^2 - 6x + 5$

$\therefore$ the slope of the tangent at any point $= \frac{dy}{dx} = 2x - 6$

a $\because \frac{dy}{dx} = 2 = 2x - 6 \quad \therefore 2x = 8$ i.e. $x = 4$

$\therefore y = (4)^2 - 6(4) + 5 = -3$

the slope of the tangent = 2 at the point (4, -3)

b $\because$ the tangent // x-axis $\therefore$ the slope of the tangent = the slope of x-axis = 0

$\frac{dy}{dx} = 2x - 6 = 0 \qquad \therefore x = 3,\ y = 9 - 18 + 5 = -4$

the tangent at the point (3, -4) is parallel to x-axis

c the tangent is perpendicular to the straight line $4y + x - 1 = 0$

whose slope is $\frac{-1}{4}$

the slope of the tangent $= -1 \div \frac{-1}{4} = 4$

$2x - 6 = 4 \qquad \therefore x = 5,\ y = 0$

$\therefore$ the tangent at the point (5, 0) is perpendicular to the straight line $4y + x - 1 = 0$

Remember

The slope of the straight line $ax + by + c = 0$ is $\frac{-a}{b}$

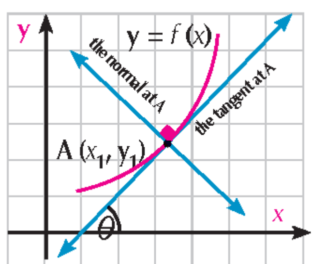
Try to solve

9 Find the points which lie on the curve $y = x^3 - 3x^2$ at which the tangent to the curve :

a Parallel to x-axis **b** perpendicular to the straight line $x + 9y + 3 = 0$

Learn Equation of the Tangent to a Curve

If (x_1, y_1) is a point lie on the curve of the function f where $y = f(x)$, and m is the slope of the tangent to the curve at this point, then the equation of the tangent at the point (x_1, y_1) is



$$y - y_1 = m(x - x_1)$$

Note: the equation of the normal to the curve at the point (x_1, y_1) is :

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

Example

(10) Find the equations of the tangent and the normal to the curve $y = 2x^2 - 5x + 1$ at the point on it whose abscissa = 2

Solution

$\because$ the point lies on the curve $\therefore$ it satisfies its equation

at $x = 2$ then $y = 2(2)^2 - 5(2) + 1 = -1$ $\therefore$ the point is (2, -1)

$\because \frac{dy}{dx} = 4x - 5$ $\therefore (\frac{dy}{dx})_{x=2} = 4 \times 2 - 5 = 3$

i.e. at the point (2, -1) the slope of the tangent = 3 and the slope of the normal = $\frac{-1}{3}$

$\therefore$ the equation of the tangent,

$y - (-1) = 3(x - 2)$

$y + 1 = 3x - 6$

$y - 3x + 7 = 0$

the equation of the normal

$y - (-1) = \frac{-1}{3}(x - 2)$

$3y + 3 = -x + 2$

$3y + x + 1 = 0$

Try to solve

(10) Find the equations of the tangent and the normal to the following curves at the indicated points:

a) $y = x^3 + x^2 - 1$, $x = -2$

b) $y = (3x - 5)^7$, $x = 2$

Exercises 3 - 3

Choose thr correct answer:

1) The rate of change of $x^3 + 4$ with respect to x when $x = 2$ equals

a) 4 b) 8 c) $\frac{1}{4}$ d) 12

2) The slope of the tangent to the curve $y = 3x - x^3$ at $x = 0$ equals

a) 3 b) zero c) - 3 d) 6

3) The tangent to the curve $y = x^2 - 8x + 2$ is parallel to x-axis at $x =$:

a) -8 b) 2 c) 4 d) zero

4) The straight line $x + y = 5$ touches the curve $y = 3x^2 + 5x + 1$ at $x =$

a) 1 b) 5 c) 3 d) -1

Complete the following:

5) $\frac{d}{dx}(2x) =$ ....

6) $\frac{d}{dx}(3x^2 + 1) =$ ....

7) $\frac{d}{dx}(x^4 - 2x^2 + 1)$ ....

8) $\frac{d}{dx}(x + \sqrt{x})$ ....

