Arabic Republic of Egypt
Ministry of Education
Book Sector

Mathematics

Student's Book

First form secondary

SECOND TERM

2025-2026

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Book Sector

Mathematics

Student's Book

First form secondary Second term

Mathematics has Practical applications in various fields including road construction, bridges and urban planning and preparing their maps which depend on parallel lines and their transversals according to the proportion between the real length and the drawing length.

Elsalam bridge connecting between the two shores of the Suez canel

2025-2026

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2025-2026

Introduction

بسم اللّه الرحمن الرحيم

we are pleased we offer this book to make it clear philosophy that has been in light construction of educational material and can be summarized as follows:

1. To emphasise that the main purpose of these books is to help the learner to solve problems and make decisions in their daily lives, and help them to participate in society.
2. Emphasis on the principle of continuity of life-long learning through work that students gain a systematic scientific thinking, and practice learning mixed with fun and suspense, relying on the development of problem- solving skills and develop the skills of the conclusion and reasoning, and the use of methods of self-learning, active learning and collaborative learning team spirit, and discussion and dialogue, and accept the opinions of others, and objectivity in sentencing, in addition to some definition of national activities and accomplishments.
3. Provide a comprehensive coherent visions of the relationship between science, technology and society (STS) reflect the role scientific progress in the development of the local community, in addition to focusing on the practice of conscious students to act effectively about to use technological instruments.
4. The development of positive attitudes towards the study of mathematics and aspect of its scientists.
5. To provide students with a comprehensive culture to use the available environmental resource.
6. Rely on the fundamentals of knowledge and develop methods of thinking, the development of scientific skills and stay away of the details and educational memorization, that's concern directed to bring concepts and general principles and research methods, problem solving and methods of thinking about the fundamental distinction mathematics from the others.

We have been especially cautions in this book the following:

★ The book has been divided into integrated and coherent units, for each one there is an introduction shows its aims, lessons, a short, and key terms, it has been divided into lessons explain the goal of study under the title"you will learn", each lesson starts with the main idea to the content of the lesson .It takes onto consideration, the presentation to the scientific article from easy to difficult and includes a set of activities that integrated with other subjects and to suit different abilities of students and take into consideration the individual differences between them and emphasizes the collaborative work, and integrated with the subject.

★ Every lesson has been presented examples from easy to difficult, it include variety of levels of thinking with drills on it under the title of " try to solve" and the lesson ends with a title of "check your understanding"

★ Each unit ends with a summary of the unit deals with concepts and instructions contained in the unit.

Eventually, we hope getting the right track for the benefits of our students as well as for our dearest Egypt hopping bright future to our dearest students. And the God of the intent behind, with leads to either way.

Contents

Unit Four

Analytic Geometry , Straight Line

Unit Five

Trigonometry

Algebra

Unit 1

Matrices

Unit objectives

By the end of this unit, the student should be able to:

- Recognize the concept of the matrix and its order.
- Recognize some special matrices (row - column
- - square - zero - diagonal - unit - symmetric and
- skew symmetric) matrix.
- Multiply a real number by a matrix .
- Recognize the equality of two matrices.
- Carry out the operations of addition, subtraction
- and multiplication on matrices.
- Verify the solutions of some problems including
- matrices using the available programs.
- Model some life problems using matrices .
- Use matrices in other domains.
- Recognize the determinant of a matrix of order 2×2 and 3×3.
- Find the value of the triangular determinant.
- Find the inverse of the square matrix of order 2×2.
- Solve two simultaneous equations using the
- inverse of a matrix.
- Solve the equations using Cramar›s rule
- Find the area of the triangle using determinants.

Key - Terms

- Matrix
- Element
- Row matrix
- Column matrix
- Square matrix
- Zero matrix
- Equal matrices
- Symmetric matrix
- Skew-symmetric matrix
- Unit matrix
- Matrix equation
- Variable matrix
- Constant matrix
- Adding matrices
- Subtracting matrices
- Matrix multiplication
- Transpose of matrix
- Determinant
- Second order determinant
- Third order determinant
- Coefficient matrix
- Inverse matrix

Lessons of the Unit

Lesson (1 - 1): Organizing data in Matrices.

Lesson (1 - 2): Adding and subtracting Matrices.

Lesson (1 - 3): Matrix Multiplication .

Lesson (1 - 4): Determinants .

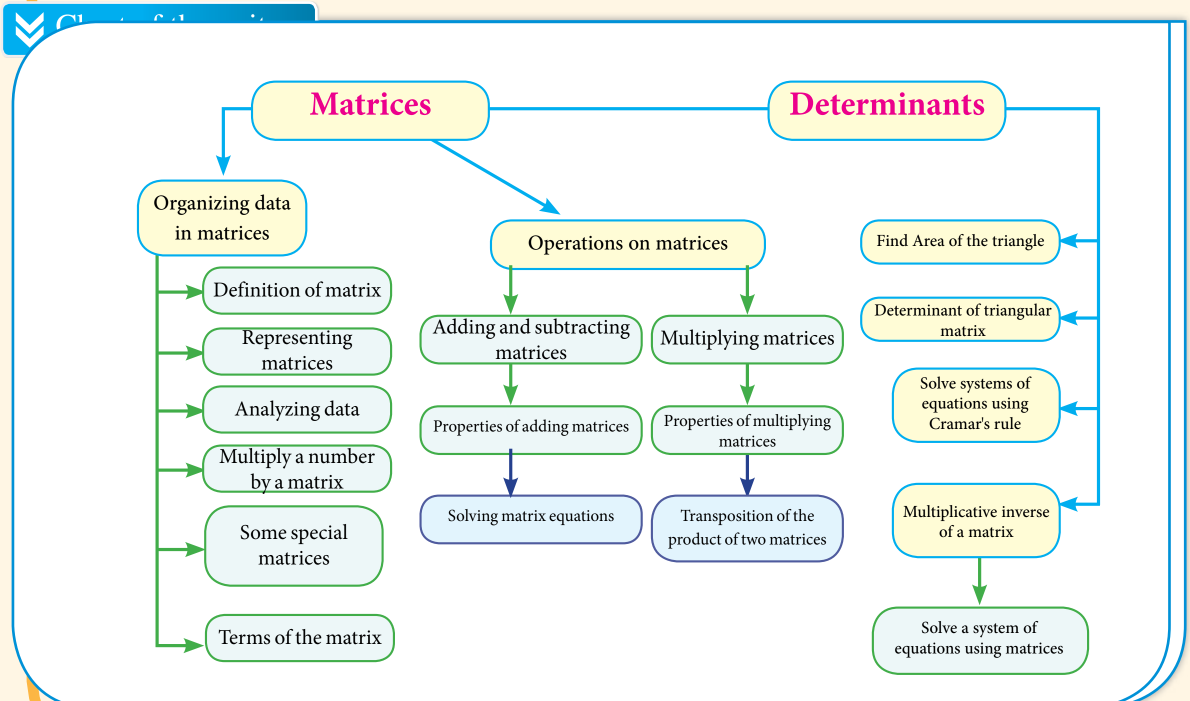
Lesson (1 - 5): Multiplicative inverse of a matrix

Materials

Scientific calculator - Excel program- Computer.

Brief History

Matrices is the plural of the word matrix and is one of mathematical concepts commonly used in modern times. It includes many branches of knowledge. We use it in science, statistics, economics, sociology, psychochology and so on. Because it displays the data and stores them in the form of rectangular array and organizes the data in this form to remember, compare and carry out the operations on them easily. Further more, matrices have an important role in mathematics especially in the branch of linear algebra. Scientist Kelly was the first to observe and use matrices in (18211895-)



1 - 1 Organizing data in Matrices

You will Learn

- What is the matrix?
- Some special matrices
- (square - row - columnzero - diagonal - unit) matrix.
- Symmetric and skew symmetric matrix
- Equality of two matrices.
- Multiply a real number by a matrix.

Key - Terms

- Matrix
- Element
- Row matrix
- Column matrix
- Square matrix
- Zero matrix
- Equal matrices
- Symmetric matrix
- Skew symmetric matrix

Materials

- Graphic calculator
- Excel program
- Computer
- Scientific calculator

Think and discuss

Industry

A three-section factory for producing some components of TV screens. Produces 4 principal parts of the screen A, B, C, and D as follows:

The first section produces 75 pieces from A, 135 pieces from B, 150 pieces from C, and 215 pieces from D daily.

The second section produces 100 pieces from A, 168 pieces from B, 210 pieces from C, and 282 pieces from D daily.

The third section Produces 80 pieces from A, 100 pieces from B, 144 pieces from C , and 64 pieces from D daily.

It is difficult to remember or compare these data on this form, in this way, so there is a question:

How can these data be arranged in order to analyse and benefit from them?

To answer this question, we can record these data in a table to know what each of the three sections in the factory produces from the different parts quickly and clearly, and also easily compare the production of the three sections of different parts.

Parts

Sections	A	B	C	D
First section	75	135	150	215
Second section	100	168	210	282
Third section	80	100	144	64

If we know that the numbers in the first row represent the production of the first section from parts A, B, C and D respectively, similarly, the numbers in the second row represent the production of the second section respectively and the numbers in the third row represent the production of the third section respectively, then we can write these data which recorded in the previous table in a simple form as follows :

	First column	Second column	Third column	Fourth column
First row	75	135	150	215
Second row	100	168	210	282
Third row	80	100	144	64

This form is called a «Matrix» , the numbers enclosed by two parentheses( ) are called «elements of the matrix»

This matrix has 3 rows and 4 columns, thus it is said that it is a matrix of order 3 * 4 or simply «a 3 * 4 matrix». You always write the number of rows first, and the number of columns second. We notice that : number of elements of the matrix = 3 *4 = 12 elements .

Now:

1-Is there another method to arrange these data to form another matrix? Explain your answer.

2- From the previous matrix, what is the element in the first row and second column? and what is the element in the second row and first column?

3- Open question: Write an example of your own in which the data included are in the form of a 2 * 3 matrix

Learn

Organizing Data in Matrices

The matrix is an arrangement of a number of elements (variables or numbers) in rows and columns enclosed by two parentheses as ( ). The elements in the matrix are arranged such that the position of each element in the matrix has a meaning. Capital letters are used to name the matrix or to symbolize it as A, B, C, X, Y, ... but small letters are used to name the elements of the matrix as a, b, x, x, y, ...

If A is a matrix of order m* n then we can express it in the form ai j where i is the number of rows and j is the number of columns.

For example, the element a_{12} lies in the first row and in the second column also a32 lies in the third row and in the second column.

In the matrix $A = \begin{pmatrix} -1 & 4 & 6 & 5 \\ 2 & -1 & 2 & 4 \\ 3 & 5 & -2 & -1 \end{pmatrix}$

The element -1 lies in the second row and in the second column, and is denoted by the symbol a_{22}

The element 6 lies in the first row and in the third column and is denoted by the symbol a_{13}

Generally:

The matrix consists of « m» rows and «n» columns and is in the form of m × n or of order m × n or of type m × n (and is read as «m» times «n») where m and n are positive integers.

Try to solve

1 Use the matrix B = $\begin{pmatrix} 1 & 5 \\ 3 & 2 \\ 5 & 7 \end{pmatrix}$ to answer each of the following:

A What is the order of the matrix B?

B What is the value of b_{12} and b_{21}?

Learn

Representing of Matrcies

If A is a m × n matrix, then it is possible to write the matrix A in the form:

$A = (a_{ij})$, $i = 1, 2, 3, ..., m$

$j = 1, 2, 3, ..., n$

We will study only the cases of matrices that m ≤ 3 and n ≤ 3

Example

1 Write all the following elements of matrices:

A $A = (a_{ij})$, $i = 1, 2$, $j = 1, 2, 3$

B $B = (b_{ij})$, $i = 1, 2, 3$, $j = 1$

C $C = (c_{ij})$, $i = 1, 2$, $j = 1, 2$

Solution

A $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ is a 2 × 3 matrix B

B $B = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$ is a 3 ×1 matrix

C $C = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a 2 × 2 matrix

Try to solve

2 Write all the following elements of matrices:

A $A = (a_{ij})$, $i = 1, 2, 3, j = 1, 2, 3$

B $B = (b_{ij})$, $i = 1, 2, j = 1$

Example

2 **Consumer:** the table opposite shows the prices in pounds for 3 kinds of sandwiches in 3 different sizes in a fast food restaurant.

	Small	Medium	Large
Fried chicken	8	12	16
Fried shrimps	9	13	17
Fried fish	7	11	15

A Arrange these data in a matrix, such that the prices are arranged ascendingly.

B Determine the order of this matrix.

C What is the value of the element a_{32}?

Solution

A

	Small	Medium	Large
Fried fish	7	11	15
Fried chicken	8	12	16
Fried shrimps	9	13	17

$$\begin{pmatrix} 7 & 11 & 15 \\ 8 & 12 & 16 \\ 9 & 13 & 17 \end{pmatrix}$$

B There are 3 rows and 3 columns, thus the matrix is of order 3×3

C The value of the element a_{32} lies in the 3rd row and in the 2nd column which is 13

Try to solve

3 The coach of a team of the basket-ball in the school recorded the scores of three players in the classes league, as follows:

Samir: played 10 games , 20 shots , 5 scores.

Hazem: played 16 games , 35 shots , 8 scores.

Karim: played 18 games , 41 shots , 10 scores.

A Arrange these data in a matrix such that the players are arranged ascendingly according to their number of scores.

B Determine the order of the matrix. What is the value of a_{23}?

Learn

Some special Matrices

A **Square matrix:** It is a matrix in which the number of its rows equals the number of its columns. For example: $\begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$ (a 2×2 square matrix)

B **Row matrix:** It is a matrix containing one row and any number of columns. For example : $(2 \ \ 4 \ \ 6 \ \ 8)$ (a 1×4 row matrix)

C **Column matrix:** It is a matrix containing one column and any number of rows. For example: $\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ (a 3×1 column matrix)

D **Zero matrix:** It is a matrix in which all of its elements are Zeros. It may be a square matrix or not. For examples:

(0) is a 1×1 zero matrix, $(0 \;\; 0)$ is a 1×2 zero matrix, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a 2×1 zero matrix, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a 2×2 square zero matrix and is denoted by $\bigcirc$.

E **Diagonal matrix:** It is a square matrix in which all elements are zeros except the elements of its diagonal then at least one of them is not equal to zero. For example: the matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (is a 3×3 diagonal matrix)

F **Unit matrix:** it is a diagonal matrix in which each element on the main diagonal has the numeral 1, while 0 exists in all other elements , it is denoted by I. for example: each of:

(1) , $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a unit matrix.

Try to solve

4 Write the type and the order of each of the following matrices.

A $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ **B** $(1 \;\; 3 \;\; 5 \;\; 7)$ **C** $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

D $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **E** $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ **F** $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

5 Write the 3×3 zero matrix

Learn

Equality of two Matrices

Two matrices A and B are equal if and only if they have the same order and the elements of the matrix A are equal to the corresponding elements in the matrix B **i.e.:** $a_{ij} = b_{ij}$. $\forall i$ and j.

Example

3 **A** The two matrices $\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 0 \\ -1 & 5 & 0 \end{pmatrix}$ **are not equal because they are different in order.**

B $\begin{pmatrix} 1 & x & 2 \\ -1 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 2 \\ -1 & 6 & y \end{pmatrix}$ **if and only if $x = -3$, $y = 5$**

C The two matrices $\begin{pmatrix} 1 & 2 \\ x & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & y \\ 3 & -1 \end{pmatrix}$ **can not be equal because the corresponding elements are not equal**

D $\begin{pmatrix} 0 & 1 & 5 \\ 1 & 7 & 0 \\ 2 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 5 \\ 1 & 7 & 0 \\ 2 & 6 & 3 \end{pmatrix}$

The two matrices are equal because they have the same order and the corresponding elements are equal.

Try to solve

6 **A** If $A = \begin{pmatrix} -0.75 & \frac{1}{5} \\ \frac{1}{2} & -2 \end{pmatrix}$, $B = \begin{pmatrix} -\frac{3}{4} & 0.2 \\ 0.5 & -2 \end{pmatrix}$ Is A = B? Explain your answer.

B If $X = \begin{pmatrix} 3 & 4 \\ 0 & -2 \end{pmatrix}$, $Y = \begin{pmatrix} -3 & 4 \\ 0 & -2 \end{pmatrix}$ Is X = Y? Explain your answer.

Example

Use the equal matrices to solve the equations

4 If: $\begin{pmatrix} 2x - 5 & 4 \\ 3 & 3y + 12 \end{pmatrix} = \begin{pmatrix} 25 & 4 \\ 3 & y + 18 \end{pmatrix}$. Find the value of x and y.

Solution

$\begin{pmatrix} 2x - 5 & 4 \\ 3 & 3y + 12 \end{pmatrix} = \begin{pmatrix} 25 & 4 \\ 3 & y + 18 \end{pmatrix}$

$\because$ The two matrices are equal $\Rightarrow$ the corresponding elements are equal:

$2x - 5 = 25$, $3y + 12 = y + 18$

$2x = 25 + 5$, $3y - y = 18 - 12$

$2x = 30$, $2y = 6$

$x = 15$, $y = 3$

the solution is $x = 15$, $y = 3$

Try to solve

7 If $\begin{pmatrix} x + 8 & -5 \\ 3 & -y \end{pmatrix} = \begin{pmatrix} 38 & -5 \\ 3 & 4y - 10 \end{pmatrix}$. Find the value of each of x , y

8 Critical thinking: If $(3x \quad x + y \quad x - z) = (-9 \quad 4 \quad -10)$. Find the values of x, y and z

9 Critical thinking: If: $\begin{pmatrix} a + b & a - b \\ a + b + c & a - b + 2d \end{pmatrix} = \begin{pmatrix} 9 & -3 \\ 7 & 5 \end{pmatrix}$. Find the values of a, b, c, d

Learn

Multipling a Real Number by a Matrix

Multiplying a real number by a matrix means multiplying each element of the elements of the matrix by that real number **i.e.:**

Product of a real number K by the m × n matrix "A' is the matrix C = KA with the same order m × n, each element in it as C_{ij} equals the corresponding element to it in the matrix A multiplied by the real number K .

i.e.: $C_{ij} = K a_{ij}$ where i = 1, 2, ..., m , j = 1, 2, ..., n

Notice that:

$K\begin{pmatrix} x & y \\ z & \ell \end{pmatrix} = \begin{pmatrix} kx & ky \\ kz & k\ell \end{pmatrix}$

For example: $-2\begin{pmatrix} 4 & 1 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} -2\times 4 & -2\times 1 \\ -2\times 5 & -2\times -1 \end{pmatrix} = \begin{pmatrix} -8 & -2 \\ -10 & 2 \end{pmatrix}$

Try to solve

10 If $A = \begin{pmatrix} 15 & -12 & 10 \\ 20 & -10 & 7 \\ -2 & 1 & 3 \end{pmatrix}$ Find $-5A$

Learn

Transpose of Matrix

In any matrix A of order m × n, if the rows are replaced by the columns and the columns are replaced by the rows in the same order, then we get a matrix of order n × m which is called the transpose of matrix A and is denoted by the symbol A^t. It is clear that $(A^t)^t = A$

Example

5 Find the transpose of each of the following matrices:

A $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 5 \end{pmatrix}$ **B** $B = (1 \quad -2 \quad 6)$ **C** $C = \begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$

Solution

A $A^t = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ -1 & 5 \end{pmatrix}$ is a matrix of order 3 × 2

B $B^t = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}$ is a matrix of order 3 × 1

C $C^t = \begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$ is a matrix of order 2 × 2

Symmetric and Skew Symmetric Matrices

If A is a square matrix, then it is called a symmetric if and only if $A = A^t$, and is called skew symmetric if and only if $A = -A^t$

Example

6 Is the matrix $B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}$ symmetric or skew symmetric?

Solution

$B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}$ $\quad B^t = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix}$

$B^t = -1 \times \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix} = -B$

$\therefore$ $B^t = -B$ and $B = -B^t$ then the matrix B is skew symmetric

Try to solve

11 Is the matrix $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 5 \\ -1 & 5 & 6 \end{pmatrix}$ Symmetric or skew symmetric?

Check your understanding

1 Find the value of x, y, z in each of the following:

A $\begin{pmatrix} x & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$ $\quad$ B $\begin{pmatrix} x & 3 & 0 \\ 0 & 1 & y \end{pmatrix} = \begin{pmatrix} 2 & z & 0 \\ 0 & 1 & 3 \end{pmatrix}$

2 Show which of the following matrices is symmetric and which is skew symmetric:

A $\begin{pmatrix} 1 & -1 & 4 \\ -1 & 2 & 6 \\ 4 & 6 & 5 \end{pmatrix}$ $\quad$ B $\begin{pmatrix} 0 & -\frac{5}{2} & -1 \\ \frac{5}{2} & 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} & 0 \end{pmatrix}$

Exercise (1 - 1)

1 A, B and C are three cities , if the distance in kilometres between any two cities is shown in the table opposite.

	A	B	C
A	0	75	80
B	75	0	56
C	80	56	0

A Write a matrix represent these data.

B Let X be the required matrix in "A" find the following:

1- What does it mean by X_{23}? ....

2- What does it mean by X_{32}? ....

3- what is the relation between x_{23}, x_{32}? ....

C Write all elements of the second row in the matrix X. ....

D Write all elements of the second column in the matrix X. What do you deduce from **C** and **D**?

....

E Find X_{kk} when K = 1, 2, 3 What do you notice? ....

F Complete each of the following:

1- X is a matrix of order ....

2- $X_{ij} = X_{ji}$ for all values ....

2 What is the number of elements in each of the following matrices:

A Matrix of order 2×3 ....

B Matrix of order 2×2 ....

C Matrix of order 3×2 ....

3 Find the values of a, b, c and d if:

A $\begin{pmatrix} 3 & -5 \\ a-3 & 3d-2 \end{pmatrix} = \begin{pmatrix} a-2 & 2b+1 \\ c & 16 \end{pmatrix}$ ....

B $\begin{pmatrix} 15 & 2b \\ 0 & 2a+c \end{pmatrix} = \begin{pmatrix} 3a & 10 \\ 2b-d & 10 \end{pmatrix}$ ....

4 Find the value of each of a and b if $\begin{pmatrix} 4 & -1 \\ 2a-1 & 3b+1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 3 & 7 \end{pmatrix}$

5 If $A = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2d & -1 \\ 3e & 4 \end{pmatrix}$ where $A = B^t$

then find the value of each of d and e.

6 If $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -3 \\ 4 & 5 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 3 & 3 \\ -4 & -5 & 5 \end{pmatrix}$

Find $A + B$, $A - B$, $A + 2B$, $B - 3A$

7 **Critical thinking:** If $A = (A_{xy})$ $\forall$ x, y $\in$ { 1, 2, 3 } Write the matrix A, given that $A_{xy} = y - x$, then find A^t

1 - 2 Adding and subtracting Matrices

You will Learn

- Adding matrices.
- Subtracting matrices.

Key - Terms

- Adding matrices
- Subtracting matrices
- Graphic calculator

Group work

Statistics: Work with your classmate and use the data in the following table:

Year	The arithmetic mean of marks			
	Science		Mathematics	
	Male	Female	Male	Female
2011	428	420	502	457
2012	425	421	501	460
2013	429	426	503	463

1- **A** Find the sum of marks of the two arithmetic means for male in each year in the table.

B Find the sum of marks of the two arithmetic means for female in each year in the table.

2- **A** Write a matrix representing the arithmetic mean for the marks of science for male and female. lable a title for the matrix and its rows and columns.

B What is the order of the matrix?

3- **A** Write a matrix representing the mean of marks of mathematics for male and female. lable a title for the matrix and its rows and columns.

B What is the order of the matrix?

4- By checking your answer for question (1) and the matrices that you wrote in questions (2) and (3) , write a third matrix representing the sum of marks of the two arithmetic means for male and females. Lable a title for the matrix and its rows and columns. What is the order of the matrix?

5- Use your observations, and any patterns you see to find a method for adding the matrices.

Learn

Adding Matrices

Sometimes, we need to add or subtract the matrices in order to get new data. To find the matrix of addition, add the corresponding elements.

i.e.: If A, B are two matrices of order $m \times n$, **then** $A + B$ is also a matrix of order $m \times n$ and each element in it is the sum of the two corresponding elements in A and B.

Example

1) If $A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 7 & -2 \\ 1 & -4 \end{pmatrix}$. Find: A + B.

Solution

$A + B = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 7 & -2 \\ 1 & -4 \end{pmatrix}$ **(by substituting A and B)**

$= \begin{pmatrix} 0+7 & 2+(-2) \\ -1+1 & 3+(-4) \end{pmatrix}$ **(Add corresponding elements)**

$= \begin{pmatrix} 7 & 0 \\ 0 & -1 \end{pmatrix}$ **(Simplify)**

Try to solve

1) If $A = \begin{pmatrix} -4 & -1 \\ -3 & -7 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -7 \\ 8 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$. Find each of the following "if possible":

A) A + B B) A + C

Learn

Properties of Adding Matrices

Let A , B and C be three matrices of order $m \times n$ and O is a zero matrix of the same order, then:

1- Closure property: A + B form a matrix of order $m \times n$

If $A = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$ is a matrix of order 2×2, $B = \begin{pmatrix} 7 & 2 \\ -2 & 0 \end{pmatrix}$ is a matrix of order 2×2,

then $A + B = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 7 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ -2 & 3 \end{pmatrix}$ is a matrix of order 2×2

2- Commutative property: A + B = B + A

If $A = \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 5 \\ -2 & 3 \end{pmatrix}$, **Show that** A + B = B + A

3- Associative property: (A + B) + C = A + (B + C)

If $A = \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 5 \\ -2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix}$. **Show that** (A + B) + C = A + (B + C)

4- Additive identity property: A + O = O + A = A

for example: $\begin{pmatrix} 1 & 2 & 3 \\ 6 & -5 & 4 \\ 7 & 8 & -9 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 6 & -5 & 4 \\ 7 & 8 & -9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 6 & -5 & 4 \\ 7 & 8 & -9 \end{pmatrix}$

5- Additive inverse property: A + (-A) = (-A) + A = O

where (-A) is the additive inverse of the matrix A

For example $\begin{pmatrix} 3 & 5 & 2 \\ 2 & 0 & -5 \end{pmatrix} + \begin{pmatrix} -3 & -5 & -2 \\ -2 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

where $\begin{pmatrix} -3 & -5 & -2 \\ -2 & 0 & 5 \end{pmatrix} = -\begin{pmatrix} 3 & 5 & 2 \\ 2 & 0 & -5 \end{pmatrix}$

Learn

Subtracting matrices

If each of the two matrices A, B of order $m \times n$, then the matrix $C = A - B = A + (-B)$ where C is a matrix of order $m \times n$ and $(-B)$ is the inverse of the matrix B with respect to the addition of matrices.

For example: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} x & y \\ z & \ell \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -x & -y \\ -z & -\ell \end{pmatrix} = \begin{pmatrix} a-x & b-y \\ c-z & d-\ell \end{pmatrix}$

Example

2 If $A = \begin{pmatrix} 7 & -4 & 11 \\ 6 & 5 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 9 & 2 \\ 8 & -7 & -3 \end{pmatrix}$. Prove that $A - B \neq B - A$.

Solution

$A - B = \begin{pmatrix} 7 & -4 & 11 \\ 6 & 5 & -1 \end{pmatrix} - \begin{pmatrix} 5 & 9 & 2 \\ 8 & -7 & -3 \end{pmatrix}$,

$= \begin{pmatrix} 7 & -4 & 11 \\ 6 & 5 & -1 \end{pmatrix} + \begin{pmatrix} -5 & -9 & -2 \\ -8 & 7 & 3 \end{pmatrix}$

$= \begin{pmatrix} 2 & -13 & 9 \\ -2 & -2 & -4 \end{pmatrix}$ **(1)**

$B - A = \begin{pmatrix} 5 & 9 & 2 \\ 8 & -7 & -3 \end{pmatrix} - \begin{pmatrix} 7 & -4 & 11 \\ 6 & 5 & -1 \end{pmatrix}$

$= \begin{pmatrix} 5 & 9 & 2 \\ 8 & -7 & -3 \end{pmatrix} + \begin{pmatrix} -7 & 4 & -11 \\ -6 & -5 & 1 \end{pmatrix}$

$= \begin{pmatrix} -2 & 13 & -9 \\ 2 & -12 & -2 \end{pmatrix}$ **(2)**

from (1), (2), we notice that: $A - B \neq B - A$ (subtraction of matrices is not commutative)

Think: Is the subtraction of matrices associative?

Example

3 If $A = \begin{pmatrix} 2 & 5 & -1 \\ 3 & -4 & 6 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 4 & 3 \\ 9 & -2 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 6 & -1 & -2 \\ -3 & 5 & 4 \end{pmatrix}$. Find the matrix $2A - 3B + 5C$

Solution

$2A = 2\begin{pmatrix} 2 & 5 & -1 \\ 3 & -4 & 6 \end{pmatrix} = \begin{pmatrix} 2\times2 & 2\times5 & 2\times-1 \\ 2\times3 & 2\times-4 & 2\times6 \end{pmatrix} = \begin{pmatrix} 4 & 10 & -2 \\ 6 & -8 & 12 \end{pmatrix}$

$3B = 3\begin{pmatrix} -1 & 4 & 3 \\ 9 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 3\times-1 & 3\times4 & 3\times3 \\ 3\times9 & 3\times-2 & 3\times5 \end{pmatrix} = \begin{pmatrix} -3 & 12 & 9 \\ 27 & -6 & 15 \end{pmatrix}$

$-3B = \begin{pmatrix} 3 & -12 & -9 \\ -27 & 6 & -15 \end{pmatrix}$

$5C = 5\begin{pmatrix} 6 & -1 & -2 \\ -3 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 5\times6 & 5\times-1 & 5\times-2 \\ 5\times-3 & 5\times5 & 5\times4 \end{pmatrix} = \begin{pmatrix} 30 & -5 & -10 \\ -15 & 25 & 20 \end{pmatrix}$

$\therefore 2A - 3B + 5C = \begin{pmatrix} 4 & 10 & -2 \\ 6 & -8 & 12 \end{pmatrix} + \begin{pmatrix} 3 & -12 & -9 \\ -27 & 6 & -15 \end{pmatrix} + \begin{pmatrix} 30 & -5 & -10 \\ -15 & 25 & 20 \end{pmatrix}$

$= \begin{pmatrix} 4+3+30 & 10-12-5 & -2-9-10 \\ 6-27-15 & -8+6+25 & 12-15+20 \end{pmatrix} = \begin{pmatrix} 37 & -7 & -21 \\ -36 & 23 & 17 \end{pmatrix}$

Try to solve

2 If $A = \begin{pmatrix} 2 & -1 \\ -3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 4 \\ 6 & -2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -3 \\ 0 & 3 \end{pmatrix}$. Find the matrix $2A - 3B + 4C$

Check your understanding

1 Find the value of each of the following:

A $\begin{pmatrix} 2 & 3 & 1 \\ 9 & 0 & -1 \end{pmatrix} + \begin{pmatrix} -4 & 2 & 1 \\ 1 & 3 & -2 \end{pmatrix}$ **B** $\begin{pmatrix} 3 & -3 \\ 0 & 4 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} -3 & 4 \\ 5 & -1 \\ 2 & 2 \end{pmatrix}$

2 If $A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & 1 \end{pmatrix}$

A Find A - B , B - A. What do you notice? **B** Verify - (A + B) = (-A) + (-B)

Exercise (1 - 2)

1. If $A = \begin{pmatrix} -2 & 0 & -1 \\ 4 & 5 & 0 \end{pmatrix}$ and $K_1 = 2$, $K_2 = -1$ then find each of the following matrices: K_1A and K_2A

2. If $A = \begin{pmatrix} -7 & 0 & -5 \\ 4 & 7 & 5 \end{pmatrix}$, $B = \begin{pmatrix} -8 & 4 \\ 0 & 7 \\ -6 & -5 \end{pmatrix}$ then find the result of the following operations if "possible" , give reasons in case of impossible solution.

   A) $A + B$ B) $A + B^t$ C) $A^t + B$

3. If $X = \begin{pmatrix} -4 & -2 \\ 3 & 6 \\ 0 & 4 \end{pmatrix}$, $Y = \begin{pmatrix} 5 & 1 \\ 0 & -2 \\ 4 & -3 \end{pmatrix}$, $Z = \begin{pmatrix} -2 & -4 \\ -3 & 2 \\ 6 & 0 \end{pmatrix}$ then find the matrix $3X - Y + Z$

4. If $A = \begin{pmatrix} 4 & 8 & -6 \\ 2 & -4 & 8 \\ 6 & 12 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -6 & 2 \\ 4 & -10 & 0 \\ -1 & 8 & -4 \end{pmatrix}$, then find the matrix X where $X = 2A - 3B$

5. **Critical thinking :** Find the values of a, b, c and d which satisfy the equation:

$$2\begin{pmatrix} a & 3 \\ 6 & b \end{pmatrix} = 3\begin{pmatrix} a & d \\ c & -2 \end{pmatrix} - 4\begin{pmatrix} c & 3 \\ 0 & a \end{pmatrix}$$

1 - 3 Multiplying matrices

You will Learn

- Multiplying matrices.
- Properties of multiplying matrices.
- Transpose of product of two matrices.

Key - Terms

- Multiplying matrices
- Transpose of matrix

- Scientific calculator

Group work

Work with your classmate and use the data in the table opposite:

	Meal (1)	Meal (2)	Meal (3)
Price of meal in pounds	3.50	2.75	2
Number of sold meals	50	100	75

1- What is the price of lunches (1)? lunches(2)? lunches (3)?

2- **A** What's the total price of all sold units of the three meals?

B Show how to use the data in the table to find the solution.

3- **A** Write the matrix of order 1×3 to represent the price of each sold meal.

B Write the matrix of order 3×1 to represent the number of sold meals.

C **Writing:** Use the words: "row - column- element" to describe the steps of using the matrices which you got to find how much the three meals are sold.

Now: To multiply matrices, multiply the elements of each row in the first matrix by the elements of each column in the second matrix, then add the products.

For example, to find the product of: $A = \begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix}$

We multiply a_{11} by b_{11}, then multiply a_{12} by b_{21} and add the

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} ? & \square \\ \square & \square \\ \square & \square \end{pmatrix} \qquad (0) \times 5 + 2 \times (-1) = -2$$

The result is the element in the first row and the first column. Repeat the same steps with the rows and columns left.

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & ? \\ \square & \square \\ \square & \square \end{pmatrix} \qquad (0)(0) + (2)(1) = 2$$

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ ? & \square \\ \square & \square \end{pmatrix} \qquad (-2)(5) + (-3)(-1) = -7$$

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -7 & ? \\ \square & \square \end{pmatrix}$$

(-2)(0) + (-3)(1) = -3

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -7 & -3 \\ 1 & \square \end{pmatrix}$$

(1)(0)+(4)(1)=4

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -7 & -3 \\ ? & \square \end{pmatrix}$$

(1)(5) + (4)(-1)= 1

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -7 & -3 \\ 1 & 4 \end{pmatrix}$$

4- Describe a model for coloured rows and columns.

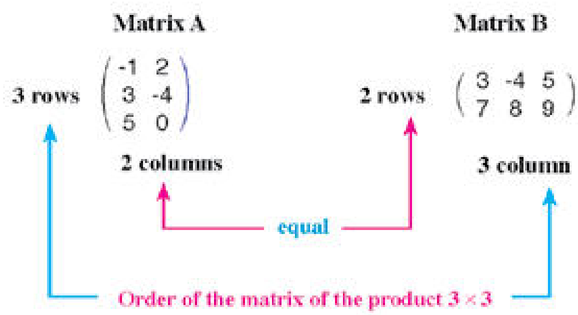
5- **A** What is the order of the original matrices in the previous example and what is the order of the matrix of the product?

B **Critical thinking:** How can we compare the order of the matrix of the product and the original matrices?

Learn

Multiplying matrices

You can multiply two matrices if and only if the number of columns of the first matrix equals the number of rows of the second matrix. When multiplying the matrix A of order $m \times n$ by the matrix B of order $n \times \ell$ then the product is the matrix AB of order $m \times \ell$ **for example:**



Example

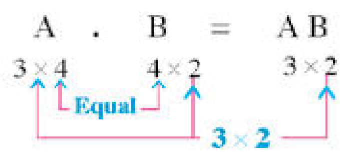
1 Determine whether the matrix of the product A B is defined or not in each case.

A If the matrix A of order 3×4 and the matrix B of order 4×2

B If the matrix A of order 5×3 and the matrix B of order 5×2

Solution

A The number of columns of the matrix A equal the number of rows of the matrix B, then matrix of the product AB is defined and of order 3×2



B The number of columns of the matrix A is not equal to the number of rows of the matrix B, then matrix of the product AB is undefined.

Try to solve

1 Determine whether the matrix of the product AB is defined or not in each case. Give a reason.

A If the matrix A of order 3×2 and the matrix B of order 2×3.

B If the matrix A of order 1×3 and the matrix B of order 1×3.

From the definition of multiplying matrices, it is possible that AB is defined while BA is undefined. Generally, if each of AB and BA are defined, then AB is not necessarily equal BA, even if they have the same order.

Example

2 If $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 3 \\ 0 & 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 1 \\ 5 & 0 & -1 \end{pmatrix}$. Find each of A B , B A. What do you notice?

Solution

∵ A is a matrix of order 3 × 3 and B of order 3 × 3, then A B is defined (because the number of columns of A equals the number of rows of B)

then the matrix of the product is of order 3 × 3

$$A\,B = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 3 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 1 \\ 5 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times2+(-1)\times3+2\times5 & 1\times1+(-1)\times4+2\times0 & 1\times0+(-1)\times1+2\times(-1) \\ -1\times2+0\times3+3\times5 & -1\times1+0\times4+1\times0 & -1\times0+0\times1+3\times(-1) \\ 0\times2+1\times3+4\times5 & 0\times1+1\times4+4\times0 & 0\times0+1\times1+4\times(-1) \end{pmatrix} = \begin{pmatrix} 9 & -3 & -3 \\ 13 & -1 & -3 \\ 23 & 4 & -3 \end{pmatrix}$$

∵ B is a matrix of order 3 × 3 and A is of order 3 × 3, B A is defined (because the number of columns of B equals the number of rows of A) then, the matrix of the product is of order 3 × 3

$$BA = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 1 \\ 5 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 3 \\ 0 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times1+1\times-1+0\times0 & 2\times-1+1\times0+0\times1 & 2\times2+1\times3+0\times4 \\ 3\times1+4\times-1+1\times0 & 3\times-1+4\times0+0\times1 & 3\times2+4\times3+1\times4 \\ 5\times1+0\times-1+(-1)\times0 & 5\times-1+0\times0+0\times1 & 5\times2+0\times3+-1\times4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 7 \\ -1 & -3 & 22 \\ 5 & -5 & 6 \end{pmatrix}$$

Notice that $AB \neq BA$. **It is possible to use the multiplication of matrices in some life situations.**

Example

3 Tourism: A tourist company has 3 hotels in Hurghada, the table opposite shows the number of different rooms in each hotel. If the daily fare of 1- bed room is 250 pounds, 2-bed room is 450 pounds and the suite is 600 pounds.

Hotel	1-bed room	2-bed room	Suite
Venus	28	64	8
Pearl	35	95	20
Diamond	20	80	15

A Write a matrix representing the number of different rooms in the three hotels, then write a matrix of prices of rooms.

B Write a matrix representing the daily income for the company. Assuming that all the rooms have been filled.

C What is the daily income for the company , assuming that all the rooms have been filled?

Solution

A **Write the matrix of number of rooms A as folows:** $A = \begin{pmatrix} 28 & 64 & 8 \\ 35 & 95 & 20 \\ 20 & 80 & 15 \end{pmatrix}$

and write the matrix of prices of rooms B as follows: $B = \begin{pmatrix} 250 \\ 450 \\ 600 \end{pmatrix}$

Notice that the matrices were written such that the number of rows in the matrix A equals the number of columns in the matrix B in order to do the multiplication operation and find the required in (B) and (C) .

B The matrix of the daily income for the company is the matrix A B = $\begin{pmatrix} 28 & 64 & 8 \\ 35 & 95 & 20 \\ 20 & 80 & 15 \end{pmatrix} \begin{pmatrix} 250 \\ 450 \\ 600 \end{pmatrix}$

$$= \begin{pmatrix} 28 \times 250 + 64 \times 450 + 8 \times 600 \\ 35 \times 250 + 95 \times 450 + 20 \times 600 \\ 20 \times 250 + 80 \times 450 + 15 \times 600 \end{pmatrix} = \begin{pmatrix} 40600 \\ 63500 \\ 50000 \end{pmatrix}$$

C The daily income for the company = 40600 + 63500 + 50000 = 154100 pounds

Learn

Properties of multiplying matrices

From the definition of adding and multiplying matrices. Assuming the necessary conditions are met for definitions : It is possible to deduce the following properties:

1- Associative property of multiplication: $(A\,B)\,C = A\,(B\,C)$ **Now,** if:

$A = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 3 & 2 \\ 2 & -1 \end{pmatrix}$. Find (A B) C, A (B C). What do you notice? Is the multiplication operation of matrices associative?

2- Multiplicative identity property $A\,I = I\,A = A$ where I is the unit matrix

Now, if $A = \begin{pmatrix} 2 & -3 \\ -1 & 5 \end{pmatrix}$. Prove that: AI= IA= A where I is the unit matrix

3- Distributive property of multiplication on addition of matrices.

$A(B + C) = A\,B + A\,C$
$(A + B)\,C = A\,C + B\,C$

Now, if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 \\ 5 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 1 \\ -3 & 0 \end{pmatrix}$

Prove that: **A** A(B + C) = AB + AC **B** (B + C) A = B A + CA

Transpose of the product of two matrices

From the definition of the transpose of the matrix and the definition of multiplying of matrices, it is possible to deduce the following property: $(A\,B)^t = B^t\,A^t$

Now, if $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 2 \\ 1 & -1 \\ 4 & 3 \end{pmatrix}$, Prove that: $(A\,B)^t = B^t\,A^t$

Check your understanding

Determine whether the matrix of the product AB is defined or not in each of the following , if it is defined, find the order of the resulted matrix:

A The matrix A is of order 3 × 1, and the matrix B is of order 2 × 3

B The matrix A is of order 3 × 3, and the matrix B is of order 2 × 2

Exercise (1 - 3)

1 If $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 3 \\ 3 & 9 \end{pmatrix}$, then find each of the following:

A A B ....

B B A ....

C (A + B)A ....

2 If $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x & 7 \\ 3 & y \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 11 & 18 \end{pmatrix}$ find the value of each of x, y:

....

3 **Critical thinking:** If A and B are two matrices and, $\bigcirc$ is the zero matrix, A B = $\bigcirc$ Does it aways mean that A = $\bigcirc$ or B = $\bigcirc$. Take $A = \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$, then show your opinion.

....

....

1 - 4 Determinants

You will Learn

- Determinant of the square matrix of the second order.
- Determinant of the square matrix of the third order.
- Determinant of the triangular matrix.
- Finding the area of the triangle using the determinants .
- Solve the system of linear equations using the determinants.
- Solve the system of linear equations using Cramar's rule

Key - Terms

- Determinan
- Second order determinant
- Third order determinant
- Principle or leading diagonal
- Other diagonal
- Coefficient matrix

- Scientific calculator .
- Graphic papers.

Think and discuss

1- What is the square matrix?

2- Write a square matrix of order 2×2 and of order 3×3

3- If A is a square matrix of order 2×2 where: $A = \begin{pmatrix} 2 & 5 \\ 1 & 7 \end{pmatrix}$, then the determinant of the matrix A is the number defined as follows:

$|A| = 2 \times 7 - 5 \times 1 = 14 - 5 = 9$

What is the determinant of each of the following matrices?

$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 5 \\ -3 & 1 \end{pmatrix}$

Learn

Determinants

If A is a square matrix of order 2×2 where:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the determinant of the matrix A is denoted by $|A|$ and is called determinant of the second order and is the number defined as follows:

Other diagonal

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\,d - b\,c$$

Principle diagonal

We notice that: the value of the determinant of the second order equals the product of the two elements of the principle diagonal minus the product of the two elements of the other diagonal.

Example

1 Find the value of each of the following determinants:

A $\begin{vmatrix} 4 & 5 \\ 3 & 7 \end{vmatrix}$ **B** $\begin{vmatrix} 0 & 5 \\ 7 & 3 \end{vmatrix}$ **C** $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ **D** $\begin{vmatrix} 1 & 0 \\ 2 & 7 \end{vmatrix}$

Solution

A $\begin{vmatrix} 4 & 5 \\ 3 & 7 \end{vmatrix} = 4 \times 7 - 3 \times 5$
$= 28 - 15 = 13$

C $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \times 1 - 0 \times 0$
$= 1 - 0 = 1$

B $\begin{vmatrix} 0 & 5 \\ 7 & 3 \end{vmatrix} = 0 \times 3 - 7 \times 5$
$= 0 - 35 = -35$

D $\begin{vmatrix} 1 & 0 \\ 2 & 7 \end{vmatrix} = 1 \times 7 - 2 \times 0$
$= 7 - 2 = 5$

Try to solve

1) Find the value of each of the following determinants :

A) $\begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix}$ B) $\begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix}$ C) $\begin{vmatrix} a & b \\ b & c \end{vmatrix}$

Learn

Third order determinant

The determinant of the matrix of order 3×3 is called a third order determinant. To find the value of the third order determinant , $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ then:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(e\,i - f\,h) - b(\,d\,i - f\,g) + c\,(d\,h - e\,g)$$

Example

2) To find the value the determinant $\begin{vmatrix} 7 & 2 & 5 \\ 3 & 4 & 1 \\ -1 & 2 & 6 \end{vmatrix}$, then:

$$\begin{vmatrix} 7 & 2 & 5 \\ 3 & 4 & 1 \\ -1 & 2 & 6 \end{vmatrix} = 7\begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -1 & 6 \end{vmatrix} + 5\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= 7\,(4 \times 6 - 2 \times 1) - 2\,(3 \times 6 - 1 \times (-1)) + 5\,(3 \times 2 - 4 \times (-1))$$

$$= 7 \times 22 - 2 \times 19 + 5 \times 10$$

$$= 154 - 38 + 50 = 166$$

Learn

Minor determinant corresponding to any element of a matrix

If the matrix A is a matrix of order 3×3 where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Then: the minor determinant corresponding to the element a_{11} is denoted by $|a_{11}|$ which is $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Notice that we get this determinant by elemenating the row and the column intersected at the element a_{11} as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Similarly:

- The minor determinant corresponding to the element a_{12} and is denoted by $|a_{12}|$ which is $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$
- The minor determinant corresponding to the element a_{13} and is denoted by $|a_{13}|$ which is $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$
- The minor determinant corresponding to the element a_{21} and is denoted by $|a_{21}|$ which is $\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

and so on. All these determinants are determinants of the second order:

Important Notes

1- If A is a square matrix of order 3 × 3 in the form:

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, and the determinant A is denoted by the symbol |A| where:

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}|a_{11}| - a_{12}|a_{12}| + a_{13}|a_{13}|$$

2- Notice that, we multiply each element in the minor determinant corresponding to it preceeding by the signs +, -, +, ... respectively and the sign of the minor determinant corresponding to the element, a_{ij} is determined by the rule:
Sign of $|a_{ij}|$ is the same as the sign of $(-1)^{i+j}$

For example: **the sign of** $|a_{12}|$ is the same as the sign of $(-1)^{1+2}$ which is negative

The sign of $|a_{13}|$ is the same as the sign of $(-1)^{1+3}$ which is positive

To determine the sign of each minor determinant corresponding to an element , we add the two orders of row and column which intersect at the element:

- If the sum of the two orders is **even,** then the sign is **positive** .
- If the sum of the two orders is **odd,** then the sign is **negative** .

We notice that, the rule of signs of the minor determinant is as follows: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

3- It is possible to expand the determinant in terms of the elements of any row or column and its minor determinant but with a suitable sign.

Example

3 To find the value of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 7 & -2 & -1 \end{vmatrix}$ using the elements of the second column.