9 $\frac{d}{dx}\left(\frac{1}{x^3}\right) =$ ....

10 $\frac{d}{dx}(5\pi) =$ ....

11 $\frac{d}{dx}\left(x^{\frac{1}{2}}\right) =$ ....

12 $\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x}}\right) =$ ....

13 $\frac{d}{dx}(5x^2 + 3x + 2) =$ ....

14 $\frac{d}{dx}\left(\sqrt{2}\,x^7 - \frac{x^5}{5} + \pi\right) =$ ....

15 Find $\frac{dy}{dx}$ for each of the following :

a $y = 3x^5$ **b** $y = \frac{3}{4}x^{-8}$ **c** $y = \frac{3}{2x^2}$ **d** $y = 4\sqrt{x}$

Find the first derivative with respect to x of each of the following.

16 $y = x^3 + 3x^2 - 5$

17 $y = \frac{1}{2}x^4 - \frac{2}{3}x^3 + 7x - 9$

18 $y = 2x^6 + 3\sqrt{x}$

19 $y = (x^2 + 3)(x^3 - 3x + 1)$

20 $y = \frac{5x - 2}{5\quad x + 1}$

21 Find the value of $\frac{dy}{dx}$ of each of the following at the indicated point:

a $y = (x^2 - 2)^7$ at $x = 0$ **b** $y = (x^2 - x + 1)^{-4}$ at $x = 1$

22 Find $\frac{dy}{dx}$ of each the following:

a $y = (x + 3)^7$ **b** $y = (2x^2 - 3)^4$

c $y = (x^3 + x - 1)^5$

e $y = \sqrt[3]{(2x^3 - 4x + 7)^2}$ **f** $y = z^2$, $z = 3x^2 + 2$

23 Find the points on the curve $y = x^3 - 6x^2 - 15x + 20$ at which the tangent is parallel to x-axis.

24 Find the points on the curve $y = x^3 - 9x^2 - 16x + 1$ at which the slope of the tangent equals 5

25 Find the equations of the tangent and the normal to the curve $y = 3x^2 - 7x + 2$ at the point (2, 0) which lies on it .

26 Find the equation of the tangent to each of the following curves at the points on it whose abscissa is shown in the front of it:

a $y = (x + 3)^3$, $x = -1$ **b** $y = \frac{3}{x - 2}$, $x = 4$

c $y = \sqrt{x + 7}$, $x = 2$ **d** $y = (x - 5)(x + 5)$, $x = -3$

Unit Three

3-4 Integration

We will learn

- The anti derivative of a function.
- The indefinite integral
- Integral of some algebraic functions.

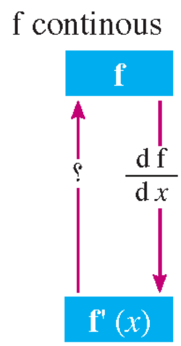
Key - term

- Antiderivative
- Integraation

matrials

- Scientific calculator.
- Computer graphing program.

Explore



From the differentiation study, you have learned: the derivative of the function f where $f(x) = x^3 + c$, c is real constant is $f'(x) = 3x^2$.

i.e. $f'(x) = \frac{d}{dx} f(x)$

in this case the function f is called the original function of the function f'

In this lesson we will study the anti derivative which means, if we know the derivative function f'. How can we get the original function f?

To find the original function whose derivative with respect to x is $5x^4$

let $f(x) = 5x^4$

Let's start with inverse way of differentiation operation

$n\,x^{n-1} = 5x^4 \qquad \therefore n - 1 = 4 \ , \ n = 5$

then $F(x) = x^5$ or $x^5 + 3$ or $x^5 - 2$

The function F is called the antiderivative of the function f

Can you discover the antiderivative F of function f if:

a $f(x) = 2x$ **b** $f(x) = 7x^6$

Learn

Antiderivative

If $y = x^2$ **then the first derivative is** $\frac{dy}{dx} = 2x$

the conclusion of y from the derivative function $\frac{dy}{dx}$ is called integral operation or anti derivative.

for example x^2 is an anti derivative of the function $2x$ notice that $2x$ has many anti derivatives such $x^2 + 1$, $x^2 + 2$, $x^2 - 3$, ... the derivative of all of them is $2x$ and the constant c is the different between them .