Notice that the signs of the minor determinant corresponding to the elements of the second column is - , + , - respectively, then the determinant:

$= -2\begin{vmatrix} 4 & 5 \\ 7 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 7 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix}$

$= -2\,(-4 - 35) + 0 + 2\,(5 - 12)$

$= 78 - 14 = 64$

Useful idea for solution

you can expand the determinant using any row or column in which the greatest number of zeros to get its value easily after taking the suitable sign

Try to solve

2 Find the value of each of the following determinants:

A $\begin{vmatrix} -1 & 7 & 5 \\ 3 & 0 & 1 \\ 4 & 0 & 6 \end{vmatrix}$ **B** $\begin{vmatrix} -2 & 3 & 7 \\ 0 & 4 & 5 \\ 0 & 0 & -3 \end{vmatrix}$ **C** $\begin{vmatrix} 3 & 4 & 0 \\ 2 & -3 & 1 \\ 5 & 0 & -2 \end{vmatrix}$ **D** $\begin{vmatrix} 2 & 0 & -3 \\ 5 & -1 & 4 \\ -2 & 0 & 3 \end{vmatrix}$

4 In any determinant if the rows replace the columns in the same order, the value of the determinant is unchanged

$\triangle = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ b_{12} & b_{22} & b_{32} \\ c_{13} & c_{23} & c_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{vmatrix}$ this can be proved by expanding the two determinants.

Example

4 Prove that $\begin{vmatrix} 2 & -3 & -1 \\ 1 & 0 & 4 \\ -2 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -2 \\ -3 & 0 & 5 \\ -1 & 4 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & -3 & -1 \\ 1 & 0 & 4 \\ -2 & 5 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 4 \\ 5 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} = 2\,(0 - 20) + 3(2 + 8) - (5 - 0) = -15$$

$$\begin{vmatrix} 2 & 1 & -2 \\ -3 & 0 & 5 \\ -1 & 4 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 5 \\ 4 & 2 \end{vmatrix} - \begin{vmatrix} -3 & 5 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} -3 & 0 \\ -1 & 4 \end{vmatrix} = 2\,(0 - 20) - (-6 + 5) - 2(-12, -0) = -15$$

So : $\begin{vmatrix} 2 & -3 & -1 \\ 1 & 0 & 4 \\ -2 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -2 \\ -3 & 0 & 5 \\ -1 & 4 & 2 \end{vmatrix}$

Try to solve

3 Prove that $\begin{vmatrix} 1 & 2 & -3 \\ 5 & 4 & 1 \\ -1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -1 \\ 2 & 4 & 3 \\ -3 & 1 & 2 \end{vmatrix}$

5 The value of the determinant vanishes in each of the following two cases

First: if all the elements of any row (column) in any determinant equal zero, then the value of the determinant = 0

The value of the determinant $\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ \text{zero} & \text{zero} & \text{zero} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = 0$

This can be proved by expanding the determinant using the elements of second row, then: $\triangle = 0$

Second: If the corresponding elements of any two rows (columns) in any determinant are equal, then the value of the determinant = zero

i.e. $\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} = 0$

Since the corresponding elements of the first and second rows are equal (prove that).

Example

5 Without expanding the determinant, prove that $\begin{vmatrix} 1 & -5 & 8 \\ 3 & 7 & 9 \\ 1 & -5 & 8 \end{vmatrix} = 0$

Solution

In the determinant, we find $R_3 = R_1$ $\therefore$ The value of the determinant = 0

Try to solve

4 Without expanding the determinant, prove that $\begin{vmatrix} 3 & -1 & 3 \\ 2 & 5 & 2 \\ -1 & 7 & -1 \end{vmatrix} = 0$

6 If there is a common factor in all the elements of any row (column) in a determinant, then this factor can be taken outside the determinant

Example

6 Without expanding the determinant, find the value of $\begin{vmatrix} 3 & 2 & -7 \\ 4 & 6 & 2 \\ 10 & 15 & 5 \end{vmatrix}$

Solution

Take 2 as a common factor of R_2 and 5 as a common factor of R_3

$$\begin{vmatrix} 3 & 2 & -7 \\ 4 & 6 & 2 \\ 10 & 15 & 5 \end{vmatrix} = 2 \times 5 \begin{vmatrix} 3 & 2 & -7 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 10 \times 0 = 0$$

Since the corresponding elements in R_2 and R_3 are equal"Try to prove that by another method".

Try to solve

5 If $\begin{vmatrix} a & d & m \\ b & e & n \\ c & f & z \end{vmatrix} = 10$ find the value of $\begin{vmatrix} 2a & 2d & 2m \\ b & e & n \\ -4c & -4f & -4z \end{vmatrix}$

7 In any determinant, if the positions of two rows (columns) are interchanged. The value of the resulting determinant equals the additive inverse of the value of the original determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = - \begin{vmatrix} a & b & c \\ x & y & z \\ d & e & f \end{vmatrix}$$

by interchanging the second and the third rows

Example

7 Without expanding the determinant, prove that: $\begin{vmatrix} 2 & -1 & 2 \\ 3 & 3 & 4 \\ 5 & 7 & 6 \end{vmatrix} + \begin{vmatrix} 3 & 3 & 4 \\ 2 & -1 & 2 \\ 5 & 7 & 6 \end{vmatrix} = 0$

Solution

By interchanging the first and the second rows in the first determinant

$$\therefore - \begin{vmatrix} 3 & 3 & 4 \\ 2 & -1 & 2 \\ 5 & 7 & 6 \end{vmatrix} + \begin{vmatrix} 3 & 3 & 4 \\ 2 & -1 & 2 \\ 5 & 7 & 6 \end{vmatrix} = -\triangle + \triangle = \text{zero}$$

Learn

Determinant of triangular Matrix

The triangular matrix is a matrix in which all its elements above or below the principal diagonal are zeros as:

$\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 2 & -4 & 0 \\ 5 & -1 & 2 \end{pmatrix}$

and We notice that: the value of the determinant of the triangular matrix equals the product of the elements of its principal diagonal .

i.e.:

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\,a_{22}\,a_{33}$$

To prove that, expand the determinant using the elements of the first row:

The determinant $= a_{11} \begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} = a_{11}\,(a_{22} \times a_{33} - a_{12} \times 0) = a_{11}\,a_{22}\,a_{32}$

Example

8 What is the value of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & 5 \\ 0 & 0 & 6 \end{vmatrix}$?

Solution

We notice that the determinant is the determinant of the triangular matrix, then the value of the determinant $= 1 \times -3 \times 6 = -18$

Try to solve

6 Find the value of each of the following determinant:

A $\begin{vmatrix} -1 & 2 & 5 \\ 0 & 3 & -4 \\ 0 & 0 & -2 \end{vmatrix}$ **B** $\begin{vmatrix} -3 & 2 & 5 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{vmatrix}$

Learn

Finding area of a triangle by using Determinants

You can use the determinant to find the surface area of the triangle in terms of the coordinates of the vertices of the triangle as follows:

Area of the triangle in which its vertices are: X (a, b), Y (c, d), Z (e, f) equals |A| where:

$$A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

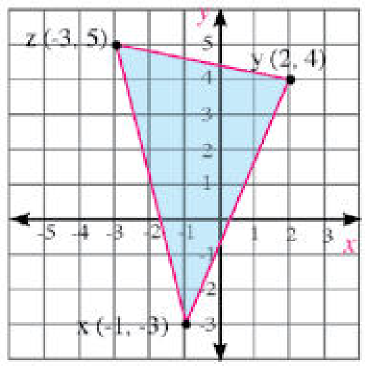
Remember

|A| means the positive value of A.

Example

9 Find the area of the triangle in which the coordinates of its vertices are (-1, -3) , (2, 4), (-3, 5)

Solution



$$A = \frac{1}{2}\begin{vmatrix} -1 & -3 & 1 \\ 2 & 4 & 1 \\ -3 & 5 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-1\begin{vmatrix} 4 & 1 \\ 5 & 1 \end{vmatrix} - (-3)\begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ -3 & 5 \end{vmatrix}\right]$$

$$= \frac{1}{2}[-1(4 - 5) + 3(2 + 3) + 1(10 + 12)]$$

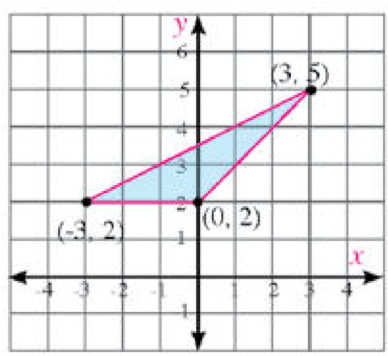
$$= \frac{1}{2}(1 + 15 + 22) = 19 \text{ square units}$$

Try to solve

7 Find the area of the triangle ABC in which A(-2, -2), B (3, 1), C (-4, 3)

Example

10 Geometry: if the coordinates of the three points in the lattice are (0, 2) (3, 5), (-3, 2), measured in metres. Find the area of the triangle in which its vertices are that points.



Solution

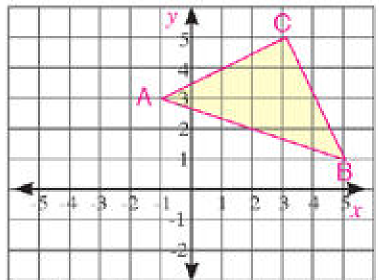
$$A = \frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 3 & 5 & 1 \\ -3 & 2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[0\begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix}\right]$$

$$= \frac{1}{2}\left|0 - 0 - 3\,(2\text{-}5)\right| = 4\,\frac{1}{2}\text{square metres}$$

Try to solve

8 Find the area of the triangle shown in the figure opposite.



Learn

Solving a system of linear equations by Cramer's rule

1- Solving a system of Linear equations in two variables

If we have a system of linear equations in two variables as follows:

$$a\,x + b\,y = m$$
$$c\,x + d\,y = n$$

Then the matrix whose elements are the coefficients of the two variables after arranging the system by a matrix of the coefficients $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and you can use the determinant to solve systems of the linear equations, if the value of the determinant of the matrix of coefficients $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and denoted by the symbol $\triangle$ (is read as delta) is not equal to zero, then the system has a unique solution. If the value of the determinant equals zero, either the system has an infinite number of solution or has no solution.

We notice that the two coefficients of the variable x form the first column to the determinant $\triangle$ and the two coefficients of the variable y form the second column to the determinant $\triangle$.

$\begin{vmatrix} m & b \\ n & d \end{vmatrix}$ is called the determinant of the variable x and is denoted by the symbol $\triangle x$ (is read as delta x), and we get it from the determinant $\triangle$ after changing the elements of the first column (coefficients of x) by the constants m and n.

also $\begin{vmatrix} a & m \\ c & n \end{vmatrix}$ is called the determinant of the variable y and is denoted by the symbol $\triangle y$ (is read as delta y), and we get it from the determinant $\triangle$ after changing the elements of the second column (coefficients of y) by the constants m and n.

Now: Let $\triangle \neq 0$, then the solution of the system is:

$$x = \frac{\triangle x}{\triangle} = \frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{m\,d - n\,b}{a\,d - c\,b} \quad , \quad y = \frac{\triangle y}{\triangle} = \frac{\begin{vmatrix} a & m \\ b & n \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{an - b\,m}{ad\text{-}c\,b}$$

Example

(11) Solve the system of the following equations using Cramar's rule.

x - 3 y = -4 $\qquad$ 2x + y = 2

Solution

$: \Delta = \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = (1 \times 1) - (2 \times -3) = 1 + 6 = 7 \neq 0$

$x = \frac{\Delta x}{\Delta} = \frac{\begin{vmatrix} -4 & -3 \\ 2 & 1 \end{vmatrix}}{7} = \frac{(-4 \times 1) - (2 \times -3)}{7} = \frac{-4 + 6}{7} = \frac{2}{7}$

$y = \frac{\Delta y}{\Delta} = \frac{\begin{vmatrix} 1 & -4 \\ 2 & 2 \end{vmatrix}}{7} = \frac{(1 \times 2) - (-4 \times 2)}{7} = \frac{2 + 8}{7} = \frac{10}{7}$

The solution set = $\{(\frac{2}{7}, \frac{10}{7})\}$

Check:

- $\frac{2}{7} - 3(\frac{10}{7}) \stackrel{?}{=} -4$

  $\frac{-28}{7} = -4$ (✓)

- $2(\frac{2}{7}) + \frac{10}{7} \stackrel{?}{=} 2$

  $2 = 2$ (✓)

Try to solve

(9) Solve the system of the following equations using Cramar's rule:

x + 2 y = 0 $\qquad$ 2 x - 3 y = 1

2- Solving systems of Linear equations in three variables

If we have a system of linear equations in three variables as follows:

$a_1 x + b_1 y + c_1 z = m \qquad a_2 x + b_2 y + c_2 z = n \qquad a_3 x + b_3 y + c_3 z = k$

Then, by a similar way as we did in case of system of linear equations in two variables:

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ = determinant of the coefficients

$\Delta x = \begin{vmatrix} m & b_1 & c_1 \\ n & b_2 & c_2 \\ k & b_3 & c_3 \end{vmatrix}$ = determinant of the variable x

we get it by changing the elements of the first column (coefficients of x) by the constants m, n, k

$\Delta y = \begin{vmatrix} a_1 & m & c_1 \\ a_2 & n & c_2 \\ a_3 & k & c_3 \end{vmatrix}$ = determinant of the variable y

we get it by changing the elements of the second column (coefficients of y) by the constants m, n, k

$$\Delta z = \begin{vmatrix} a_1 & b_1 & m \\ a_2 & b_2 & n \\ a_3 & b_3 & k \end{vmatrix} = \text{determinant of the variable } z$$

we get it by changing the elements of the third column (coefficients of z) by the constants m, n, k

Now , If $\Delta \neq$ zero, then: $x = \frac{\Delta x}{\Delta}, y = \frac{\Delta y}{\Delta}, z = \frac{\Delta z}{\Delta}$

Example

12 Solve the system of the following linear equations using Cramar's rule.

$-x + 3y + z = 0$ $\qquad$ $3x - 2y - z = 1$ $\qquad$ $x + y + 2z = 0$

Solution

$$\Delta = \begin{vmatrix} -1 & 3 & 1 \\ 3 & -2 & -1 \\ 1 & 1 & 2 \end{vmatrix} = -1(-4+1) - 3(6-1) + 1(-3+2) = 3 - 15 - 1 = -13$$

$$\Delta_x = \begin{vmatrix} 0 & 3 & 1 \\ 1 & -2 & -1 \\ 0 & 1 & 2 \end{vmatrix} = -1(6-1) = -5 \qquad \Delta_y = \begin{vmatrix} -1 & 0 & 1 \\ 3 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix} = 1(-1 \times 2 - 1 \times 1) = -3$$

$$\Delta_z = \begin{vmatrix} -1 & 3 & 0 \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1(-1 \times 1 - 1 \times 3) = 4$$

$\therefore x = \frac{\Delta x}{\Delta} = \frac{-5}{-13} = \frac{5}{13}, \qquad y = \frac{\Delta y}{\Delta} = \frac{-3}{-13} = \frac{3}{13},$

$z = \frac{\Delta z}{\Delta} = \frac{4}{-13}$

The solution set $= \{(\frac{5}{13}, \frac{3}{13}, \frac{-4}{13})\}$

Check:

- $-(\frac{5}{13}) + 3(\frac{3}{13}) + (\frac{-4}{13}) \stackrel{?}{=} 0$
  $0 = 0$ (✓)
- $3(\frac{-5}{13}) + 2(\frac{3}{13}) - (\frac{-4}{13}) \stackrel{?}{=} 1$
  $1 = 1$ (✓)
- $1(\frac{5}{13}) + 1(\frac{3}{13}) + 2(\frac{-4}{13}) \stackrel{?}{=} 0$
  $0 = 0$ (✓)

Try to solve

10 Solve the system of the following linear equations using Cramar's rule:

$x + y - z = 2$ $\qquad$ $x + 2y + z = 7$ $\qquad$ $3x - y + z = 10$

Exercise (1 - 4)

① Find the value of each of the following determinants:

A $\begin{vmatrix} 7 & 5 \\ 3 & 2 \end{vmatrix}$ ....

B $\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$ ....

C $\begin{vmatrix} 6 & -3 \\ 19 & -7 \end{vmatrix}$ ....

D $\begin{vmatrix} a+x & a \\ b+y & b \end{vmatrix}$ ....

E $\begin{vmatrix} x+1 & x^2+1 \\ y+1 & y^2+1 \end{vmatrix}$ ....

F $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 4 & 4 \\ 0 & 7 & 8 \end{vmatrix}$ ....

G $\begin{vmatrix} 0 & 42 & 3 \\ 2 & 18 & 7 \\ 0 & 28 & 3 \end{vmatrix}$ ....

H $\begin{vmatrix} 3 & -4 & -3 \\ 2 & 0 & -31 \\ 5 & 0 & 2 \end{vmatrix}$ ....

I $\begin{vmatrix} 13 & 3 & 23 \\ 30 & 7 & 5 \\ 0 & 0 & 1 \end{vmatrix}$ ....

② Solve the following linear equations by Cramar's rule:

A $2x - 3y = 5$
$3x + 4y = -1$

B $x + y = 5$
$2x + 5y = 16$

C $x + 3y = 5$
$2x + 5y = 8$

D $3x + 2y = 5$
$2x + y = 3$

E $3x = 1 - 4y$
$5x + 12 = 7y$

F $2x = 3 + 7y$
$y = 5 - x$

G $2x + y - 2z = 10$
$3x + 2y + 2z = 1$
$5x + 4y + 3z = 4$

③ **Geometry:** Find the area of the triangle A B C in which A(2, 4), B (−2, 4), and C(0, −2).

....

④ Find the area of triangle X Y Z in which X (3, 3), Y (−4, 2), and Z (1, −4).

....

⑤ Use the determinants to prove that the points (3, 5), (4, −1), (5, 7). are collinear

....

....

6 If $\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = 12$, then $\begin{vmatrix} z & x & y \\ f & d & e \\ c & a & b \end{vmatrix} =$

a -12 b -6 c 6 d 12

7 If $\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = 15$, then $\begin{vmatrix} a & b & c \\ x & y & z \\ d & e & f \end{vmatrix} =$

a -30 b -15 c zero d 15

8 $\begin{vmatrix} a & a & a \\ a & b & c \\ c & c & c \end{vmatrix} =$

a zero b a c c b c d ab c

9 If n = $\begin{vmatrix} 1 & 0 & 3 \\ 2 & 3 & 5 \\ 1 & 4 & 2 \end{vmatrix}$, m = $\begin{vmatrix} 3 & 0 & 9 \\ 4 & 6 & 10 \\ 5 & 20 & 10 \end{vmatrix}$ then m =

a n b 10n c 20n d 30n

Without expanding the determinant , find the value:

10 $\begin{vmatrix} 5 & -1 & 10 \\ 4 & 2 & 8 \\ -5 & 2 & -10 \end{vmatrix}$

11 $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & -1 \\ -4 & 6 & 2 \end{vmatrix}$

1 - 5 Multiplicative inverse of a matrix

You will Learn

- Finding the multiplicative inverse of a matrix of order 2×2
- Solve a system of two linear equations using the inverse matrix.

Key - Terms

- Multiplicative inverse of a matrix
- Identity matrix
- Matrix equation
- Variable matrix
- Constant matrix
- Scientific Calculator

Group work

Group work

Work with your classmate

1- Find each of the following products:

A $\begin{pmatrix} 5 & 6 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ B $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -5 & 6 \\ 4 & 2 \end{pmatrix}$

2- Describe any patterns you see in your answer in number (1).

3- Find each of the following products:

A $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ B $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$

4- Describe any patterns you see in your answer in number (3).

5- Critical thinking: How can you join your answers in No (1) ,(3) ?

Learn

The multiplicative inverse of a 2×2 matrix:

If we have two square matrices A and B, and each of them is of order 2×2 and $A\,B = B\,A = I$ (unit matrix)

then the matrix B is called multiplicative inverse of the matrix A and also the matrix A is the multiplicative inverse of the matrix B.

If the matrix A has a multiplicative inverse, then we denoted it by the symbol A^{-1} where: $A\,A^{-1} = A^{-1}\,A = I$

> **Remember**
>
> *1- Identity matrix in the multiplication operation is the unit matrix I which is the square matrix, all elements of its principal diagonal are 1 and the rest elements are zeros.*
>
> *2- For any two real numbers then each of them is the multiplicative inverse of the other if their product is the identity element of multiplication (1)*

Some matrices do not have multiplicative inverse. We will help you deduce if the matrix of order 2×2 has a multiplicative inverse or not, and how to find this inverse if existed. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the multiplicative inverse of the matrix A existed when the determinant of A equals $\Delta \neq 0$

let the matrix A^{-1} be the multiplicative inverse of the matix A, and the determinant of A equals $\Delta \neq 0$ then:

$$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example

1 If $A = \begin{pmatrix} -1 & 0 \\ 8 & -2 \end{pmatrix}$. Prove that for the matrix A, there is a multiplicative inverse, then find it.

Solution

Determinant of $A = \begin{vmatrix} -1 & 0 \\ 8 & -2 \end{vmatrix} = -1 \times -2 - 8 \times 0 = 2$

$\therefore \Delta \neq 0$ i.e. for the matrix A, there is a multiplicative inverse.

$$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 0 \\ -8 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -4 & \frac{-1}{2} \end{pmatrix}$$

Remember

If $\Delta \neq 0$, then for the matrix A, there is a multiplicative inverse determined as follows:

a) exchange the elements of the principal diagonal of the matrix A.

B) Change the sign of each element of the other diagonal of the matrix A

C) multiply the resulted matrix after carrying out the stepes in (a), (b) by the number $\frac{1}{\Delta}$ then we will get A^{-1}

Try to solve

1 If $A = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$. Prove that for the matrix A, there is a multiplicative inverse, then find it.

2 Is there a multiplicative inverse for the matrix $B = \begin{pmatrix} -5 & 5 \\ -3 & 3 \end{pmatrix}$. Explain your answer .

Example

2 Find the values of a which make the matrix $\begin{pmatrix} a & 2 \\ 8 & a \end{pmatrix}$ have a multiplicative inverse.

Solution

The matrix has no multiplicative inverse when the determinant of the matrix equals zero.

i.e. $\begin{vmatrix} a & 2 \\ 8 & a \end{vmatrix} = 0$

$a^2 - 8 \times 2 = 0$

$a^2 - 16 = 0$

There are two values of a which are 4, -4 (roots of the equation $a^2 - 16 = 0$)

Make the given matrix has no multiplicative inverse.

$\therefore$ when $a \in R - \{-4, 4\}$, there is a multiplicative inverse for the given matrix.

Try to solve

3 Find the values of x which make the matrix $\begin{pmatrix} x & 9 \\ 4 & x \end{pmatrix}$ has no multiplicative inverse.

Example

3 If $X = \begin{pmatrix} 1 & x \\ 0 & -1 \end{pmatrix}$. Prove that $X^{-1} = X$

Solution

$\Delta = \begin{vmatrix} 1 & x \\ 0 & -1 \end{vmatrix} = -1 \neq 0$ $\quad\quad \therefore X^{-1} = \frac{1}{-1}\begin{pmatrix} -1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & -1 \end{pmatrix} = X$

Try to solve

4 If $B = \begin{pmatrix} x & -xy \\ 0 & y \end{pmatrix}$. Prove that $B^{-1} = \begin{pmatrix} \frac{1}{x} & 1 \\ 0 & \frac{1}{y} \end{pmatrix}$ given that $xy \neq 0$

Activity

Cryptography

You can use any matrix and its multiplicative inverse to code the message. Use the inverse of a matrix to decode the message. We write the message **"on trip"** as matrices of order 2×1 to become the numbers present consequently.

on $\begin{pmatrix} 15 \\ 14 \end{pmatrix}$ **tr** $\begin{pmatrix} 20 \\ 18 \end{pmatrix}$ **ip** $\begin{pmatrix} 9 \\ 16 \end{pmatrix}$ (1)

a	1	h	8	o	15	v	22
b	2	i	9	p	16	w	23
c	3	j	10	q	17	x	24
d	4	k	11	r	18	y	25
e	5	l	12	s	19	z	26
f	6	m	13	t	20		
g	7	n	14	u	21		

When matrix multiplication is used and the matrix for example

$D = \begin{pmatrix} 6 & 2 \\ 2 & 1 \end{pmatrix}$ is used, then the message will become these matrices:

$\begin{pmatrix} 118 \\ 44 \end{pmatrix} \begin{pmatrix} 156 \\ 58 \end{pmatrix} \begin{pmatrix} 86 \\ 34 \end{pmatrix}$ (2)

Notice that: the cryptography matrix C^{-1} could be found as follows:

$\because C = \begin{pmatrix} 6 & 2 \\ 2 & 1 \end{pmatrix}$, $\Delta = \begin{vmatrix} 6 & 2 \\ 2 & 1 \end{vmatrix} = 6 - 4 = 2 \neq 0$

Then $C^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -2 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 \\ -1 & 3 \end{pmatrix}$

when multiplying the matrix C^{-1} by the matrices in (2), you get the matrices in (1) and you can decode the message .

Now:

1- Write the message **"on time"** and code it using multiplication of matrices and the matrix $C = \begin{pmatrix} 6 & 2 \\ 2 & 1 \end{pmatrix}$

2- Write a message of your own and code it using multiplication of matrices. (use a cryptography matrix of your own).

Learn

Solving two simultaneous equations by using Inverse Matrix

if we have two linear equations as follows:

$a_1 x + b_1 y = k_1$ $\qquad$ $a_2 x + b_2 y = k_2$

then we can write them in the following form:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

If we suppose that:

$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $C = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$

Then we can write the two equations in the form of only matrix equation as follows:

$AX = C$	where A is the matrix coefficients, X is the matrix of variables, and C is the matrix of constants.

If the determinant A ≠ 0 **i.e.** $\Delta = a_1 b_2 - a_2 b_1 \neq 0$

Then it is possible to solve the equation A X = C as follows:

$A^{-1}(AX) = A^{-1}C$ (multiply both sides of the equation from the left by A^{-1})

$\therefore (A^{-1}A)X = A^{-1}C$ (associative property)

$IX = A^{-1}C$ (multiplicative inverse of the matrix A)

$\therefore$ $\boxed{X = A^{-1}C}$

It is clear that we can find the two variables X, y in terms of the numerical constants $a_1, b_1, a_2, b_2, k_1, k_2$.

Example

4) Solve the system of the following simultaneous equations using the matrices:

$3x + 2y = 5$ $\quad$ $2x + y = 3$

Solution

The matrix equation A X = C is written where

$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $C = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

the determinant $A = \Delta = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = 3 - 4 = -1 \neq 0$

then , for the matrix A, there is a multiplicative inverse and the solution is K $X = A^{-1}$ C **where:**

$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$

$\therefore X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

i.e. x = 1 , y = 1

The solution set {(1, 1)}

Check: $3(1) + 2(1) \stackrel{?}{=} 5$
$5 = 5$ (✓)
$2(1) + 1 \stackrel{?}{=} 3$
$3 = 3$ (✓)

Try to solve

5) Solve each system of the following linear equations using the matrices.

A) $3x + 7y = 2$
$2x + 5y = 1$ (Check your answer)

B) $x + 3y - 5 = 0$
$2x = 8 - 5y$ (Check your answer)

Check your understanding

1) If $B = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$, A B =I. Find the matrix A.

2) If $A = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$, $AB = \begin{pmatrix} 4 & -2 \\ 0 & 7 \end{pmatrix}$. Find the matrix B

Exercise (1 - 5)

1 Show the matrices which have inverses, and the matrices which have not inverses in the following. if there is an inverse find it.

A $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ .... **B** $\begin{pmatrix} 2 & 6 \\ -1 & 3 \end{pmatrix}$ .... **C** $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ .... **D** $\begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix}$ ....

2 What are the values of a which make to each of the following matrices a multiplicative inverse.

A $\begin{pmatrix} a & 1 \\ 6 & 3 \end{pmatrix}$ .... **B** $\begin{pmatrix} a & 9 \\ 4 & a \end{pmatrix}$ .... **C** $\begin{pmatrix} a & 4 \\ 2 & a-2 \end{pmatrix}$ .... **D** $\begin{pmatrix} a-1 & -2 \\ 1 & a-1 \end{pmatrix}$ ....

3 If $X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ then prove that $X^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{pmatrix}$

....

4 Find the matrix A if: $A \begin{pmatrix} -2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

....

5 If $X = \begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$, Prove that $= (XY)^{-1} = Y^{-1}X^{-1}$

....

6 Solve the following linear equations using matrices, then check your answer:

A $4x + 3y = 26$, $5x - y = 4$

....

....

B $2x = 3 + 7y$, $y = 5 - x$

....

....

General Exercises

First: Complete each of the following:

1 If $A = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $B = (2 \;\; 5)$ then $(BA)^t =$ ....

2 If $\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -3 & x \end{pmatrix} = I$, then $x =$ ....

3 If $A = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$ then $A^2 =$ ....

Second: Multiple choice questions:

4 If A and B are two matrices where $AB = \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix}$ then $B^t A^t =$ ....

A $\begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix}$ B $\begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix}$ C $\begin{pmatrix} 5 & -1 \\ 4 & 3 \end{pmatrix}$ D $\begin{pmatrix} 5 & 4 \\ -1 & 3 \end{pmatrix}$

5 If: $\begin{vmatrix} 2-x & 2 \\ -3 & x+2 \end{vmatrix} = 1$ then x equals ....

A -3 B 3 C ± 3 D ± 4

6 If $A = \begin{pmatrix} 5 & 2 & -1 \\ 4 & 0 & 1 \end{pmatrix}$

A What is the order of the matrix A? ....

B Write the elements of the first row in the matrix A. ....

C Write the elements of the third column in the matrix A. ....

D Write the elements: $A_{11}, A_{22}, A_{13}, A_{12}$ ....

7 What is the number of elements in each of the following matrices?

A matrix of order 3×2 ....

B Square matrix of order 2×2. ....

8 Solve each of the following equations:

A $\begin{pmatrix} x+5 \\ x-y \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$ ....

B $\begin{pmatrix} a-3b & b \\ b+a & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$ ....

9 Solve the equation: $A + \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 5 \end{pmatrix}$.

....

....

10 Find a, b, c and d where

A $\begin{pmatrix} -3 & -2 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -3 \end{pmatrix}$ ....

B $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ ....

11 If: $X = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 4 & -2 \\ 0 & -1 & -6 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & -1 \\ 0 & 2 & 1 \end{pmatrix}$, then find:

A $2X + 3Y - 2I$ ....

B $X - (Y - 5I)$ ....

12 Find the values of x and y where: $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

13 Show which of the following matrices has a multiplicative inverse , then find it

A $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ **B** $\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$ **C** $\begin{pmatrix} a & -b \\ a & b \end{pmatrix}$ **D** $\begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix}$

....

14 Solve each system of the following linear equations using matrices

A $x + y = 3$, $x - y = 5$ ....

B $2x - 3y - 3 = 0$, $5x + 4y - 19 = 0$ ....

C $y = 11 - 5x$, $x = 3 - 5y$ ....

Unit 2

Algebra

Linear Programing

Unit objectives

By the end of this unit, the student should be able to:

- Solve first degree inequalities in one variable and represent the solution graphically.
- Solve first degree inequalities in two variables and determine the region of solution graphically.
- Solve the system of linear inequalities graphically.
- Solve life problems on systems of linear inequalities.
- Use linear programming to solve life mathematical problems.
- Record the data of a mathematical life problem in a suitable table, and transfer these data in the form of linear inequalities, then determine the region of solution graphically.
- Determine the objective function in terms of the coordinates and determine the points which belong to the solution set, giving the optimum solution to the objective function.

Key - Terms

- Linear Inequality
- Boundary line
- Dashed boundary line
- Solid boundary line
- Linear Inequality in two variables
- System of linear Inequalities
- Feasible region
- Graph
- Linear programing
- Constrains
- Optimum solution

Lessons of the Unit

Lesson (2 - 1): Linear Inequalities.

Lesson (2 - 2): Solving Systems of Linear Inequalities Graphically.

Lesson (2 - 3): Linear Programing and Optimization.

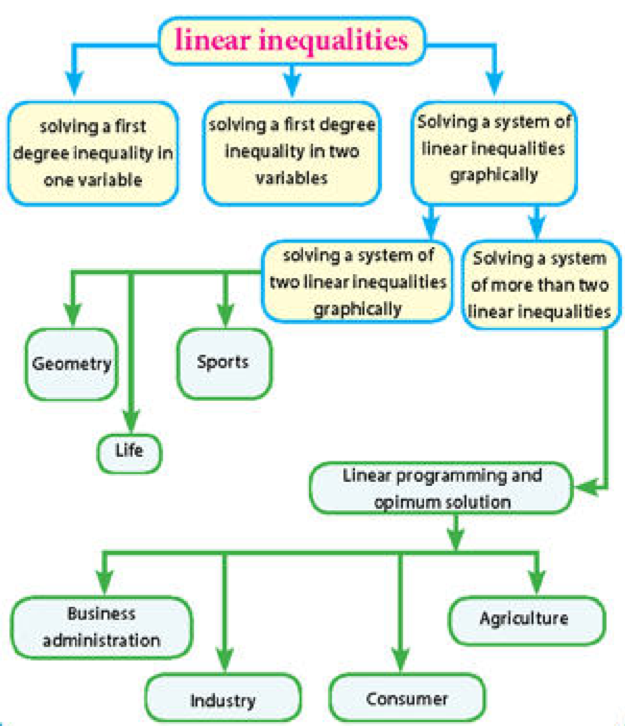
Materials

Lattice 10×10
squared papers - coloured pencils - electronic sites
such as www.phschool.com

Brief History

When the analysis of a problem leads to find the maximum or minimum value of a linear expression, it should be subject to its variables to the set of linear inequalities. We may get the solution by using the techniques of the linear programming. Historically, it has appeared problems of the linear programming as a result of the need to solve problems related to the salaries of members of the armed forces during World War Two, George Dantzig is one of those who worked in the solution of such problems and reached to the general formula to solve the problems and show a method of solving it called Simplex method, for the linear programming there are applications in each of the industry, trade, business administration, agriculture, health and others. For example, success requires in business administration that using the linear programming to achieve the maximum possible profit or to achieve the least possible cost. In this unit, we will learn methods of solving problems for linear programming which contain only two variables and its applications are in different life situations.

Chart of the unit



2 - 1 Linear Inequalities

You will learn

- Solving first degree inequality in a variable.
- Solving first degree inequality in two variables and determining the region of solution graphically.

Key-terms

- Linear inequality
- Boundary line
- Dashed boundary line
- Solid boundary line
- Linear inequality in one variable
- Linear inequality in two variables

Materials

- Lattice 10×10
- Squared papers.
- Coloured pencils.

Group work

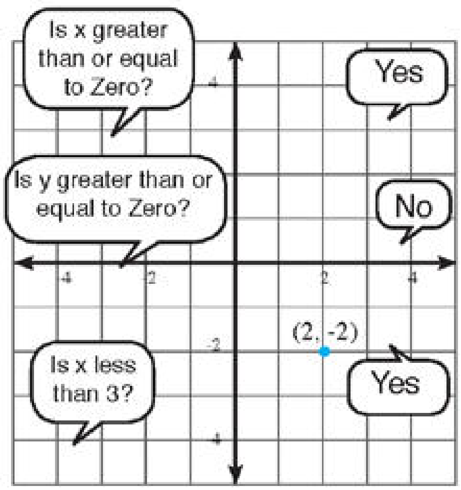
Materials: lattice 10×10

1- With your classmate, play the game"What is the point?"

Play objective:

Determining the position of a point on the coordinate plane by asking the least number of questions.

How to play?



- The player "A" chooses a point on the coordinate plane, does not The player "B" know it (secret point) , each of its coordinates is an integer from -5 to 5
- The player "B" askes questions containing the words "less than" or " greater than" the player A answers each question only by "yes ' or" no" .
- The player A records the number of questions while the player B names the secret point.
- The players exchange their rules to complete only one game from the play.

How to win?

The player who determine the point by asking of questions less is the winner of the game, and the player who win by the first three games is the winner.

2- How many questions do you need to determine the position of the secret point?

3- If you are lucky, how many questions do you need to ask to determine the position of the secret point? Explain your answer by giving examples.

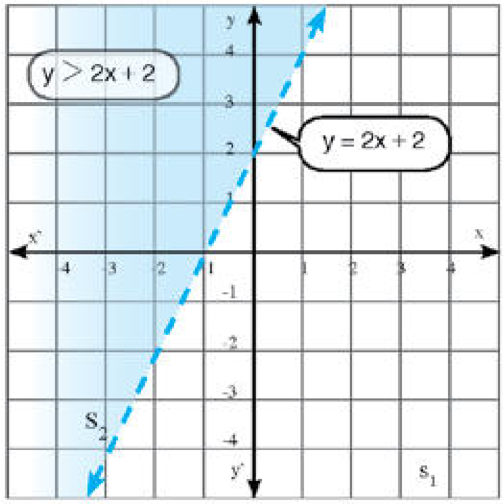
4- How do the inequalities help you to determine the secret point?

5- Suggest a strategy to win this game.

Learn: Solving linear inequalities in two variables

The first degree inequality in two variables is similar to the first degree equation in two variables and the difference between them is placing the symbol of the inequality instead of the symbol of the equality for example: $y > 2x + 2$ is a linear inequality and $y = 2x + 2$ is a linear equation related to it.

Graphical representation of the inequality $y > 2x + 2$ is shown by the shaded region in the figure opposite.



Notice that each point in the coloured region satisfies the inequality, and the graphical representation of the straight line $y = 2x + 2$ is the boundary of the region which represents the solution and the straight line is drawn dashed because it does not satisfy the inequality. If the inequality contains the symbol $\leqslant$ or $\geqslant$ then the points which lie on the boundary line will satisfy the inequality, then the representation of the straight line is a solid line.

Example

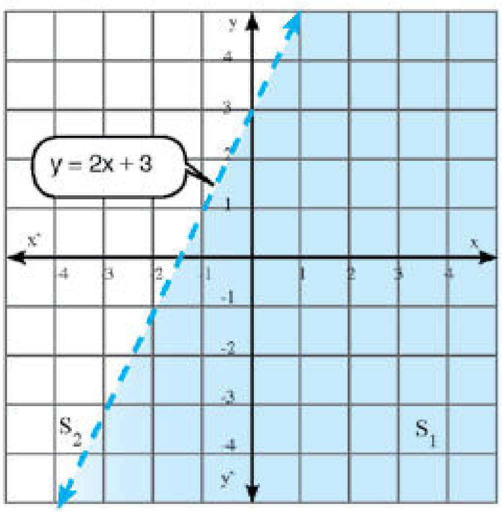
1 Represent graphically the solution set of the inequality: $y < 2x + 3$

Solution

Step (1): Draw the boundary line $y = 2x + 3$

Notice that the points of the boundary line are not solutions of the inequality, thus the straight line is drawn dashed.

x	0	−1	−2
y	3	1	−1



Step (2): choose one of the points in one side of the drawn line and substitute in the right hand side, if it satisfies the inequality, we colour this side (the solution set), and if it does not satisfy the inequality, we colour the other side and is then the solution set.

Note
The boundary line divides the plane into three sets of points.
1- Set of points of the boundary line.
2- Set of the points of the plane which lie on one side of the boundary line and is called half plane and is denoted by the symbol (S_1).
3- Set of points of the plane which lie on the other side of the boundary line and is called half plane and is denoted by the symbol (S_2).

Choose the point (0, 0) and which does not lie on the boundary line but lies on one side.

$y < 2x + 3$ (the original inequality)

$0 \overset{?}{<} 2(0) + 3$ (substitute the point (0, 0))

$0 < 3$ (true)

Shade the region which contains the point (0, 0), where the solution set is half the plain at which the point (0, 0) belongs.

Check:
Graphical representation shows that the point (2, 3) lies in the region of solution.

$y < 2x + 3$ (the original inequality)

$3 \overset{?}{<} 2(2) + 3$ (substitute the point (2, 3))

$3 < 7$ (true) then the solution is true.

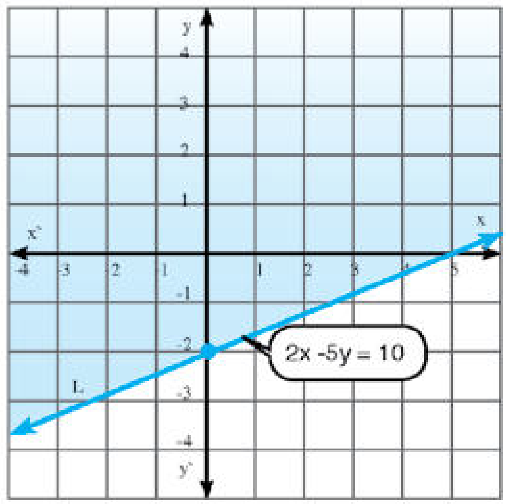
Example

2 Represent graphically the solution set of the inequality: $2x - 5y \leqslant 10$

Solution

Step (1): represent graphically the boundary line (L). $2x - 5y = 10$ by a solid line (because the inequality relation $\leqslant$).

x	0	5	$2\frac{1}{2}$
y	−2	0	−1



You can draw the boundary line, write the straight line: $2x - 5y = 10$ in the form: $y = mx + c$
where m is the slop and c is the y – intercept from the y-axis.

then: $-5y = -2x + 10 \quad \therefore \; y = \frac{2}{5}x - 2$

Step (2): test the point (0,0) which lies on one side of the boundary line.

$2x - 5y \leqslant 10$ (the original inequality)

$2(0) - 5(0) \overset{?}{\leqslant} 10$ (substitute the point (0, 0))

$0 \leqslant 10$ (True)

Colour the region which contains the point (0, 0), where the solution set is half the plane which the point (0, 0) lies ∪ the set of points on the boundary line L.

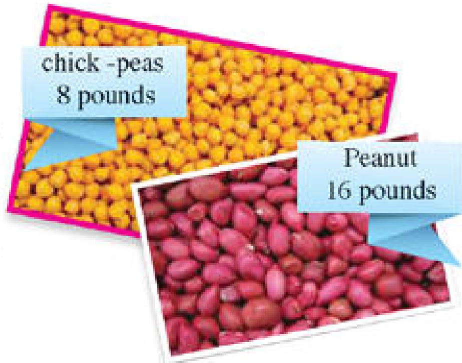
Try to solve

1 Represent graphically the solution set of each of the following inequalities

A $2x - y \geqslant 6$ **B** $y < 5x - 5$ **C** $y - 2x < 2$

Example

3 **Life applications:** **Food shopping**: Suppose you decided not to spend more than 48 pounds to buy peas and peanuts necessary for your trip for you and your family to the zoo in Giza. How many kilograms can you buy from every kind?



Solution

Define: let x be the Number of kilograms you can buy from chick- peas.
y be the Number of kilograms you can buy from peanut.

Connect: Price of buying chick-peas + Price of buying peanut ⩽ the maximum purchase (see the figure).

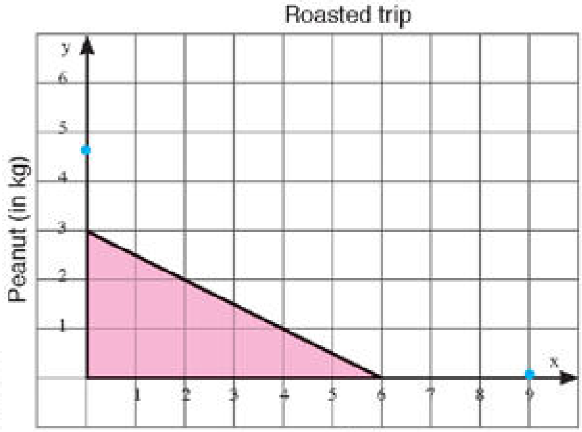
Write: $8x + 16y \leqslant 48$

Draw the boundary line $8x + 16y = 48$, and is represented by the solid straight line (because the inequality relation ⩽).

Use the first quadrant only from the coordinate plane, where that you can not buy a negative quantity of roasted peanuts.

x	0	6	2
y	3	0	2

Roasted trip



Test the point (0, 0)

$8(0) + 16(0) \overset{?}{\leqslant} 48$

$0 \leqslant 48$ (True)

Colour the region which contains the point (0, 0).

The graphic representation shows all possible solutions. For example, if you buy 2 kg of peas, then you can not buy more than 2 kg of peanuts. Now, are 2 kg of peas and 1 kg of peanut a solution to example ?

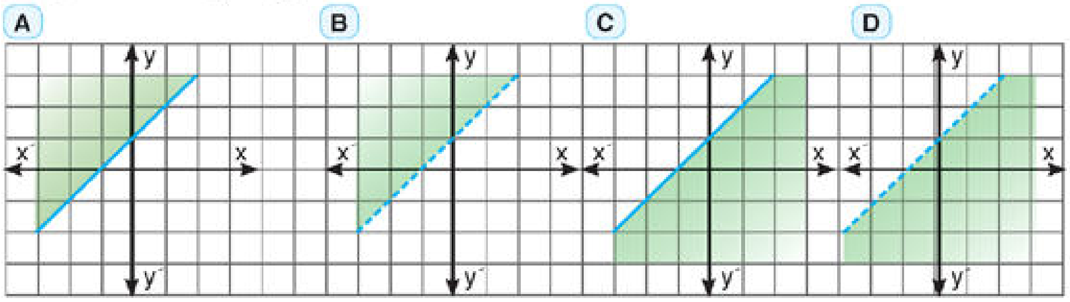
Check your understanding

1 **Critical thinking:** When we represent the inequality $y \geqslant \frac{2}{5}x - 2$ graphically, do you shade the region up or down the straight line, $y = \frac{2}{5}x - 2$? How did you know that?

2 **Consumer:** A library sells two kinds of notebooks, the price of the first kind is 6.25 pounds. The price of the second kind is 7.5 pounds. If Ahmed wanted to buy some notebooks such that he can not pay more than 25 pounds. How many notebook can he buy from each kind?

Exercise (2 - 1)

1. Join each inequality with the graph which represents its solution set (test the point (0, 0) in each inequality).



1- $y \leqslant x + 1$ **2-** $y < x + 1$ **3-** $y > x + 1$ **4-** $y \geqslant x + 1$

2. Test which of the points is the solution of the inequality:

A $y \geqslant 2x + 3$ [(0, 1) , (3, 9) , (–1, 0)] ....

B $y < 2x + 3$ [(0, 1) , (3, 9) , (–1, 0)] ....

3. Find the solution set of each of the following inequalities:

A $y \leqslant x + 2$ B $y > 2x - 3$ C $x + 3y \leqslant 6$

....

....

Solving Systems of Linear Inequalities Graphically

2 - 2

Group work

Work with your classmate.

1- Represent graphically the solution set of the inequality $x \geqslant 2$ in the orthogonal coordinate plane, and colour the feasible region in yellow.

2- Represent graphically the solution set of the inequality $y < -1$ in the same orthogonal coordinate plane, then colour the feasible region in green.

3- Determine the common area which contains yellow and green together.

4- What does the region you determined in (3) represent?

5- Choose three distinct points, each of which represents a solution for the two inequalities together. Explain your answer.

Learn

System of linear inequalities

Two or more linear inequalities form together a system of linear inequalities, and the ordered pair (x_1, y_1) is a solution for this system if it satisfies all its inequalities.

Try to solve

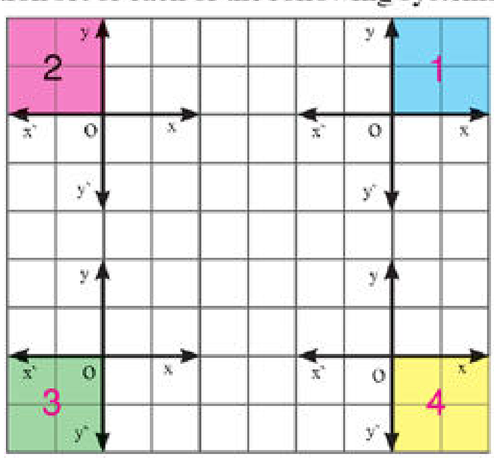
1 You can describe each of the four quadrants in the orthogonal coordinate plane by using a system of the linear inequalities. From the figure opposite, determine the number of the quadrant which represents the solution set of each of the following systems

A $x > 0$, $y > 0$

B $x > 0$, $y < 0$

C $x < 0$, $y > 0$

D $x < 0$, $y < 0$



You will learn

- Solving a system of linear inequality graphically.
- Solving life problems on systems of linear inequalities.