$\therefore \frac{d}{dx}(x^2 + c) = 2x$ where (c) is constant.

Definition

The function F is called antiderivative of the function f if $F'(x) = f(x)$ for all x in the domain of f.

Example

1. Prove that the function F where $F(x) = \frac{1}{2}x^4$ is antiderivative of the function f where $f(x) = 2x^3$.

Solution

Find the derivative of the function F then $F'(x) = \frac{1}{2} \times 4x^3 = 2x^3$

$\therefore F'(x) = f(x)$ **then the function** (F) is antiderivative of the function f

Try to solve

1. Show that the function F where $F(x) = \frac{1}{2}x^6$ is antiderivative of the function f where $f(x) = 3x^5$

Critical thinking:

If each of F_1, F_2 is an antiderivative of the function f. What is the relation between F_1, F_2?

The indefinite Integral

The set of all antiderivatives of the function f is called the indefinite integral of this function and denoted by $\int f(x)\,dx$ [**is read as integral of** $f(x)$ **with respect to** x]

Definition

If $F'(x) = f(x)$ then $\int f(x)\,dx = F(x) + c$

where c is an arbitrary constant.

Notice that: $\frac{d}{dx}(x^3 + c) = 3x^2$ $\quad\therefore \int 3x^2\,dx = x^3 + c$

$\frac{d}{dx}(2x^7) = 14x^6$ $\quad\therefore \int 14x^6\,dx = 2x^7 + c$

To calculate the value of c we need to know the integral value at a specific value of the independent variable x and this is outside your study.

Example

2. **Verify each of the following:**

a. $\int x^7\,dx = \frac{1}{8}x^8 + c$

b. $\int (7x^6 + \frac{4}{x^3})\,dx = x^7 - \frac{2}{x^2} + c$

Solution

a $\because \frac{d}{dx}(\frac{1}{8}x^8 + c) = x^7$ $\therefore \int x^7\, dx = \frac{1}{8}x^8 + c$

b $\because \frac{d}{dx}(x^7 - \frac{2}{x^2} + c) = \frac{d}{dx}(x^7 - 2x^{-2} + c)$

$= 7x^6 - 2(-2)x^{-3} = 7x^6 + \frac{4}{x^3}$

$\therefore \int (7x^6 + \frac{4}{x^2})\, dx = x^7 - \frac{2}{x^2} + c$

Try to solve

2) **Verify each of the following:**

a $\int x^{-4}\, dx = \frac{-1}{3}x^{-3} + c$ **b** $\int x\sqrt{1+x^2}\, dx = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c$

Example

3) Using the principle definition of integration, find the antiderivative of the function f where:

a $f(x) = 5x^4$ **b** $f(x) = 18x^{\circ}$ **c** $f(x) = -3x^{-4}$

Solution

a $F(x) = \int 5x^4\, dx = x^5 + c$

b $F(x) = \int 18x^{\circ}\, dx = \int 3(6x^5)\, dx = 3x6 + c$

c $F(x) = \int -3x^{-4}\, dx = x^{-3} + c$

Try to solve

3) Using the definition of integration, find the anti derivative of each of : $20x^4$, $-5x^{-4}$

Finding the anti derivatives of functions by using the previous definition requires a lot of time and effort. So some integral standard forms can be used which facilitate the process of finding the anti derivative.

Rule (1) :

$\int x^n\, dx = \frac{1}{n+1}x^{n+1} + c$ **where c is constant , n is a rational number , n ≠ -1**

Example

Find:

4) **a** $\int x^5\, dx$ **b** $\int x^{-3}\, dx$ **c** $\int x^{\frac{2}{5}}\, dx$ **d** $\int \frac{1}{\sqrt[5]{x^3}}\, dx$

Solution

a) $\int x^5\,dx = \frac{x^{5+1}}{5+1} + c = \frac{1}{6}x^6 + c$

b) $\int x^{-3}\,dx = \frac{x^{-3+1}}{-2} + c = -\frac{1}{2}x^{-2} + c$

c) $\int x^{\frac{2}{5}}\,dx = \frac{1}{\frac{7}{5}}x^{+1\frac{2}{5}} + c$

$= \frac{5}{7}x^{\frac{7}{5}} + c$

d) $\int \frac{1}{\sqrt[5]{x^3}}\,dx = \int x^{\frac{-3}{5}}\,dx$

$= \frac{1}{\frac{2}{5}}x^{1+\frac{-3}{5}} + c = \frac{5}{2}x^{\frac{2}{5}} + c$

Try to solve

4) Find:

a) $\int x^8\,dx$ b) $\int x^{\frac{2}{3}}\,dx$ c) $\int x^{\frac{7}{9}}\,dx$ d) $\int \sqrt[7]{x^5}\,dx$