Key-terms

- System of linear inequalities
- Feasible region
- Graph

Materials

- Graph papers.
- Coloured pencils

Learn Solving a system of liner inequalities graphically

Solving a system of linear inequalities means finding all ordered pairs which satisfy the inequalities in this system. To determine all points (ordered pairs) which form a solution of the system, colour (shade) the feasible region, each one of the inequalities in the same coordinate plane, then the common region among the regions of solution of the inequalities is a feasible region of this system.

Example

1 Solve the system of the following linear inequatities graphically: $y \geqslant 2x + 6, \quad y + 3x < -1$

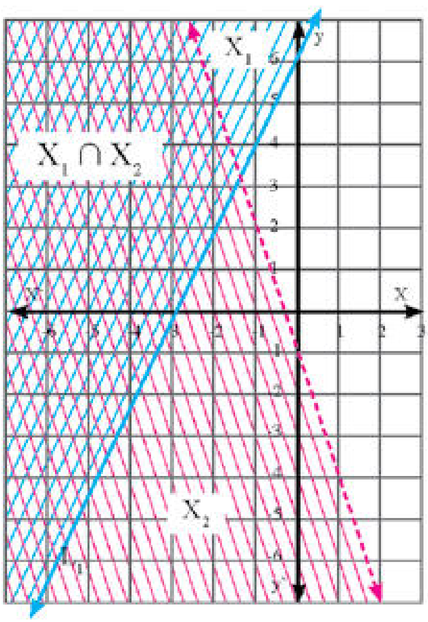
Solution

Step (1): Represent the solution set of each inequality in the system graphically, and colour the feasible region.

For the first inequality: $y \geqslant 2x + 6$

Draw the boundary line $y = 2x + 6$ (solid line)

x	0	-3	-2
y	6	0	2



The point (0, 0) does not satisfy the inequality

∴ The solution set X_1 is half the plane at which the origin point $\cup L_1$

For the second inequality: $y + 3x < -1$

Draw the boundary line $y + 3x = -1$ (dashed line)

x	0	-1	-2
y	-1	2	5

The point (0, 0) does not satisfy the inequality

∴ The solution set X_2 is half the plane at which the origin (0, 0) does not lie.

Step (2): Determine the common region among the regions of solution of the system of inequalities, which is the common coloured region, and also represents the feasible region of the system then the solution set of the two inequalities together is $X_1 \cap X_2$

Check: Notice that the point (– 4, 2) belongs to the feasible region of the system, thus you can use a test point and check the solution by substituting , (x, y) by the point (– 4, 2) in both inequalities:

$y \geqslant 2x + 6$ | $y + 3x < -1$

$2 \overset{?}{\geqslant} 2(-4) + 6$ | $2 + 3(-4) \overset{?}{<} -1$

$2 \geqslant -2$ (True) | $-10 < -1$ (True)

Try to solve

(2) Solve the following system graphically: $3x + 5y \geqslant 15$, $y < x - 1$

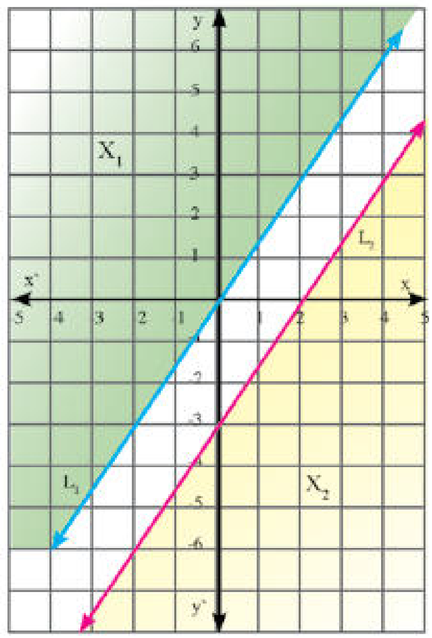
Example

(2) Solve the following linear inequalities graphically: $4y \geqslant 6x$

$-3x + 2y \leqslant -6$

Solution

Step (1): Represent the solution set of each inequality in the system graphically and colour the feasible region for the first inequality.



For the first inequality: $4y \geqslant 6x$

Draw the boundary line $4y = 6x$ (solid line)

x	0	2	-2
y	0	3	-3

The point (0, 0) lies on the boundary line, thus it is tested by using another point on one of the two sides of the boundary line, for example (-3, 2)

then: $4(2) \geqslant 6(-3)$

i.e. $8 \geqslant -12$ (True)

The solution set X_1, which is half the plane at which the point (-3, 2) lies $\cup L_1$

For the second inequality: $-3x + 2y \leqslant -6$

Draw the boundary line $-3x - 2y = -6$ (solid line)

x	0	2	-2
y	-3	0	6

The point (0, 0) does not satisfy the inequality

∴The solution set X_2, which is half the plane of which the point (0, 0) does not lie $\cup L_2$

Step (2): Determine the common region among the regions of solution of the system of inequalities , which represents the feasible region of the system .

Notice that the two straight lines L_1, L_2 are parallel and there is no common region between the coloured regions as in the figure.

∴ The solution set of the two inequalities together = ϕ

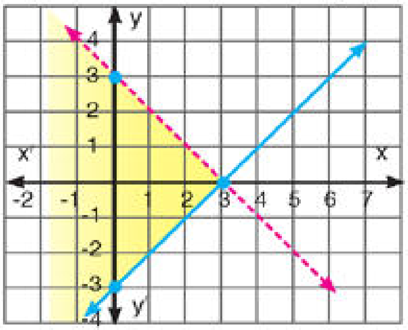
Try to solve

(3) Find the solution of the following system of linear inequalities graphically: $y \leqslant x$

$y \geqslant x + 1$

Exercise (2 - 2)

1 Which of the following systems has the shown shaded region as a solution in the figure opposite:



A $x + y \leqslant 3$
$y > x - 3$

B $x + y > 3$
$y \leqslant x - 3$

C $x + y \geqslant 3$
$y < x - 3$

D $x + y < 3$
$y \geqslant x - 3$

2 Solve each system of the following linear inequalities graphically:

A $x \geqslant 4$
$y > x + 2$
$x + 2y \geqslant -2$

B $y - x > 0$
$2x + 2y \geqslant 12$
$y < 6 + 2x$

....

....

....

2 - 3 Linear programing and Optimization

You will learn

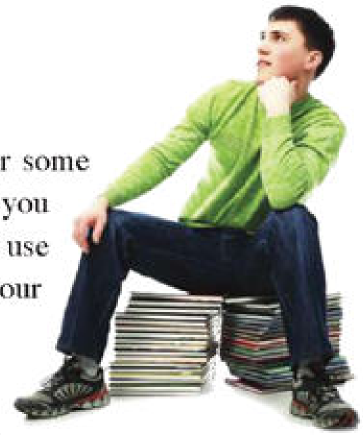
- Finding the maximum and minimum value to a function in a certain region.
- Use the linear programing to solve some problems.
- Record the data of mathematical life problem in a suitable table and transfer these data in the form of linear inequality then determine the region of solution and determining the objective function and its optimum solution

- linear programing
- Aonstrains
- Bounded
- Unbounded
- Optimum solution

Materials

- Graphic papers.
- Coloured pencils.

Group work



Suppose that you are offered a job for some time, and you think about the time that you can customize for this work. You can use mathematics to help you organize your thinking and take the right decision.
Work with your classmate:

1- **A** Write a list of ways that you spend your time during the week.
B Organize your list so that it is not more than 10 ways.

2- Make a personal assessment of last week.
A Select the time for the ways you determined in (1).
B What is the suitable time you see for work in a job some time?
C Discuss: What can you quit or not quit in your schedule?

Learn

Linear Programing

You can answer questions such as the questions discussed above, using a process called the linear programing.
then the first step is to write the linear program of the problem and consists of:

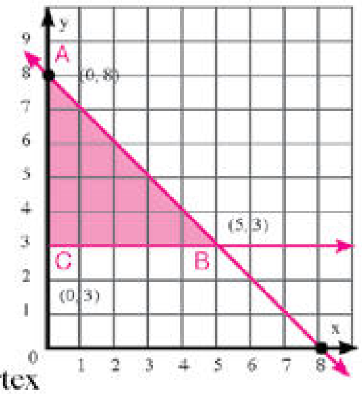
1- Objective function (which the problem under study aims to calculate the maximum or minimum value) and it is linear function in the form:
$P = ax + by$ where a, b are real numbers not equal to zero together.

2- Set of restrictions imposed by the nature of the problem, which is in the form of linear inequalities in two variables represent the upper or lower limits of the factors that control variables of the problem.

3- Restrictions imposed by the scientific fact of the problem on the variables when you can not take these variables negative values.

Example

1. Use the linear programing to find the values of x, y which make the value of the function $P = 3x + 2y$ the maximum value then minimum value under restrictions
$x \geqslant 0$, $y \geqslant 0$, $x + y \leqslant 8$, $y \geqslant 3$



Solution

Step (1): Draw the restrictions (represent the inequalities graphically)

Step (2): Find the coordinates of the vertices of the feasible region.

From the figure, we notice that the vertices of the feasible region are:

A(0, 8), B (5, 3), C (0, 3)

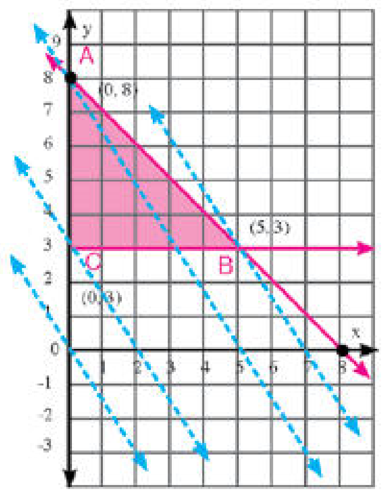
Step (3): Find the value of the function $P = 3x + 2y$ at each vertex

Form the following table:

The point	x	y	3x + 2y	value of the function P	
A (0, 8)	0	8	3 (0) + 2 (8)	16	
B (5, 3)	5	3	3 (5) + 2 (3)	21	⟶ **Maximum value**
C (0, 3)	0	3	3 (0) + 2 (3)	6	⟶ **Minimum value**

The maximum value of the function equals 21 at the point (5, 3), and the minimum value of the function equals 6 at the point (0, 3)

Think: Why does the maximum value or the minimum value of the objective function satisfy at one of the vertices of the feasible region?



To know the answer of this question:

1- Put $P = 0$ in the objective function $P = 3x + 2y$ then we get $3x + 2y = 0$ represents a straight line passes through the origin point and the point, (2, -3).

2- If you draw a set of straight lines intersect the feasible region and parallel to the straight line passing through the origin point, then:
the first of these lines passes through the point C (0, 3) and its equation is $3x + 2y = 6$ i.e. $P = 6$

3- The value of P at all points which belong to the second straight line passing through the point A (0, 8) equals 16, and P continue in increasing tell it reaches to the last line which intersects the feasible region of the system and passing through the point B (5, 3) , then we get that $P = 3 \times 5 + 2 \times 3 = 21$

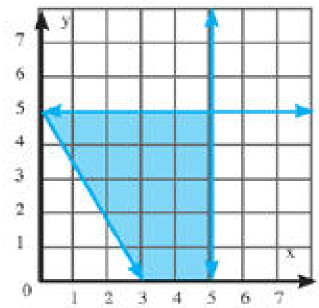
Thus, the minimum value to the objective function equals 6 at the point (0, 3) which is one of the vertices of the feasible region, and also the maximum value to the objective function equals 21 at the point (5, 3) , which is one of the vertices of the feasible region also.

From the previous, we deduce that : the maximum value and the minmum value if there exist to the objective function, then they satisfy at the vertices of the polygon which surrounded the region of possible solutions to the inequalities which form the set of restrictions of problem or at the points of intersection of lines which determine the region of possible solutions.

Try to solve

1. Use the linear programing, find each of the minimum value and the maximum value for the function $P = x + y$ under restrictions: $x \geqslant 0$, $y \geqslant 0$, $y \geqslant 2x - 2$, $y \leqslant -x + 8$

2. **In the figure opposite:** Find the values of x and y Which make the value of the function $P = 2x + 5y$ minimum.



Learn

Real life applications of linear programing

The linear programing is a mathematical method to give the best decision of solving a problem or it is the optimal solution to satisfy a certain object as satisfying least cost or greatest profit for a certain project, to bound by the terms and constraints of production and market mechanisms or problem under study, it is possible to satisfy that as follows:

1- Analysis of the situation or the problem to determine the variables, and to identify constraints and put it in the form of a system of linear inequalities.

2- Write objective function to be achieved in the problem under study (which is a linear function).

3- Represent the system of linear inequalities graphically.

4- Determining the vertices of the feasible region.

5- Substituting by coordinates of the vertices in the objective function, then test the maximum value or minimum value according to the required in the problem.

Example

2. **Business Administration** One of the sea food shop sells two types of cooked fish A and B, and the requests from the shop owner are not less than 50 fish, as he does not consume more than 30 fish from the type (A) , or more than 35 fish from the type (B). If the price of a fish from type A is 4 pounds and 3 pounds from type B. How much fish from each of the two types A and B must be used to achieve the lowest cost possible to buy?

Solution

1- Let the number of fish from type (A) be x and the number of fish from type (B) be y

and $x \geqslant 0$ (He will buy fish from type A)

$y \geqslant 0$ (He will buy fish from type B)

$x + y \geqslant 50$ (He needs at least 50 fish)

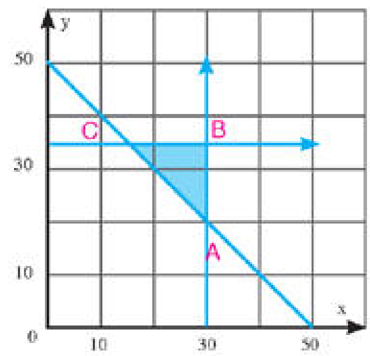
$x \leqslant 30$ (He uses not more than 30 fish from type A)

$y \leqslant 35$ (He uses not more than 35 fish from type B)

	First type	Second Type	Maximum
	x	y	50
The purchase price	4	3	$4x + 3y$

2- Write the objective function : the purchase price is minimum : $P = 4x + 3y$

3- Represent the system of the inequalities graphically as in the figure opposite.



4- Determine the vertices of the feasible region which are: A (30, 20), B (30, 35), C (15, 35).

5- Substitute by the coordinates of the vertices in the objective function to determine the minimum possible price of purchase, as shown in the following table:

The point	x	y	4x + 3y	Value of the function P
A(30, 20)	30	20	4 (30) + 3 (20)	180
B (30, 35)	30	35	4 (30) + 3 (35)	225
C (15, 35)	15	35	4 (15) + 3 (35)	165

→ Least possible value of purchase price

The shop owner must buy 15 fish from type A and 35 fish from type B such that the purchase price be minimum.

Try to solve

3 **Industry:** A small factory produces metal furniture 20 cupboard weekly at most of two different kinds A and B. If the profit from kind "A' is 80 pounds, and profit from kind B is 100 pounds. the factory sells from kind A at least 3 times what it sells from the second kind. Find number of cupboard from each kind to satisfy greatest possible profit to the factory.

Example

3 Health baby food factory produces two types of food with special specifications. If the first type contains 2 units of vitamin (A), 3 units of vitamin (B), and the second contains 3 units of vitamin A and 2 units of vitamin (A), 2 units of vitamin B. If the child in its own food if the child needs at least 120 units of vitamin (A), 100 units of vitamin (B), and the cost of the type (A) is 5 pounds, and the cost of type (B) is 4 pounds. What quantity to be purchased from each of the two types to achieve what the child needs in own food at the lowest possible cost?

Solution

1- Let: x be the number of goods of the first type and y be the number of goods of the second type:

$x \geqslant 0 \quad , \quad y \geqslant 0$

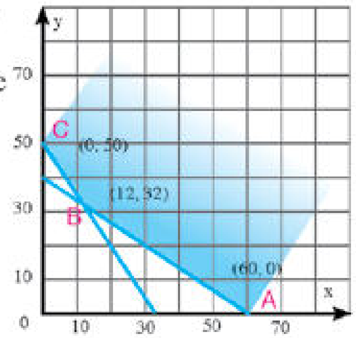
$2x + 3y \geqslant 120$

$3x + 2y \geqslant 100$

Item	No of goods of the first type	No of goods of the second type	minimum of units
Vitamin A	2x	3y	120
Vitamin B	3x	2y	100
Costs	5 pounds	4 pounds	

2- Objective function is the lowest possible cost: $P = 5x + 4y$

3- Represent the system of the linear inequalities as in the figure opposite.



4- The vertices of the feasible region are:
A (60, 0), B (12, 32), C (0, 50).

5- Substitute by the coordinates in the objective function to determine the least possible cost:

Point	x	y	5x + 4y	Value of the function P
A (60, 0)	60	0	5 (60) + 4 (0)	300
B (12, 32)	12	32	5 (12) + 4 (32)	188
C (0, 50)	0	50	5 (0) + 4 (50)	200

→ Least possible cost

then the cost is to be minimum at B, the number of goods of the first type is 12 and the number of goods of the second type is 32.

Exercise (2 - 3)

1. Choose the correct answer from the given answers:

   A. The point which belongs to the solution of the enqualities: $x > 2$, $y > 1$, $x + y \geqslant 3$ is: ....

   [(3, 1) , (1, 2) , (3, 2) , (1, 3)]

   B. The point at which the function $P = 40x + 20y$ has a maximum value is : ....

   [(0, 0) , (0, -4) , (15, 10) , (25, 0)]

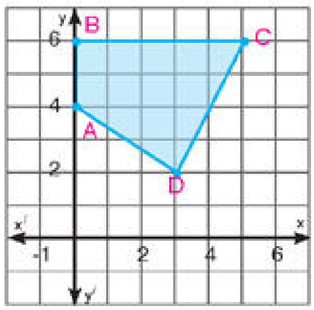
   C. The point at which the function $P = 35x + 10y$ has a minimum value is: ....

   [(0, 0) , (0, 10) , (0, 40) , (20, 10)]

2. Use the graph opposite to find the values of x, y which make the value of the objective function $p = 3x + 2y$ has a minimum value , then find this value .

   ....

   ....



3. Represent each of the following systems graphically, then find the maximum or minimum value to the objective function according to the given.

   A. $x + y \leqslant 5$
   $y \geqslant 1$
   $x \geqslant 2$
   has minimum value to the objective function $P = 2x + 3y$

   B. $2x + y \leqslant 6$
   $x \geqslant 1$
   $y \geqslant 2$
   has maximum value to the objective function $P = 2x + 3y$

4. **Industry**: Suppose you manufacture and sell skin moisturizer, if manufacturing a unit of the normal moisturizer requires $2cm^3$ of oil, $1cm^3$ of cocoa butter, and manufacturing a unit of the excellent moisturizer requires $1cm^3$ of oil, $2cm^3$ of coca butter. You will gain 10 pounds for every unit of the normal kind, 8 pounds for every unit of the excellent kind. If you had 24 cm^3 of oil, $18cm^3$ of cocoa butter. What is the number of units you can manufacture from each kind to get a maximum possible profit and what is this profit?

   ....

   ....

   ....

5. **Tourism:** One of the tourism companies set up an air bridge to transport 1600 tourists and 90 tons of luggage with the least cost. If there are two types of aeroplanes A and B and the number of available aeroplanes from the first type was 12 aeroplanes, and 9 aeroplanes from the second type and the full load to the aeroplane of the first type A was 200 persons, 6 tons of luggage and the full load to the aeroplane of the second type B was 100 persons, 15 tons of luggage. The rent of the aeroplane of the type A was 320000 pounds, and the rent of the type B was 150000 pounds. How many aeroplanes from each type could the company rent?

   ....

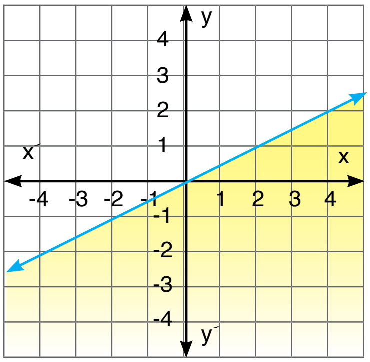
General Exercises

1 Represent graphically the solution set of each of the following inequalities:

A $y \leqslant 2x + 1$ **B** $y \geqslant -x - 4$ **C** $y < -2x + 3$

2 **Multiple choice:**

Choose the inequality which represents the shaded region as its solution:



A $y \geqslant \frac{1}{2}x$ **B** $y > \frac{1}{2}x$

C $y \leqslant \frac{1}{2}x$ **D** $y < \frac{1}{2}x$

3 Solve each system of the following linear inequalities graphically:

A $3y > 4x$, $2x - 3y > -6$

B $x + y > 6$, $y < -5x + 12$, $y < 5x + 12$

4 Find the maximum value of the function $P = 2x + y$ where:

$x \geqslant 0$, $y \geqslant 0$, $2x + 3y \leqslant 18$, $-4x + y \geqslant -8$.

5 Represent the solution set of each of the following inequalities graphically:

A $x + y \geqslant 2$ **B** $x + 3y \leqslant 6$ **C** $\frac{1}{2}x + \frac{3}{2}y \geqslant \frac{3}{4}$

D $4y > 6x + 2$ **E** $2x + 3y < 6$ **F** $4x - 2y \geqslant 8$

6 Solve each system of the following linear inequalities graphically:

A $y \geqslant 2x - 1$
$x \geqslant 2$

B $3x + 2y \leqslant 12$
$x - y \leqslant 3$

C $y > 4x - 1$
$y \leqslant -x + 4$

D $x \geqslant 0$
$y \geqslant 0$
$x + 2y > 4$
$4x + y > 9$

E $x \geqslant 0$
$y \geqslant 0$
$y - x \leqslant 1$
$4x + 3y \leqslant 12$

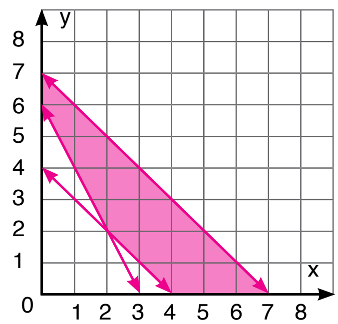
F $x \geqslant 0$
$y \geqslant 0$
$x + y < 8$
$x + 2y < 11$

7 In the figure opposite: find the values of x, y which make the function $P = \frac{1}{2}x + y$ has a minimum value.

....

....

....



Unit 3

Analytic Geometry

Vectors

Unit objectives

By the end of the unit, the student should be able to:

- Recognize the scalar quantity, the vector quantity, and the directed line segment and expresses it in terms of its two ends in the coordinate plane.
- Recognize the position vector and put it in the polar form.
- Find the norm of the vector and zero vector.
- Recognize and solve exercises on equivalent vectors.
- Recognize the unit vector and expresses it in terms of the fundamental unit vectors.
- Recognize parallel and perpendicular vectors.
- Multiply a vector by a real number.
- Add two vectors using the triangle rule (coordinates - parallelogram rule) - subtract two vectors.
- Prove some geometric theorems using vectors.
- Solve applications in the plane geometry on vectors

Key - Terms

- Scalar Quantities
- Vector Quantities
- Vector
- Distance
- Displacement
- Position Vector
- Orderd Pair
- Absolute value
- Norm
- Equivalent Vector
- Adding vectors
- The triangle Rule
- Parallelogram Rule
- Subtracting Vectors

Lessons of the Unit

Lesson (3 - 1): Scalars, Vectors and Directed Line Segment.

Lesson (3 - 2): Vectors.

Lesson (3 - 3): Operations on Vectors.