Rule (2) :

$\int a\,.\,f(x)\,dx = a\int f(x)\,dx$ **where** f is a constant real number

Example

5) a) $\int 3x^3\,dx = 3\int x^3\,dx = 3\times\frac{1}{4}x^4 + c = \frac{3}{4}x^4 + c$

b) $\int 8x^{-5}\,dx = 8\int x^{-5}\,dx = 8\times\frac{1}{-4}x^{-4} + c = -2x^{-4} + c$

Corollary

$\int a\,dx = a\,x + c$

then:

$\int 5\,dx = 5x + c$, $\int -9\,dt = -9t + c$

$\int dx = x + c$, $\int \sqrt{7}\,dz = \sqrt{7}\,z + c$

Try to solve

5) Find each of :

a) $\int 3x^4\,dx$ b) $\int -2z\,dz$ c) $\int -x\,dx$ d) $\int -\frac{dx}{5}$

Rule (3) :

$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

Example

6 Find: a $\int (4x + 3x^2)\, dx$ b $\int \frac{x^2 - 2x - 8}{x + 2}\, dx$

Solution

a $\int (4x + 3x^2)\, dx$

$= \int 4x\, dx + \int 3x^2\, dx$

$= 4\int x\, dx + 3\int x^2\, dx$

$= \frac{4}{2}x^2 + 3 \times \frac{1}{3}x^3 + c$

$= 2x^2 + x^3 + c$

b $\int \frac{x^2 - 2x - 8}{x + 2}\, dx$

$= \int \frac{(x - 4)(x + 2)}{x + 2}\, dx$

$= \int (x - 4)\, dx$

$= \frac{1}{2}x^2 - 4x + c$

Try to solve

6 Find:

a $\int (3x^2 + 2x - 1)\, dx$ b $\int (\frac{1}{x^2} + \sqrt{x} + 3)\, dx$

c $\int 2x(x + 3)\, dx$ d $\int \frac{4x^2 - 9}{2x - 3}\, dx$

Rule (4) :

$$\int [(ax + b)^n\, dx = \frac{1}{a(n + 1)}(ax + b)^{n+1} + c, \quad n \neq -1$$

Critical think:

1- Can you verify the previous rule by using the definition of anti deteriorative ? Explain.

Example

7 Find:

a $\int (3 - 2x)^5\, dx$ b $\int (2x - 7)^{-3}\, dx$

c $\int \frac{7}{\sqrt{3x - 4}}\, dx, \ x > \frac{4}{3}$

Solution

a $\int (3x-2)^5\,dx = \frac{1}{3(5+1)}(3x-2)^{5+1} + c = \frac{1}{18}(3x-2)^6 + c$

b $\int (2x-7)^{-3}\,dx = \frac{1}{2(-3+1)}(2x-7)^{-3+1} + c = -\frac{1}{4}(2x-7)^{-2} + c$

c $\int \frac{7}{\sqrt{3x-4}}\,dx = \int 7(3x-4)^{-\frac{1}{2}}\,dx = \frac{7}{3(-\frac{1}{2}+1)}(3x-4)^{-\frac{1}{2}+1} + c$

$= \frac{7}{3(-\frac{1}{2}+1)}(3x-4)^{\frac{1}{2}} = \frac{14}{3}\sqrt{3x-4} + c$

Try to solve

7) Find: **a** $\int 9(4-3x)^2\,dx$ **b** $\int \frac{15}{(3x-5)^6}\,dx$

Exercises 3 - 4

Find:

1) $\int x^2\,dx$
2) $\int x^7\,dx$
3) $\int 8x\,dx$
4) $\int -4x^3\,dx$
5) $\int 9x^8\,dx$
6) $\int 12x^{-4}\,dx$
7) $\int 5\,dx$
8) $\int (5\sqrt[3]{x}\,dx$
9) $\int -3x^7\,dx$
10) $\int -\frac{8}{5}x^3\,dx$
11) $\int \frac{7}{3}t^6\,dt$
12) $\int \frac{12}{5}f^5\,df$
13) $\int (x+1)\,dx$
14) $\int (5-2t)\,dt$
15) $\int (t^3 - 6t^4)\,dt$
16) $\int (x^3 + x^2 + x)\,dx$
17) $\int 2(3x^2 + 7)\,dx$
18) $\int x(x+3)\,dx$
19) $\int 7x^2(x^4 - 1)\,dx$
20) $\int (x-1)(x+1)\,dx$
21) $\int (x-2)(2-x)\,dx$
22) $\int (2x+3)(x-1)\,dx$
23) $\int x^2(2x + \frac{1}{x})\,dx$
24) $\int \frac{3x^2 - 4x}{x}\,dx$
25) $\int \frac{x^7 + 5x^6 - x^3}{x^3}\,dx$
26) $\int \frac{x^2 - 1}{x - 1}\,dx$
27) $\int \frac{x^3 - 27}{x - 3}\,dx$
28) $\int (x+4)^3\,dx$
29) $\int 7(2x-7)^6\,dx$
30) $\int (8-3t)^4\,dt$
31) $\int 6(x-3)^{-4}\,dx$
32) $\int (2x-3)^{\frac{2}{3}}\,dx$
33) $\int 15\sqrt{(3x-2)^5}\,dx$

General Exercises

Choose the correct answer:

1) if the average rate of change of $f = 3.2$ when x changes from 7 to 7.2 then the variation of $f = ...$

a) -6.4 b) 6.4 c) 1.6 d) -1.6

2) $\frac{d}{dx}(2 - 3x)^{-2}\, dx$

a) $12x^{-2} - 27x^{-4}$ b) $\frac{1}{3}(2 - 3x)^{-1}$ c) $6(2 - 3x)^{-3}$ d) $-2(2 - 3x)^{-3}$

3) The slope of the tangent to the curve of the function $y = (2x - 3)^5$ at $x = 2$ equals

a) 1 b) $\frac{1}{12}$ c) 5 d) 10

4) $\int (x^2 - 3)\, dx =$

a) $2x$ b) $x^3 - 3x$ c) $\frac{1}{3}x^3 - 3x + c$ d) $2x - 3 + c$

Answer the following:

5) Find the first derivative of each of the following functions:

a) $y = 3x^2 - 5x + 2$ b) $y = x(x^2 - 3x + 5)$

c) $y = (x - 2)(x + 2)$ d) $y = \frac{x - 1}{x + 1}$

6) Find $\frac{dy}{dx}$ of each of the following :

a) $y = z^2$, $z = x^4 - 6$ b) $y = 7z^3$, $z = \frac{1 + x}{1 - x}$

7) Find :

a) $\int (5x^4 - 3x^2 + 3)\, dx$ b) $\int 2x(x^2 + 3)\, dx$

c) $\int (2x + 3)^5\, dx$ d) $\int \frac{x^3 - 8}{x - 2}\, dx$

8) Find the equation of the tangent to the curve $y = (x - 2)(x + 1)$ at the points of intersection of the curve with x-axis.

9) Find the points on the curve $y = x^3 - 6x^2 - 15x + 20$ at which the tangent is parallel to x-axis.

10) Find the measure of the positive angle with the tangent to the curve $y = x^2 + \frac{1}{x} - 1$ makes with the positive direction of x-axis at $x = 1$.

11. Find the measure of the angle which the tangent of each of the following curves makes with the positive direction of x-axis:

a. $y = x^2 - 3x + 2$ at $(2, 0)$

b. $y = x^3 - x^2 - 4$ at $(1, -4)$

c. $y = (x - 1)(2x + 1)$ at the point $(0, -1)$

d. $y = x(x - 1)(x + 9)$ at the point $(0, 0)$

12. Find the slope of the tangent to the curve $y = \frac{x}{x + 2}$ at the point $(-1, -1)$ on it.

13. Find the slope of the tangent to the curve $y = \frac{x^2 + x + 1}{x^2 + 1}$ at $x = 1$