Materials

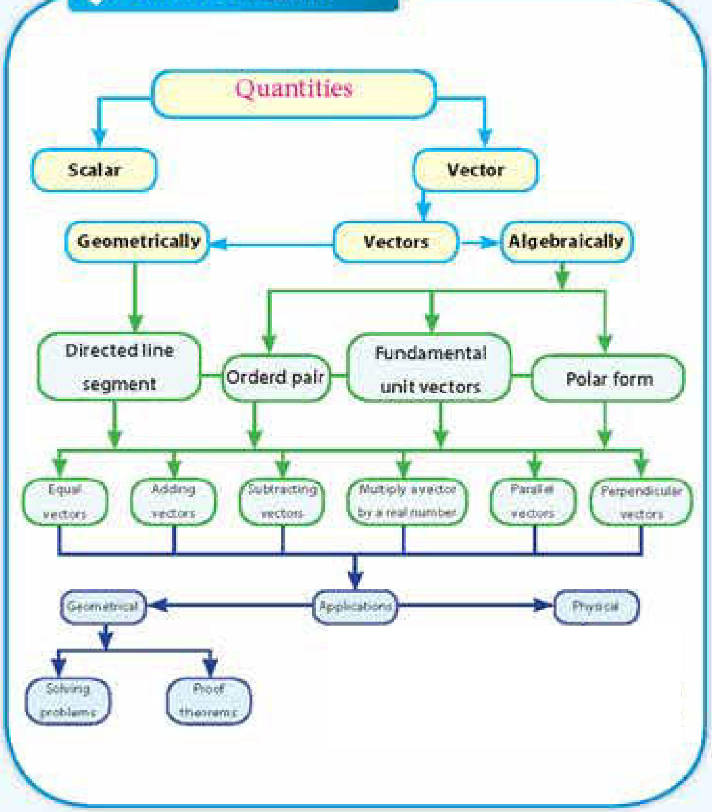
Computer – Data show – Graphic program – squared papers – Geometric instruments – threads – weights – pins

Brief History

Arabs put the first building block of analytic geometry. They used Algebra in solving some geometric problems and also used Geometry in solving Algebraic equations. Thabit Ibn Qorra (835 – 900AD) presented geometrical solutions to some equations and the Kenedy in his writings linked between algebra and geometry.

With the beginning of the seventeenth century, Fermat (1601 – 1665 AD) and Rene Descartes (1596 – 1650 AD) contributed in simplifying algebraic methods to solve geometric problems due to the plane geometry having two dimensions. They expressed almost every thing in any geometric figure in terms of two variabled lengths. They denoted them by x, y. In addition to some constant quantities contained in the figure which give a new shape to geometry known as the analytic geometry (coordinate). It used deduction theorems, facts, and prove its correctness algebraically. Also it was one of the factors to help the appearance of calculus by Newton (1642 – 1727 AD) and Leibinz (1646 – 1716AD), and innovation of Gibbs (1839 – 1903 AD) to analyze the vectors in three dimensions.

Chart of the unit



3 - 1 Scalars, Vectors and Directed Line Segment

You will learn

- Classifying and discriminating scalar and vector quantities.
- Concept of directed line segment, its direction and its norm.
- Recognize the equivalent directed line segments.
- Constructiing of a directed line segment equivalent to another one in a coordinate plane.
- Expressing a directed line segment in terms of its ends in the coordinate plane.

Key-terms

- Scalar quantity
- Vector quantity
- Distance
- Displacement
- Direction

Materials

- Geometric tools .
- Computer
- Graphic programs
- Projector

Introduction

There are quantities that you do not require to know the number expressing their value such as length, area, volume, mass, density, population and so on. On the contrary, there are other quantities to describe them mentioning the number expressing their value is not enough. For example, to know the wind speed is not enough for air traffic, but wind direction is to be determined. As you know that wind movement is measured by its magnitude, direction and the affecting force acted on an object which its effect differs on it not only by its magnitude but also by its direction, too. As a result we can conclude that we have two types of quantities.

Scalar quantities

Scalar quantities are determined completely by their magnitude only such as length, area ...

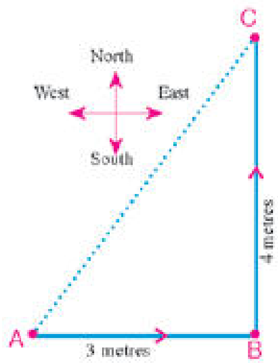
Vector quantities

Vector quantities are determined completely by their magnitude and their direction such as velocity, force. ...

Think and discuss

If a body moved from point A a distance 3 metres east, then changed its direction and moved 4 metres north and stopped at point C .

- What is the distance covered by the body during its movement?
- What is the distance between the body and the point A which is the starting point of its movement?



Notice that:

- **Distance** is a scalar quantity which is the result of AB + BC or CB + BA.
- **Displacement** is the distance between the starting and ending points only and in direction from A to C. i.e to describe the displacement, its magnitude AC and its direction from A to C must be determined.

Displacement is a vector quantity which is the distance covered in a certain direction.

Try to solve

1 **In the figure opposite:** Calculate the distance and displacement covered when a body moves from the point A to the point C, then returns to point B.

Direction

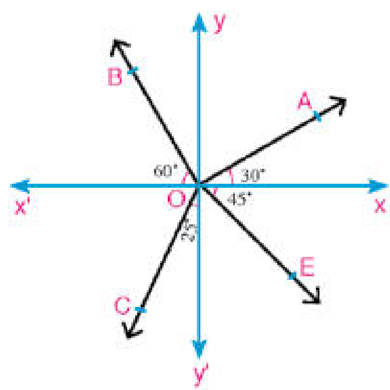
1- Each ray in the plane determines a direction. In the figure opposite:

$\overrightarrow{OX}$ determines the east direction, $\overrightarrow{OX'}$ determines the west direction,

$\overrightarrow{OY}$ determines the north direction and $\overrightarrow{OY'}$ determine the south direction.

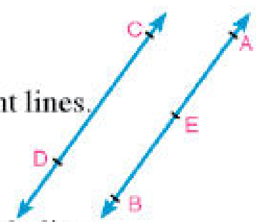
What are the directions determined by:

$\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$?



2- If $\overleftrightarrow{AB}$ // $\overleftrightarrow{CD}$, E ∈ $\overline{AB}$, then:

- $\overrightarrow{EA}$ and $\overrightarrow{BE}$ have the same direction and are carried on one straight line.
- $\overrightarrow{EA}$ and $\overrightarrow{DC}$ have the same direction and are carried on two parallel straight lines.
- $\overrightarrow{EA}$ and $\overrightarrow{EB}$ have opposite directions and are carried on one straight line.
- $\overrightarrow{EA}$ and $\overrightarrow{CD}$ have opposite directions and are carried on two parallel straight lines.



Generally:
- The two rays which have the same or opposite directions are carried on one straight line or two parallel straight lines and vice versa.
- The two rays different in direction are not carried on one straight line or two parallel straight lines.

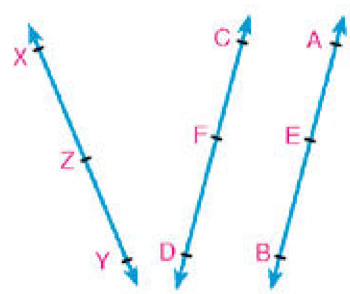
Try to solve

2 **In the figure opposite:** $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel and each is not parallel to $\overleftrightarrow{YX}$, E ∈ $\overline{AB}$, F ∈ $\overline{CD}$, Z ∈ $\overline{XY}$.

Show whether the two rays have the same, opposite or different direction in each of the following.

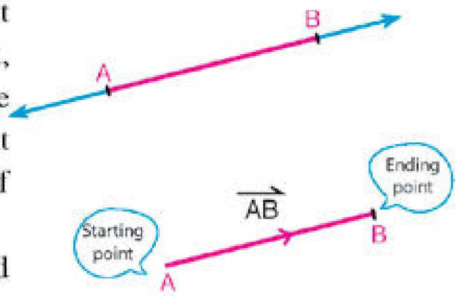
A $\overrightarrow{AB}$, $\overrightarrow{DF}$ **B** $\overrightarrow{AB}$, $\overrightarrow{XY}$ **C** $\overrightarrow{CD}$, $\overrightarrow{EB}$

D $\overrightarrow{ZY}$, $\overrightarrow{ZX}$ **E** $\overrightarrow{CF}$, $\overrightarrow{ZX}$ **F** $\overrightarrow{ZX}$, $\overrightarrow{ZY}$



The Directed Line Segment

The points A and B are the two ending points of $\overline{AB}$ or $\overline{BA}$. If we determine one of these two points as a starting point to the segment and the other one as an ending point to it, then for the line segment, there is a direction which is the same direction of the ray that carries this line segment and its starting point is the same as the starting point of the line segment.



If we determine the point A as a starting point to $\overline{AB}$ and the point B as its ending point, then we describe this segment as a directed line segment from A to B and is denoted by the symbol $\overrightarrow{AB}$.

Think and discuss

- Is $\overline{AB} \equiv \overline{BA}$? is $\overrightarrow{AB} \equiv \overrightarrow{BA}$? Explain your answer.
- Are $\overrightarrow{AB}$ and $\overrightarrow{BA}$ different or opposite in direction? Why?

Definition 1

The directed line segment: is a line segment which has a starting point, an ending point and a direction.

Try to solve

3 A, B and C are three points on a plane. Write all directed line segments determined by these points.

Definition 2

The norm of the directed line segment: norm of $\overrightarrow{AB}$ is the length of $\overline{AB}$ and is denoted by the symbol $\| \overrightarrow{AB} \|$.

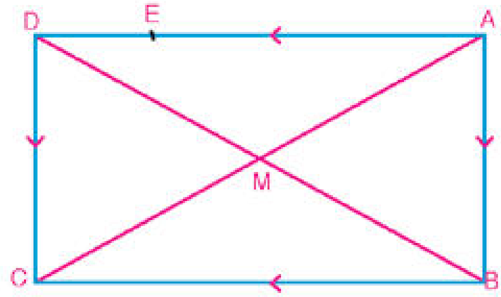
Notice that: $\| \overrightarrow{AB} \| = \| \overrightarrow{BA} \| = AB$

Definition 3

Equivalent directed line segments: Two directed line segments are said to be equivalent if they have the same norm and same direction.

Example

1 **In the figure opposite:** ABCD is a rectangle, its diagonals are intersecting at M. $E \in \overline{AD}$ then:



$\overline{AB} // \overline{CD}$, AB = CD, $\overline{BC} // \overline{AD}$, BC=AD and

MA = MC = MB = MD

A $\because \| \overrightarrow{AB} \| = \| \overrightarrow{DC} \|$, $\overrightarrow{AB}$ and $\overrightarrow{DC}$ have the same direction

$\therefore$ $\overrightarrow{AB}$ is equivalent to $\overrightarrow{DC}$

B $\because \| \overrightarrow{AM} \| = \| \overrightarrow{MC} \|$, $\overrightarrow{AM}$ and $\overrightarrow{MC}$ have the same direction

$\therefore$ $\overrightarrow{AM}$ is equivalent to $\overrightarrow{MC}$

C $\because \| \overrightarrow{MA} \| = \| \overrightarrow{MB} \|$, $\overrightarrow{MA}$ and $\overrightarrow{MB}$ have different direction

$\therefore$ $\overrightarrow{MA}$ is not equivalent to $\overrightarrow{MB}$

D $\because \| \overrightarrow{AE} \| \neq \| \overrightarrow{CB} \|$, $\overrightarrow{AE}$ and $\overrightarrow{BC}$ have the same direction

$\therefore$ $\overrightarrow{AE}$ is not equivalent to $\overrightarrow{BC}$

Try to solve

4 ABCD is a parallelogram, its diagonals are intersecting at M.

First: Determine the directed line segments (if existed) which are equivalent to:

A $\overrightarrow{AB}$ **B** $\overrightarrow{CD}$ **C** $\overrightarrow{BC}$ **D** $\overrightarrow{AM}$ **E** $\overrightarrow{MD}$

Second: Show why the following directed line segments are not equivalent?

A $\overrightarrow{AM}$, $\overrightarrow{AC}$ **B** $\overrightarrow{BA}$, $\overrightarrow{DC}$ **C** $\overrightarrow{BM}$, $\overrightarrow{DM}$

Logical thinking:

1- If $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$. What do you deduce?

2- How many directed line segments could be drawn on the plane such that each of them is equivalent to $\overrightarrow{AB}$?

3- From point C on the plane, How many directed line segments could be drawn and each is equivalent to $\overrightarrow{AB}$?

Notice that:

There is a unique directed line segment which can be drawn from point C (for example: $\overrightarrow{CD}$) such that $\overrightarrow{CD}$ is equivalent to $\overrightarrow{AB}$.

Example

2 **The directed line segments in the orthogonal coordinate plane:**

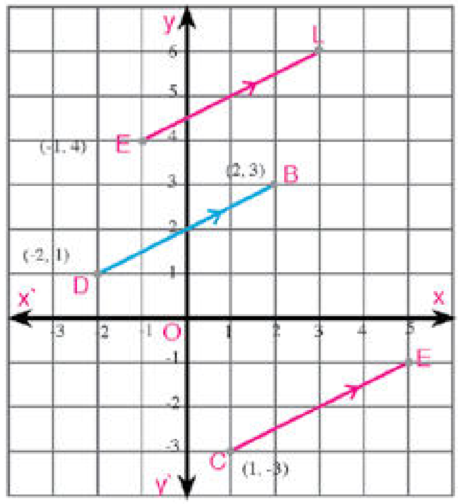
In an orthogonal coordinate plane, determine the points A(-2, 1), B(2, 3), C(1, -3), D(-1, 4), then draw $\overrightarrow{CE}$ and $\overrightarrow{DL}$ each of which is equivalent to $\overrightarrow{AB}$. Find the coordinates of E and L.

Solution

To draw $\overrightarrow{CE}$ equivalent to $\overrightarrow{AB}$, $\overrightarrow{CE}$ and $\overrightarrow{AB}$ should have the same direction the same norm.

i.e. $\overrightarrow{EC} // \overrightarrow{AB}$, $\| \overrightarrow{CE} \| = \| \overrightarrow{AB} \|$ = length of $\overline{AB}$.

- Draw $\overrightarrow{CE} // \overrightarrow{AB}$ (slope of $\overrightarrow{AB}$ = slope of $\overrightarrow{CE} = \frac{1}{2}$)
- Use the compasses to determine the length of $\overline{EC}$ = length of $\overline{AB}$ or by calculating the number of horizontal and vertical squares, then we get E (5, –1). similarly, draw $\overrightarrow{DL}$,then we get: L (3, 6)



Notice that:

Translation preserves parallelism of straight lines and lengths of line segments. Consider point C is the image of point A by the translation (1 – (–2), –3 – 1) = (3, -4)

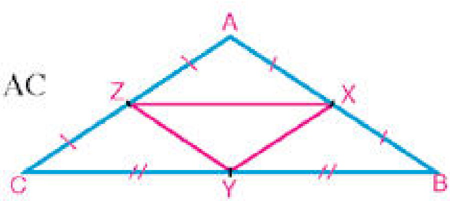
∴ To draw $\overrightarrow{CE}$ is equivalent to $\overrightarrow{AB}$, we find that $\overrightarrow{CE}$ is the image of $\overrightarrow{AB}$ by the translation (3, -4) and the coordinates of E = (2 + 3, 3 + (-4)) = (5, –1)

Use the translation: Determine the coordinates of point R which make $\overrightarrow{OR}$ equivalent to $\overrightarrow{AB}$

Check your understanding

In the figure opposite: ABC is a triangle in which AB = AC

X, Y and Z are midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ respectively



First: Which of the following statements are true?

A $\| \overrightarrow{XY} \| = \| \overrightarrow{ZY} \|$.

B $\overrightarrow{XY}$ is equivalent to $\overrightarrow{ZY}$.

C $\overrightarrow{BY}$ is equivalent to $\overrightarrow{ZX}$.

Second: Write all directed line segments (if found) which are equivalent to:

A $\overrightarrow{BX}$

B $\overrightarrow{AZ}$

C $\overrightarrow{XZ}$

D $\overrightarrow{CY}$

E $\overrightarrow{XY}$

F $\overrightarrow{ZY}$

Exercise (3 - 1)

1 **Complete the following statements to be true:**

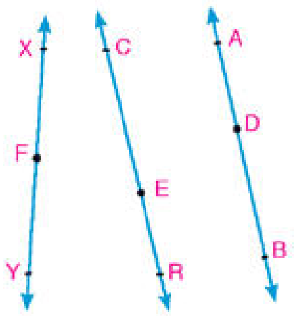
A To defined Scalor quantity completely you should know ....

B To defined Vector quantity completely you should know ....

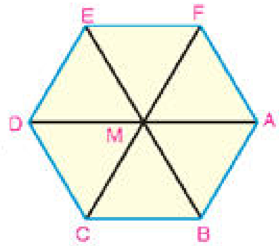
C The directed line segment is a line segment which has ...., ..., ...

D Two directed line segments are equivalent if they have ....

2 **In the figure opposite:** $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CE}$ and each of them is not parallel to $\overleftrightarrow{XY}$. join each of the following statements with a suitable one .



A $\overrightarrow{AD}$, $\overrightarrow{AB}$	**1-** have the same direction	**B** $\overrightarrow{FX}$, $\overrightarrow{XY}$
C $\overrightarrow{DA}$, $\overrightarrow{ER}$	**2-** have different direction	**D** $\overrightarrow{CE}$, $\overrightarrow{AB}$
E $\overrightarrow{BD}$, $\overrightarrow{YF}$	**3-** have opposite direction	**F** $\overrightarrow{CR}$, $\overrightarrow{XY}$



3 **In the figure opposite** ABCDEF, is a regular hexagon, its centre is M, Complete the following:

A $\overrightarrow{AB}$ is equivalent to .... and equivalent to .... and equivalent to ....

B $\overrightarrow{MD}$ is equivalent to .... and equivalent to .... and equivalent to ....

C $\overrightarrow{CD}$ is equivalent to .... and equivalent to .... and equivalent to ....

4 ABCD is a square , its diagonals are intersecting at M, write all directed equivalent line segments determined in the figure.

....

5 In a coordinate orthogonal plane : If A(4, –3) , B (4, 4), C (–3, –1), $\overrightarrow{BA}$, $\overrightarrow{CD}$, $\overrightarrow{OM}$, $\overrightarrow{NO}$ are equivalent directed line segments , where O is the origin. Find the coordinates of each of D, M , and N.

....

6 On the lattice: A(2, 3) , B (−3, 1), C (5, −1)

A Draw $\overrightarrow{CD}$, such that it is equivalent to $\overrightarrow{AB}$ and determine the coordinates of D.

....

....

....

B Determine the coordinates of the point M which is the mid-point of $\overrightarrow{BC}$, then determine the directed line segments which are equivalent to each of :

First: $\overrightarrow{BM}$ **Second:** $\overrightarrow{AM}$ **Third:** $\overrightarrow{AC}$ **Fourth:** $\overrightarrow{DB}$

....

....

....

....

C Is the figure ACDB a parallelogram? Explain your answer.

....

....

....

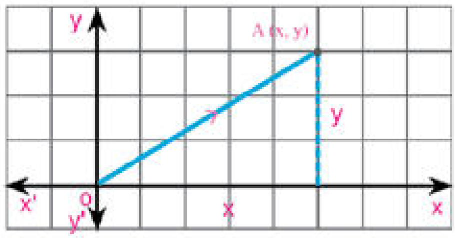
....

Vectors

3 - 2

Introduction

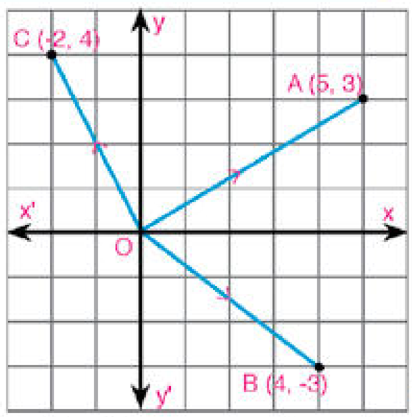
It is possible to determine the position of point A in the orthogonal coordinate plane by knowing the ordered pair (x, y) corresponding to it, where each point in the coordinate plane has a unique position with respect to the origin point O.



Position Vector

Definition 4 **The position vector of a given point with respect to the origin point is the directed line segment which its starting point is the origin point and the given point is its ending point.**

Example



1 In the figure opposite:

A (5, 3), B(4, –3), C (–2, 4) then:

- $\overrightarrow{OA}$ is the position vector of the point A with respect to the origin point O, and corresponding to the ordered pair (5, 3) and is written as $\overrightarrow{OA}$ = (5, 3).
- $\overrightarrow{OB}$ is the position vector of point B with respect to the origin point where $\overrightarrow{OB}$ = (4, –3) and also $\overrightarrow{OC}$ = (–2, 4)

Note: All position vectors have the same starting point (O) then it is possible to denote the position vector $\overrightarrow{OA}$ by the symbol $\overrightarrow{A}$ and the position vector $\overrightarrow{OB}$ by the symbol $\overrightarrow{B}$ and so on, then:

$\overrightarrow{A}$ = (5, 3) , $\overrightarrow{B}$ = (4, –3) , $\overrightarrow{C}$ = (–2, 4).

Norm of the vector:

Is the length of the line segment representing to the vector.

If: $\overrightarrow{R}$ = (x, y)

Then: $\|\overrightarrow{R}\| = \sqrt{x^2 + y^2}$

You will learn

- Find the position vector of a given point with respect to the origin in the orthogonal coordinate plane.
- Put a vector in the polar form.
- Find the norm of a vector and recognize the zero vector.
- Concept of equivalent two vectors and solving exercises on it.
- Multiplication operation of a real number on R^2.
- Express the vector in terms of the fundamental unit vector.
- Condition of two parallel vectors.
- Condition of two perpendicular vectors
- Multiply a vector by a real number and geometric representation to it.

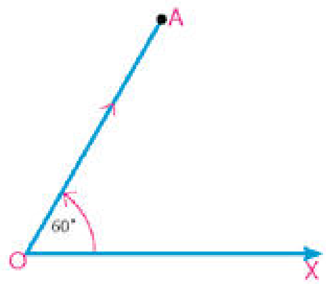
Key - Terms

- Vector
- Position Vector
- Orderd Pair
- Absolute Value
- Norm
- Equivalent Vectors
- Addition of Vectors
- Multiplication
- Polar Form
- Unit Vector
- Magnitude

Try to solve

(1) In the orthogonal coordinate plane, if A(2, -1), B(5, 0), C(-2, -3). Find the position vector of each of them with respect to the origin point O, and draw the directed line segment representing it in the coordinate plane.

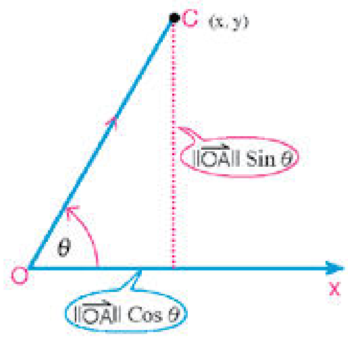
Think and discuss



The figure opposite shows a directed line segment $\overrightarrow{OA}$, its norm equals 4cm, and its direction makes 60° with the positive direction of the x-axis.

How could you find the position vector of point A with respect to the origin point O in the orthogonal coordinate plane?

Polar form of position Vector



In the figure opposite: the vector $\overrightarrow{OA}$ makes θ with the positive direction of the x-axis and its norm equals $\|\overrightarrow{OA}\|$.It is possible to express it as follows:

$\overrightarrow{OA} = (\|\overrightarrow{OA}\|, \theta)$ Polar form of the vector.

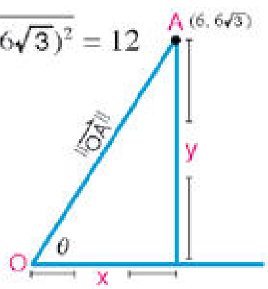
the coordinates of point A in the orthogonal coordinate plane are:

$$x = \|\overrightarrow{OA}\| \cos\theta \quad , \quad y = \|\overrightarrow{OA}\| \sin\theta \quad , \quad \tan\theta = \frac{y}{x}$$

Example

(2) In the orthogonal coordinate plane, If A(6, $6\sqrt{3}$). Find the polar form of the position vector of point A with respect to the origin point O .

Solution



$\because \overrightarrow{OA} = (6, 6\sqrt{3})$ $\quad \therefore \|\overrightarrow{OA}\|$ = length of $\overline{OA} = \sqrt{(6)^2 + (6\sqrt{3})^2} = 12$

$, \tan\theta \simeq \frac{y}{x} = \frac{6\sqrt{3}}{6} = \sqrt{3}$, $\theta \in]0, \frac{\pi}{2}[$

$\therefore \theta = \tan^{-1}(\sqrt{3}) = 60^\circ = \frac{\pi}{3}$ $\quad \therefore \overrightarrow{OA} = (12, \frac{\pi}{3})$

Try to solve

(2) **A** If $\overrightarrow{OA} = (8\sqrt{3}, 8)$, find the polar form of the vector $\overrightarrow{OA}$.

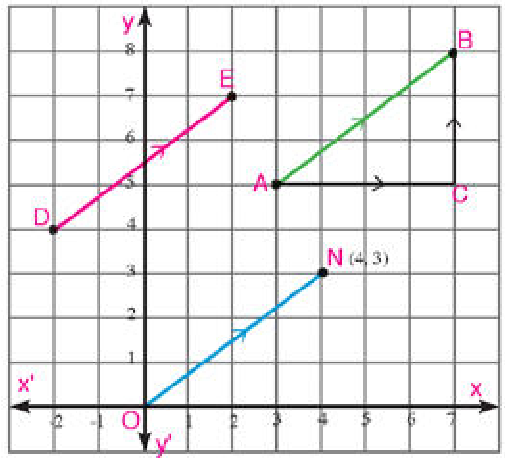
B If $\overrightarrow{OC} = (12\sqrt{2}, \frac{3\pi}{4})$ is a position vector of the point C with respect to the origin point O, find the coordinates of the poin C.

Think: What is the position vector of the origin point O (0, 0) in the orthogonal coordinate plane?

The zero vector: the vector $\overrightarrow{O} = (0, 0)$ is defined as the zero vector $\overrightarrow{O}$ and $\|\overrightarrow{O}\| = \|\overrightarrow{O}\| = 0$, and the zero vector has no direction.

Equivalent Vectors

Let a body move from A to B after covering 4 units to the right and 3 units up, then $\overrightarrow{AB}$ represents the displacement vector to the body from A to B.



It is possible to represent $\overrightarrow{AB}$ in the orthogonal coordinate plane by an infinite number of parallel directed line segments equivalent to $\overrightarrow{AB}$ and one of them is the position vector $\overrightarrow{ON}$.

i.e: $\overrightarrow{AB} = \overrightarrow{DE} = ... = \overrightarrow{ON} = (4, 3)$

Then: $\|\overrightarrow{AB}\| = \|\overrightarrow{DE}\| = ... = \|\overrightarrow{ON}\|$

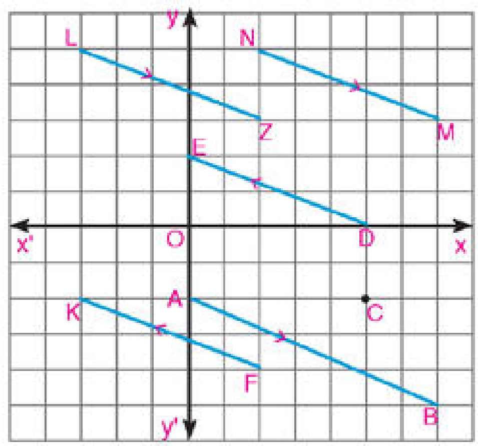
$= \sqrt{(4)^2 + (3)^2} = 5$ length units.

Try to solve

3 **In the figure opposite:**

A Determine the position vector of point C with respect to the origin point O , then find its norm.

B Determine all vectors such that each of which is equivalent to $\overrightarrow{OC}$.



You noticed that the vectors are related to the elements of the set of ordered pairs (x, y) where $(x, y) \in R^2$, then we can define the vectors as follows:

Definition 5

The vectors: The elements of the set R^2 with the addition and multiplication by a real number defined on it are called vectors.

The vectors are denoted by the symbols $\overrightarrow{M}$, $\overrightarrow{N}$, $\overrightarrow{F}$, $\overrightarrow{R}$ .... as:

$\overrightarrow{M} = (2, 3)$, $\overrightarrow{N} = (-7, 2)$, $\overrightarrow{F} = (0, 5)$ .... and so on

Adding two Vectors Algebraically

For every $\overrightarrow{A} = (x_1, y_1) \in R^2$, $\overrightarrow{B} = (x_2, y_2) \in R^2$

Then: $\overrightarrow{A} + \overrightarrow{B} = (x_1 + x_2, y_1 + y_2) \in R^2$

For example: $(3, -2) + (5, 7) = (3 + 5, -2 + 7) = (8, 5)$

Add to your knowledge

We denote the Cartesian product $R \times R$ by the symbol R^2 and is read as: R two

Properties of addition operation on vectors:

Enclosure property	for every $\overrightarrow{A}, \overrightarrow{B} \in R^2$ then $\overrightarrow{A} + \overrightarrow{B} \in R^2$
Commulative property	for every $\overrightarrow{A}, \overrightarrow{B} \in R^2$ then $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{B} + \overrightarrow{A}$
Associative property	for every $\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C} \in R^2$ then $\overrightarrow{A} + (\overrightarrow{B} + \overrightarrow{C}) = (\overrightarrow{A} + \overrightarrow{B}) + \overrightarrow{C} = \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}$
Identity element property	for every $\overrightarrow{A} \in R^2$ there exists $\overrightarrow{O} = (0, 0) \in R^2$ where: $\overrightarrow{A} + \overrightarrow{O} = \overrightarrow{A} = \overrightarrow{O} + \overrightarrow{A}$
Additive inverse property	for every $\overrightarrow{A}$ $(x, y) \in R^2$ there exists $-\overrightarrow{A} = (-x, -y) \in R^2$ where: $\overrightarrow{A} + (-\overrightarrow{A}) = \overrightarrow{O} = (-\overrightarrow{A}) + \overrightarrow{A}$
Elimination property	for every $\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C} \in R^2$ if $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{A} + \overrightarrow{C}$ then $\overrightarrow{B} = \overrightarrow{C}$

Multiplying a vectore by a real number

for every $\overrightarrow{A} = (x, y) \in R^2$, , $K \in R$: $K\overrightarrow{A} = K(x, y) = (Kx, Ky) \in R^2$

for example: $3(2, -5) = (6, -15)$, $\frac{1}{2}(4, 9) = (2, \frac{9}{2})$, $4(0, 0) = (0, 0)$, $-2(3, -4) = (-6, 8)$

Properties of multiplication operation on vectors:

Distributive property	**First:** for every $\overrightarrow{A}, \overrightarrow{B} \in R^2$, $K \in R$ then: $K(\overrightarrow{A} + \overrightarrow{B}) = K\overrightarrow{A} + K\overrightarrow{B}$ **Second:** for every $\overrightarrow{A} \in R^2$, $K_1, K_2 \in R$ then: $(K_1 + K_2)\overrightarrow{A} = K_1\overrightarrow{A} + K_2\overrightarrow{A}$
Association property	for every $\overrightarrow{A} \in R^2$, $K_1, K_2 \in R$ then: $(K_1 K_2)\overrightarrow{A} = K_1(K_2\overrightarrow{A})$
Elemination property	for every $\overrightarrow{A}, \overrightarrow{B} \in R^2$, $K \in R^*$ If $K\overrightarrow{A} = K\overrightarrow{B}$ then: $\overrightarrow{A} = \overrightarrow{B}$ and vice versa

Notice that : If $\overrightarrow{M} = (x_1, y_1)$ is equivalent to $\overrightarrow{N} = (x_2, y_2)$

Then: $x_1 = x_2$, $y_1 = y_2$ (equality of two ordered pairs), then we can say that the two vectors $\overrightarrow{M}$ and $\overrightarrow{N}$ are equal.

Example

(3) If $\overrightarrow{A} = (6, -2)$, $\overrightarrow{B} = (4, 3)$

A Find $2\overrightarrow{A} - 3\overrightarrow{B}$

B Express $\overrightarrow{B} = (11, 5)$ in terms of $\overrightarrow{A}$, and $\overrightarrow{B}$

Solution

A $2\overrightarrow{A} - 3\overrightarrow{B} = 2(6, -2) - 3(4, 3) = (12, -4) + (-12, -9) = (0, -13)$

B Let $\overrightarrow{C} = K_1\overrightarrow{A} + K_2\overrightarrow{B}$, where $K_1, K_2 \in R$

$= K_1(6, -2) + K_2(4, 3) = (6K_1, -2K_1) + (4K_2, 3K_2)$

$= (6K_1 + 4K_2, -2K_1 + 3K_2)$

From the property of equality of two ordered pairs we get:

$6K_1 + 4K_2 = 11$ (1) , $-2K_1 + 3K_2 = 5$ (2)

from (1), (2) we get: $K_1 = \frac{1}{2}$, $K_2 = 2$ $\therefore \overrightarrow{C} = \frac{1}{2}\overrightarrow{A} + 2\overrightarrow{B}$

Try to solve

4) If $\overrightarrow{A} = (2, -6)$, $\overrightarrow{B} = (-2, 5)$, $\overrightarrow{C} = (-6, 14)$

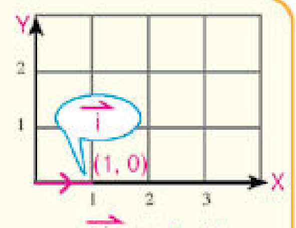
A) Find: $2\overrightarrow{A}$, $-\overrightarrow{B}$, $\frac{1}{2}\overrightarrow{C}$, $\overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C}$ B) Express $\overrightarrow{C}$ in terms of $\overrightarrow{A}$ and $\overrightarrow{B}$.

Unit vector: the unit vector is a vector whose norm is unity.

Expressing the vector in terms of the fundamental unit vectors.

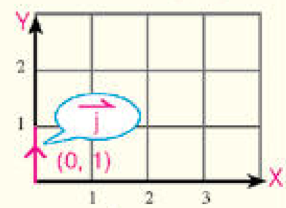
Definition 6

- **The fundamental unit vector $\overrightarrow{i}$:** it is the directed line segment with a starting point to the origin point and its norm is the unity and its direction is the positive direction of the x-axis.



$\overrightarrow{i} = (1, 0)$

- **The fundamental unit vector $\overrightarrow{j}$:** it is the directed line segment with a starting point to the origin point and its norm is the unity and its direction is the positive direction of the y-axis.



$\overrightarrow{j} = (0, 1)$

If $\overrightarrow{M} = (x, y)$

$\therefore \overrightarrow{M} = (x, 0) + (0, y)$ from definition of addition.

$= x\,(1, 0) + y\,(0, 1)$ from definition multiplication.

$= x\,\overrightarrow{i} + y\,\overrightarrow{j}$

Then: $\|\overrightarrow{M}\| = \sqrt{x^2 + y^2}$

Example

4) Express each of the following vectors in terms of the fundamental unit vectors:

A) $\overrightarrow{M} = (2, 7)$ B) $\overrightarrow{N} = (4, -3)$ C) $\overrightarrow{L} = (-5, 0)$ D) $\overrightarrow{Z} = (0, -\frac{3}{2})$

Solution

A) $\overrightarrow{M} = 2\,\overrightarrow{N} + 7\,\overrightarrow{j}$ B) $\overrightarrow{N} = 4\,\overrightarrow{i} - 3\,\overrightarrow{j}$

C) $\overrightarrow{L} = -5\,\overrightarrow{i}$ D) $\overrightarrow{Z} = \frac{-3}{2}\,\overrightarrow{j}$

Try to solve

5 Express each of the following vectors in terms of the fundamental unit vectors, then find its norm:

A $\overrightarrow{M} = (-3, 4)$ **B** $\overrightarrow{N} = (5, -12)$ **C** $\overrightarrow{L} = (-3, -6)$ **D** $\overrightarrow{Z} = (-7, 0)$

Example

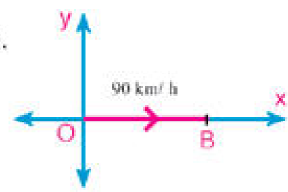
5 Find in terms of the two fundamental unit vectors. The vector which expresses each of the following:

A The uniform speed of a car covered 90 km per hour in the east direction.

B a force of magnitude 50 newtons acts on a particle in the direction 30° north of east.

Solution

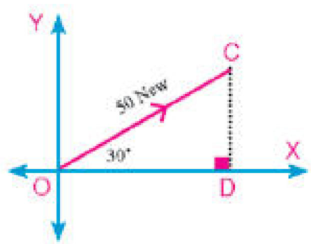
A Let the position vector of the speed of the car be $\overrightarrow{OB} = (x, y)$.



$\therefore x = 90 \ , \ y = 0$

$\overrightarrow{B} = 90 \overrightarrow{i}$

B Let the position vector of the given force be $\overrightarrow{OC} = (x, y)$



$\therefore x = 50 \cos 30° = 25\sqrt{3}$,

, $y = 50 \sin 30° = 25$

$\overrightarrow{C} = 25\sqrt{3}\ \overrightarrow{i} + 25\ \overrightarrow{j}$

Try to solve

6 Find in terms of the two fundamental unit vectors the vector which expresses each of the following:

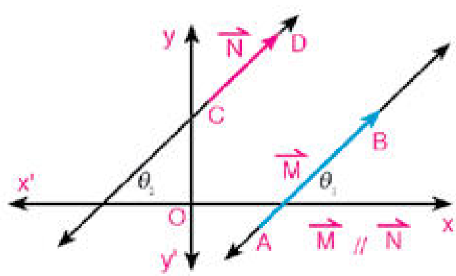
A The displacement of a body a distance 60 cm in the south direction.

B A force of magnitude 30 kg. wt acts on a particle in the direction 60° north of west.

Parallel and Perpendicular Vectors

For every $\overrightarrow{M}$ and $\overrightarrow{N}$ are two non zero vectors

where $\overrightarrow{M} = (x_1, y_1)$, $\overrightarrow{N} = (x_2 , y_2)$

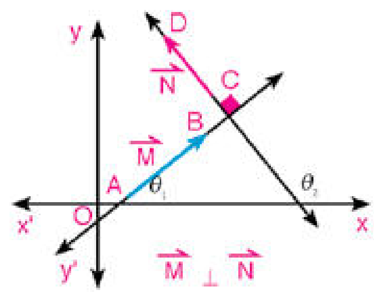


1- If $\overrightarrow{M} \ // \ \overrightarrow{N}$

then: $\tan \theta_1 = \tan \theta_2$, $\frac{y_1}{x_1} = \frac{y_2}{x_2}$

and $x_1 y_2 - x_2 y_1 = 0$ and vice versa

2- If $\overrightarrow{M} \perp \overrightarrow{N}$



then: $\tan\theta_1 \times \tan\theta_2 = -1$

$$\frac{y_1}{x_1} \times \frac{y_2}{x_2} = -1$$

and $x_1\, x_2 + y_1\, y_2 = 0$ and vice versa

Notice that:

If $\overrightarrow{A} = (2, 4)$, $\overrightarrow{B} = (-6, 3)$, $\overrightarrow{C} = (4, 8)$

then: $\overrightarrow{A} \perp \overrightarrow{B}$ because: $2 \times -6 + 4 \times 3 = -12 + 12 = 0.$

$\overrightarrow{A} \parallel \overrightarrow{C}$ because: $2 \times 8 - 4 \times 4 = 16 - 16 = 0.$

$\overrightarrow{B} \perp \overrightarrow{C}$ because: $-6 \times 4 + 3 \times 8 = -24 + 24 = 0.$

Example

6 If $\overrightarrow{A} = (2, 5)$, $\overrightarrow{B} = (K, -4)$, find the value of K when:

A $\overrightarrow{A} \parallel \overrightarrow{B}$.

B $\overrightarrow{A} \perp \overrightarrow{B}$.

Solution

A When $\overrightarrow{A} \parallel \overrightarrow{B}$,then the condition of parallelism is: $2 \times -4 - 5 \times K = 0$

$\therefore -8 - 5K = 0$, $K = -\frac{8}{5}$

B $\overrightarrow{A} \perp \overrightarrow{B}$, then the condition of perpendicularity is: $2 \times K + 5 \times -4 = 0$

$\therefore 2K - 20 = 0$, $K = 10$

Try to solve

7 If $\overrightarrow{A} = (-4, 6)$, $\overrightarrow{B} = (6, -9)$, $\overrightarrow{C} = (3, 2)$. Prove that: $\overrightarrow{A} \parallel \overrightarrow{B}$, $\overrightarrow{B} \perp \overrightarrow{C}$, $\overrightarrow{C} \perp \overrightarrow{A}$

Notice that

If $\overrightarrow{M} = (x, y)$, $K \in R$

then: $K\overrightarrow{M} = K(x, y) = (Kx, Ky)$

If $\overrightarrow{M}$ is a non zero vector, $K \neq 0$ **then:** $\overrightarrow{M} \parallel K\overrightarrow{M}$

and: $\|K\overrightarrow{M}\| = |K| . \|\overrightarrow{M}\|$

where the direction of $K\overrightarrow{M}$ is the same as the direction of $\overrightarrow{M}$ for every $K > 0$

the direction of $K\overrightarrow{M}$ is opposite to the direction of $\overrightarrow{M}$ for every $K < 0$

For example:

If $\overrightarrow{M} = (2, 1)$

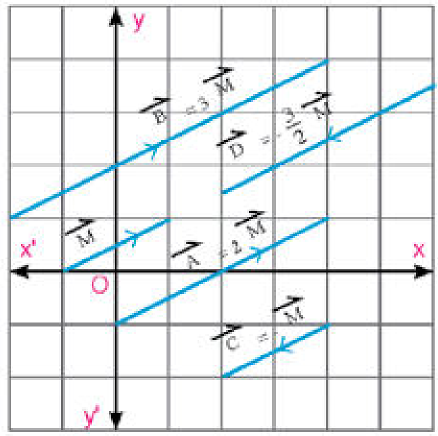
then: $\overrightarrow{A} = 2\overrightarrow{M} = 2(2, 1) = (4, 2)$

$\overrightarrow{B} = 3\overrightarrow{M} = 3(2, 1) = (6, 3)$

$\overrightarrow{C} = -\overrightarrow{M} = -(2, 1) = (-2, -1)$

$\overrightarrow{D} = -\frac{3}{2}\overrightarrow{M} = -\frac{3}{2}(2, 1) = (-3, -\frac{3}{2})$

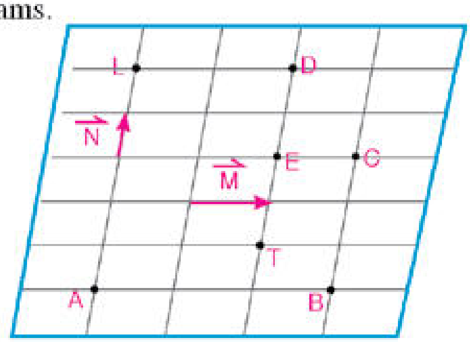
as in the figure opposite.



Try to solve

8 The lattice opposite represents congruent parallelograms.

First: express each of the following directed line segments in terms of $\overrightarrow{M}$ and $\overrightarrow{N}$



A $\overrightarrow{AB}$ **B** $\overrightarrow{CB}$ **C** $\overrightarrow{CE}$

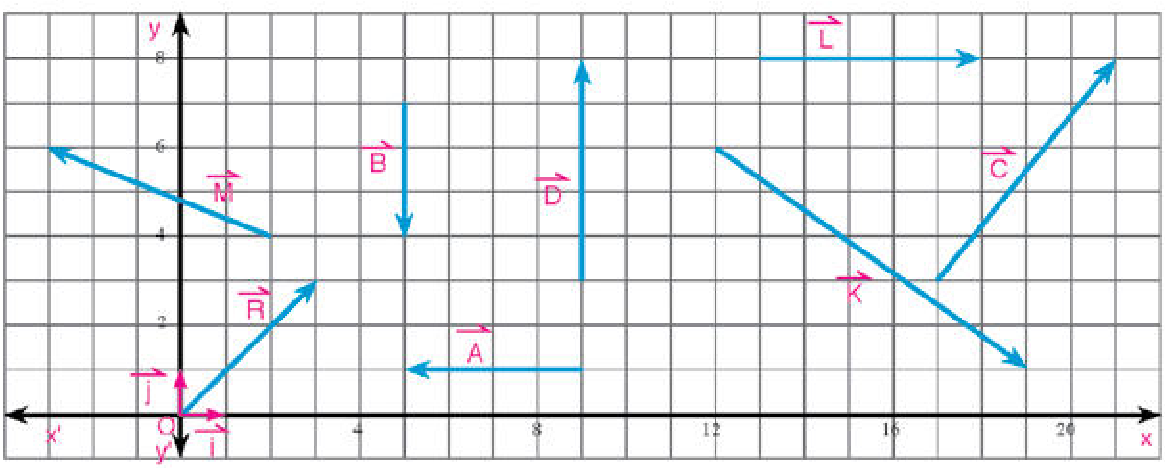
D $\overrightarrow{BC}$ **E** $\overrightarrow{BA}$ **F** $\overrightarrow{TE}$

G $\overrightarrow{DL}$ **H** $\overrightarrow{DE}$ **I** $\overrightarrow{LA}$

Second: deduce that $\overrightarrow{AB} = -\overrightarrow{BA}$ and interpret that geometrically.

Check your understanding

The following figure shows a representation of some vectors in the orthogonal coordinate plane. Write each vector in terms of the two fundamental unit vectors.



Exercise (3 - 2)

1. On the lattice : A(3, −4) , B (−12, 5) , C (−3, −6), Find the position vector for each of the points A , B , C with respect to the origin O (0, 0), then find the norm of each of them.

2. Express each of the following vectors in terms of the fundamental unit vectors. Then find the norm of each of them:

   A. $\overrightarrow{M} = (-4, -3)$
   B. $\overrightarrow{N} = (8, -6)$
   C. $\overrightarrow{F} = (-5, -12)$
   D. $\overrightarrow{A} = (0, 2\sqrt{2})$
   E. $\overrightarrow{B} = (-3\sqrt{3}, 0)$
   F. $\overrightarrow{C} = (\sqrt{2}, -3\sqrt{2})$

3. Find the polar form of each of the following vectors:

   A. $\overrightarrow{M} = 8\sqrt{3}\,\overrightarrow{i} + 8\,\overrightarrow{j}$
   B. $\overrightarrow{N} = 3\sqrt{2}\,\overrightarrow{i} + 3\sqrt{2}\,\overrightarrow{j}$

4. If $\overrightarrow{A} = (3, -2)$, $\overrightarrow{B} = (-2, 5)$, $\overrightarrow{C} = (0, 11)$:

   A. Write each of the following vectors in terms of the fundamental unit vectors
   $2\overrightarrow{B}$, $3\overrightarrow{C}$, $\overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C}$, $\frac{1}{2}(\overrightarrow{B} + \overrightarrow{C})$

   B. Express $\overrightarrow{C}$ in terms of $\overrightarrow{A}$ and $\overrightarrow{B}$

5. Find in terms of the fundamental unit vectors, the vector which expresses:

   A. A uniform speed of magnitude 60 km/h in west direction.
   B. A force of magnitude 20 kgm wt . acts on a body in the direction 30° south of east
   C. A displacement of a body a distance 40cm in the direction north west

6 If $\vec{M} = (5, 1)$, $\vec{N} = (4, -20)$, $\vec{L} = (-10, -2)$ Prove that:

A $\vec{M} \perp \vec{N}$ **B** $\vec{M} \parallel \vec{L}$ **C** $\vec{N} \perp \vec{L}$

7 If $\vec{M} = 3\vec{i} + 4\vec{j}$, $\vec{N} = -6\vec{i} - 8\vec{j}$

$\vec{L} = a\vec{i} - 8\vec{i}$, $\vec{F} = 4\vec{i} + b\vec{j}$

A Prove that $\vec{M} \parallel \vec{N}$

....

B Find $a \in R$, if $\vec{M} \parallel \vec{L}$

....

C Find $b \in R$, if $\vec{F} \perp \vec{N}$

....

D Is $\vec{F} \perp \vec{M}$? explain your answer

....

8 The lattice opposite of congruent parallelograms. Express each of the following directed line segments in terms of $\vec{M}$ and $\vec{N}$.

A $\overrightarrow{AB}$ .... **B** $\overrightarrow{BY}$ ....

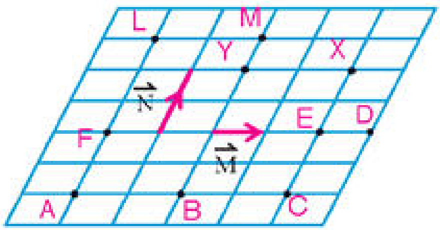
C $\overrightarrow{EC}$ .... **D** $\overrightarrow{DE}$ ....

E $\overrightarrow{XE}$ .... **F** $\overrightarrow{XY}$ ....

G $\overrightarrow{YM}$ .... **H** $\overrightarrow{LM}$ ....

I $\overrightarrow{BM}$ .... **J** $\overrightarrow{EF}$ ....

K $\overrightarrow{FL}$ .... **L** $\overrightarrow{FD}$ ....



Operations on Vectors

3 - 3

First : Adding vectors geomitricaly

You will learn

- Adding vectors and their geometric representation.
- Triangle rule of adding vectors.
- Parallelogram rule of adding vectors.
- Subtracting vectors and their graphic representation.
- Expressing the directed line segment in terms of the position vectors of its ends.

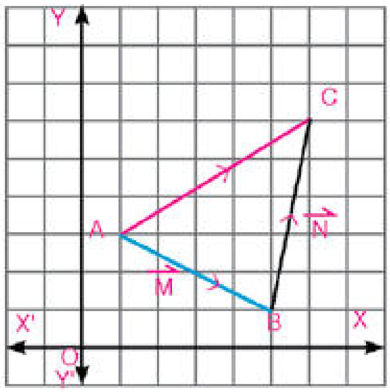
Think and discuss

If $\overrightarrow{AB}$ represents the vector $\overrightarrow{M}$ and $\overrightarrow{BC}$ represents the vector $\overrightarrow{N}$ where:

$\overrightarrow{M} = (4, -2)$, $\overrightarrow{N} = (1, 5)$

Write what equal to $\overrightarrow{M} + \overrightarrow{N}$.

Write the vector which represents $\overrightarrow{AC}$

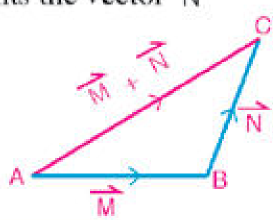


What do you notice? What do you deduce?

Triangle Rule of Adding two vectors

If $\overrightarrow{AB}$ represents the vector $\overrightarrow{M}$ and $\overrightarrow{BC}$ represents the vector $\overrightarrow{N}$ **Where** point B is the end point of the vector $\overrightarrow{M}$ and is the starting point of the vector $\overrightarrow{N}$.

Then: the vector $\overrightarrow{M} + \overrightarrow{N}$ is represented by the directed line segment $\overrightarrow{AC}$



i.e.: $\overrightarrow{M} + \overrightarrow{N} = \overrightarrow{AC}$, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

This relation is known as "shal relation"

- Adding of Vectors
- Subtracting Vectors
- Triangle Rule
- Parallelogram Rule

Example

1. A ship covers 300 metres east, then 400 metres north to exit from the port. Calculate the displacement of the ship until it exits from the port.

Solution

1- Take a suitable drawing scale: every 1 cm represents 100 metres.
∴ 3 cm represents 300 metres, 4 cm represents 400 metres.

2- Draw the path way of the trip by the drawing scale using your geometric instruments , then the displacement vector $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

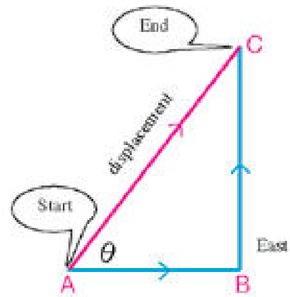
- Geometric instruments .
- Squared papers for drawing

3- Measure the length of $\overline{AC}$ using the ruler (AC = 5 cm)

4- Norm of the displacement = length of drawing × drawing scale = 5 × 100 = 500 metres.

5- Direction of the displacement : $\theta = \tan^{-1}(\frac{4}{3}) \simeq 53^\circ$ to the nearest degree.

∴ The ship is at 500 metres from the point of sailing in the direction 53° north of East.

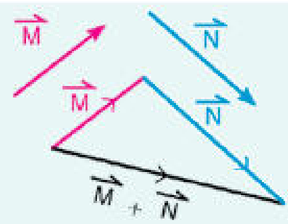


Try to solve

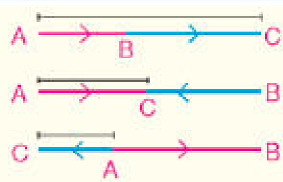
1) A truck moved from site A a distance 80 km in the west direction, then 120 km in the direction 60° north of west to reach site B. Find the magnitude and direction of the displacement $\overrightarrow{AB}$.

Important Notes:

1- Any two vectors $\overrightarrow{M}$ and $\overrightarrow{N}$ could be added (finding their resultant) by constructing two consecutive vectors equivalent to the two vectors $\overrightarrow{M}$ and $\overrightarrow{N}$ as in the figure opposite .



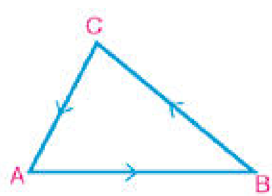
2- Shal rule of adding two vectors is true if the points A, B and C belong to a straight line.
In the three figures opposite , then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



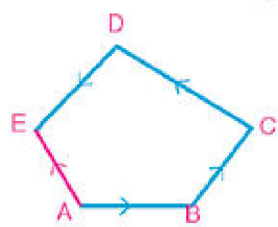
3- $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \overrightarrow{0}$ (Identity element of addition operation of vectors)
∴ $\overrightarrow{BA}$ is the additive inverse of the vector $\overrightarrow{AB}$
i.e. $\overrightarrow{BA} = -\overrightarrow{AB}$

Think: Deduce that the following statements are true:

1- In △ ABC: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{0}$

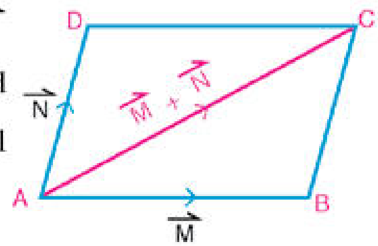


2- In the figure ABCDE: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$



Parallelogram Rule of Adding Two Vectors

If $\overrightarrow{AB}$ represents the vector $\overrightarrow{M}$ and $\overrightarrow{AD}$ represents the vector $\overrightarrow{N}$ i.e. for the two vectors $\overrightarrow{M}$ and $\overrightarrow{N}$, the same starting point to find $\overrightarrow{M} + \overrightarrow{N}$, complete the parallelogram ABCD, draw its diagonal $\overrightarrow{AC}$ then $\overrightarrow{AD}$ is equivalent to $\overrightarrow{BC}$. (why?)



$\therefore \overrightarrow{M} + \overrightarrow{N} = \overrightarrow{AB} + \overrightarrow{AD}$

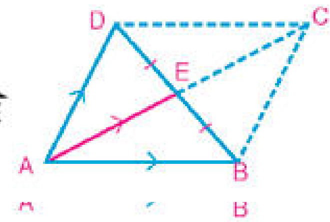
$= \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ i.e.: $\boxed{\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}}$

This rule is known as parallelogram rule of adding two vectors.

Think: Deduce that the following statements are true:

1- $\overrightarrow{M} + \overrightarrow{N} = \overrightarrow{N} + \overrightarrow{M}$

2- In $\triangle$ ABD, if E is the midpoint of $\overline{BD}$ then: $\overrightarrow{AB} + \overrightarrow{AD} = 2\,\overrightarrow{AE}$

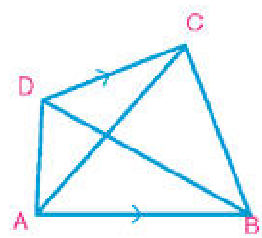


Try to solve

Example

2 In any quadrilateral ABCD, prove that: $\overrightarrow{AB} + \overrightarrow{DC} = \overrightarrow{AC} + \overrightarrow{DB}$

Solution



In $\triangle$ ABC: $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$ (1)

In $\triangle$ DCB: $\overrightarrow{DC} = \overrightarrow{DB} + \overrightarrow{BC}$ (2)

From (1) , (2) We get:

$\overrightarrow{AB} + \overrightarrow{DC} = \overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{DB} + \overrightarrow{BC}$

$= \overrightarrow{AC} + \overrightarrow{DB} + \overrightarrow{CB} + \overrightarrow{BC}$ (Commutative property).

$= (\overrightarrow{AC} + \overrightarrow{DB}) + (\overrightarrow{CB} + \overrightarrow{BC})$ (Associative property).

$= \overrightarrow{AC} + \overrightarrow{DB} + \overrightarrow{O}$ (Additive inverse).

$= \overrightarrow{AC} + \overrightarrow{DB}$ (Identity element property).

Try to solve

2 ABCD is a quadrilateral in which $\overrightarrow{BC} = 3\,\overrightarrow{AD}$. Prove that:

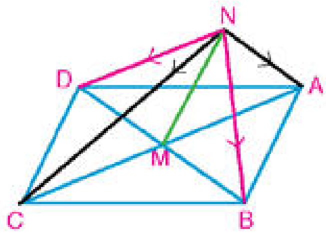
A ABCD is a trapezium. **B** $\overrightarrow{AC} + \overrightarrow{DB} = 4\,\overrightarrow{AD}$.

Example

3 ABCD is a parallelogram , its diagonals are intersecting at M. N is a point in the same plane. Prove that:

A $\overrightarrow{AB} + \overrightarrow{AD} + 2\,\overrightarrow{CM} = \overrightarrow{O}$ **B** $\overrightarrow{NA} + \overrightarrow{NC} = \overrightarrow{NB} + \overrightarrow{ND}$

Solution



A $\because \overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ (1) parallelogram rule.

$2\ \overrightarrow{CM} = \overrightarrow{CA}$ (2) (CM = MA).

add (1) and (2), we get

$\overrightarrow{AB} + \overrightarrow{AD} + 2\ \overrightarrow{CM} = \overrightarrow{AC} + \overrightarrow{CA}$

$\because \overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{O}$ $\therefore \overrightarrow{AB} + \overrightarrow{AD} + 2\ \overrightarrow{CM} = \overrightarrow{0}$

B Draw $\overline{NM}$

In $\triangle$ NAC: $\because$ M is the midpoint of $\overline{AC}$ $\therefore \overrightarrow{NA} + \overrightarrow{NC} = 2\ \overrightarrow{NM}$ (3).

In $\triangle$ NBD: $\because$ M is the midpoint of $\overline{BD}$ $\therefore \overrightarrow{NB} + \overrightarrow{ND} = 2\ \overrightarrow{NM}$ (4).

From (3) , (4) we get: $\overrightarrow{NA} + \overrightarrow{NC} = \overrightarrow{NB} + \overrightarrow{ND}$.

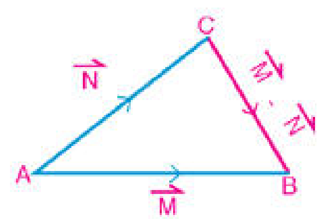
Try to solve

3 ABCD is a parallelogram in which E is the midpoint of $\overline{CB}$. Prove that:

$\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{DC} = 2\ \overrightarrow{AE}$

Seconad : Subtracting Vectors geometricaly

In the figure opposite , ABC is a triangle:



$\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{AB} + (-\ \overrightarrow{AC}\ )$ (Definition of subtaction).

$= \overrightarrow{AB} + \overrightarrow{CA}$ (Additive inverse).

$= \overrightarrow{CA} + \overrightarrow{AB}$ (commutative property).

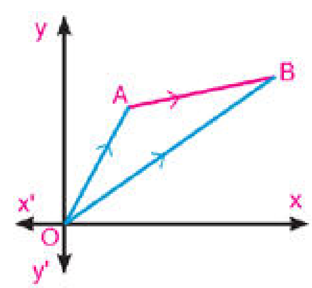
$= \overrightarrow{CB}$ (Triangle rule).

i.e. $\boxed{\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}}$

If $\overrightarrow{AB}$ represents the vector $\overrightarrow{M}$ and $\overrightarrow{AC}$ represents the vector $\overrightarrow{N}$

then: $\overrightarrow{CB}$ represents $\overrightarrow{M} - \overrightarrow{N}$ as $\overrightarrow{BC}$ represents $\overrightarrow{N} - \overrightarrow{M}$

Expressing the directed line segment $\overrightarrow{AB}$ in terms of the position vectors of its ends:



If A (x_1, y_1) , B (x_2, y_2).

then: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ (subtraction rule).

where $\overrightarrow{OB}$ and $\overrightarrow{OA}$ are the two position vectors of the points B and A respectively.

$\therefore$ $\boxed{\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}}$

For example: If A (7, –1) , B (2, 5) then $\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (2, 5) - (7, -1) = (-5, 6)$

Example

4 ABCD is a parallelogram where A(2, −1) , B (7, 1) , C (4, 4). Find the coordinates of the point D.

Solution

$\because \overline{AD} \parallel \overline{CB}$, $AD = BC$ $\qquad \therefore \overrightarrow{AD} = \overrightarrow{BC}$

and $\overrightarrow{D} - \overrightarrow{A} = \overrightarrow{C} - \overrightarrow{B}$ $\qquad \therefore D = \overrightarrow{A} + \overrightarrow{C} - \overrightarrow{B}$

i.e $\overrightarrow{D} = (2, -1) + (4, 4) - (7, 1) = (-1, 2)$ $\qquad \therefore$ the coordinates of the point D are (−1, 2)

Try to solve

4 ABCD is a quadrilateral in which A(−1, −2), B (9, 0), C(8, 4), D (0, 2).

Prove that: **A** $\overrightarrow{AB} = \overrightarrow{DC}$. **B** $\overrightarrow{AB} \perp \overrightarrow{BC}$.

Example

5 If $3\overrightarrow{N} - 2\overrightarrow{AB} = 3\overrightarrow{CB} + 5\overrightarrow{AB}$. Prove that $\overrightarrow{N} = \overrightarrow{CA}$.

Solution

$3\overrightarrow{N} = 3\overrightarrow{CB} + 5\overrightarrow{BA} + 2\overrightarrow{AB}$ (add $2\overrightarrow{AB}$ to both sides).

$3\overrightarrow{N} = 3\overrightarrow{CB} + 5\overrightarrow{BA} - 2\overrightarrow{BA}$ (additive inverse of vectors).

$3\overrightarrow{N} = 3\overrightarrow{CB} + 3\overrightarrow{BA}$ (subtraction operation).

$3\overrightarrow{N} = 3(\overrightarrow{CB} + \overrightarrow{BA}) = 3\overrightarrow{CA}$ $\qquad \therefore \overrightarrow{N} = \overrightarrow{CA}$.

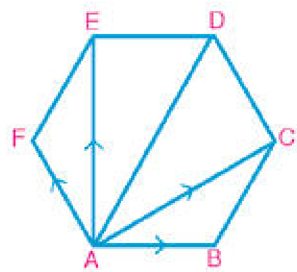
Try to solve

5 If $2\overrightarrow{M} + 3\overrightarrow{AB} = 2\overrightarrow{CB} - \overrightarrow{BA}$ Prove that $\overrightarrow{M} = \overrightarrow{CA}$.

Check your understanding

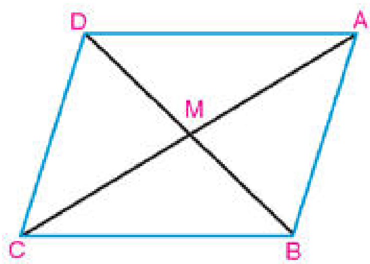
In the figure opposite: ABCDEF is a regular hexagon. Prove that:

$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AE} + \overrightarrow{AF} = 2\overrightarrow{AD}$.



Exercise (3 - 3)

1 **In the figure opposite:** ABCD is a parallelogram , M is the intersection point of its diagonals. Complete:



A $\overrightarrow{AB}$ = ...

B $\overrightarrow{BC}$= ....

C $\overrightarrow{AB} + \overrightarrow{BC}$ = ...

D $\overrightarrow{AC} + \overrightarrow{CD}$ = ...

E $\overrightarrow{BD} + \overrightarrow{DC}$ = ...

F $\overrightarrow{CA} + \overrightarrow{AD}$ = ...

G $\overrightarrow{AB} + \overrightarrow{AD}$ = ...

H $\overrightarrow{BC} + \overrightarrow{DC}$ = ...

I $\overrightarrow{DA} + \overrightarrow{DC}$ = ...

J $\overrightarrow{AM} + \overrightarrow{MC}$ = ...

K $\overrightarrow{AB} + \overrightarrow{AD}$= 2 ....

L $\overrightarrow{AD} + \overrightarrow{CD}$ = 2 ....

M $\overrightarrow{MA} + \overrightarrow{MB}$ = ...

N $\overrightarrow{AB} + 2\,\overrightarrow{BM}$ = ...

2 In any triangle XYZ, Prove that: $\overrightarrow{XY} + \overrightarrow{YZ} + \overrightarrow{ZX} = \vec{0}$

....

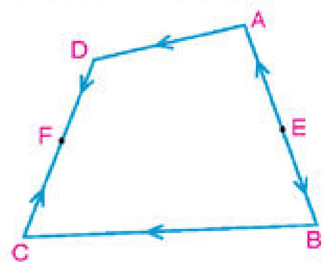
....

3 In any quadrilateral ABCD, prove that: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$.

....

....

4 **In the figure opposite:** ABCD is a quadrilateral, $E \in \overline{AB}$, $F \in \overline{CD}$.
prove that: $\overrightarrow{EB} + \overrightarrow{BC} + \overrightarrow{CF} = \overrightarrow{EA} + \overrightarrow{AD} + \overrightarrow{DF}$



....

....

5 ABCD is a quadrilateral, if $\overrightarrow{AC} + \overrightarrow{DB} = 2\overrightarrow{AB}$ prove that : ABCD is a parallelogram.

....

....

6 ABC is a triangle in which D is the mid-point of $\overline{AB}$, E is the mid–point of $\overline{AC}$.
Prove that : $\overrightarrow{AE} + \overrightarrow{CD} = \overrightarrow{EB} + \overrightarrow{DA}$.

....

....

7 In the triangle ABC : D , E and F are the mid-points of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ reprectively .Prove that : $\overrightarrow{AE} + \overrightarrow{BF} = \overrightarrow{DC}$

....

....

8 ABCD is a trapezium in which $\overline{AD} \mathbin{//} \overline{BC}$, $\frac{AD}{BC} = \frac{2}{3}$. Prove that: $\overrightarrow{AC} + \overrightarrow{BD} = \frac{5}{2}\overrightarrow{AD}$

9. If $\overrightarrow{A} = 3\overrightarrow{i} - 2\overrightarrow{j}$, $\overrightarrow{B} = -\overrightarrow{i} - 4\overrightarrow{j}$

Find:

A. $\overrightarrow{A} + \overrightarrow{B}$ ....

B. $\overrightarrow{A} - \overrightarrow{B}$ ....

C. $\|\overrightarrow{A} + \overrightarrow{B}\|$ ....

D. $2\overrightarrow{A} + 3\overrightarrow{B}$ ....

E. $\overrightarrow{A} - 3\overrightarrow{B}$ ....

F. $-3\overrightarrow{A}$ ....

10. ABCD is a parallelogram, in which A(3, 0) , B (0, 4) , D(−2, −1). Find the coordinates of the point C.

....

....

11. ABCD is a trapezium in which A(−2, −3) , B (4, −1) , C (2, 5), D(−1 , k).

A. If $\overrightarrow{AB} \,/\!/\, \overrightarrow{DC}$, find the value of k.

....

B. Prove that: $\overrightarrow{CB} \perp \overrightarrow{AB}$.

....

C. Find the area of the trapezium ABCD.

....

General Exercises

1 On the lattice O (0, 0)is the origin point. Determine the points A(–4, 0) , B (0, –3), C (3, 1) , D (2, 8), then find:

A The position vector of the points A, B, C in terms of the fundamental unit vectors .
B The position vector of the point D with respect to the origin point O in the polar form.
C The norm of the directed line segment $\overrightarrow{AB}$.
D The value of K which makes $\overrightarrow{AD} = K\,\overrightarrow{BC}$.

....

....

....

....

2 Find in terms of the fundamental unit vectors the vector which expresses:

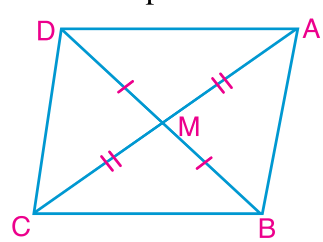
A A force of magnitude 20 newtons acts at a particle vertically in the north direction..........
B A displacement of a body moves a distance 50 cm in 30° north of the west direction ....
C The uniform velocity of a car covers a distance 70 km/h in the west direction..........

3 If $\overrightarrow{A} = (4, -6)$, $\overrightarrow{B} = (-6, 9)$, $\overrightarrow{C} = (-3, -2)$

A Prove that: $\overrightarrow{A} \,//\, \overrightarrow{B}$, $\overrightarrow{B} \perp \overrightarrow{C}$, $\overrightarrow{C} \perp \overrightarrow{A}$
B **Find:** $2\overrightarrow{A} + \overrightarrow{B}$, $\overrightarrow{B} - 2\overrightarrow{C}$, $\frac{1}{2}\overrightarrow{A} + B - 3\overrightarrow{C}$

....

4 **In the figure opposite:** All the following statements express $\overrightarrow{AC}$ Except the statement:

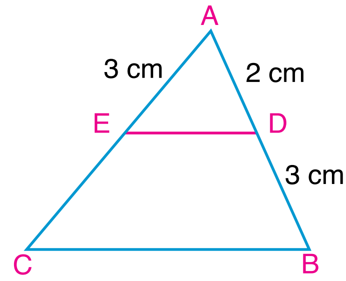


A $2\,\overrightarrow{AM}$
B $\overrightarrow{AD} + \overrightarrow{DC}$
C $\overrightarrow{AB} + \overrightarrow{BD}$
D $\overrightarrow{BC} + \overrightarrow{DC}$

5 The vector $\overrightarrow{M} = (12\sqrt{2} , \frac{\pi}{4})$ in terms of the fundamental unit vectors expresses the vector: ....

A $6\,\overrightarrow{i} + 6\,\overrightarrow{j}$
B $12\,\overrightarrow{i} - 12\,\overrightarrow{j}$
C $-6\,i - 12\,\overrightarrow{j}$
D $12\,\overrightarrow{i} + 12\,\overrightarrow{j}$

Short answer questions:



⑥ **In the figure opposite:** $\overline{DE} // \overline{BC}$

Find the numerical values of K , L, M , N if :

- **A** $\overrightarrow{BD} = K\ \overrightarrow{DA}$ ....
- **B** $\overrightarrow{CE} = L\ \overrightarrow{CA}$ ....
- **C** $\overrightarrow{BC} = M\ \overrightarrow{ED}$ ....
- **D** $\overrightarrow{AD} + \overrightarrow{DE} = N\ \overrightarrow{AC}$ ....

⑦ If ABCD is a parallelogram, in which A (2, −2) , B (4, −2) , C (2, 3) Find the coordinates of the point D.

....

⑧ On the lattice , $\overrightarrow{AB} = (-2, 3)$, $\overrightarrow{CB} = (-6, -4)$, $2\ \overrightarrow{B} + \overrightarrow{AC} = (6, 11)$. Find:

First: the coordinates of each of the points A , B and C

Second: Area of the triangle ABC (using vectors).

Unit 4
Analytic Geometry
Straight Line

Unit objectives

By the end of this unit, the student should be able to:

- Find the coordinates of the division point of a line segment internally or externally if the ratio of the division is known.
- Find the ratio by which the line segment is divided internally or externally if the coordinates of the division point is known.
- Recognize the different forms of the equation of the straight line.
- Find the vector equation, parametric equations and cartesian equation of the straigth line.
- Find the general form of the equation of the straight line.
- Find the equation of the straight line in terms of the intercepted parts of the two axes.
- Find the measure of the acute angle between two straight lines.
- Find the length of a perpendicular drawn from a point to a straight line.

Key - Terms

- point of division
- direction vector of a Straight line
- Vector equation
- parametric Equation
- Cartesian Equation
- General Equation
- Angle between two straight lines
- Length of a perpendicular

Lessons of the Unit

Lesson (4 - 1): Division of a line segment.
Lesson (4 - 2): Equation of the straight line.
Lesson (4 - 3): Measure of the angle between two straight lines.
Lesson (4 - 4): The length of the perpendicular from a point to a straight line .

Materials

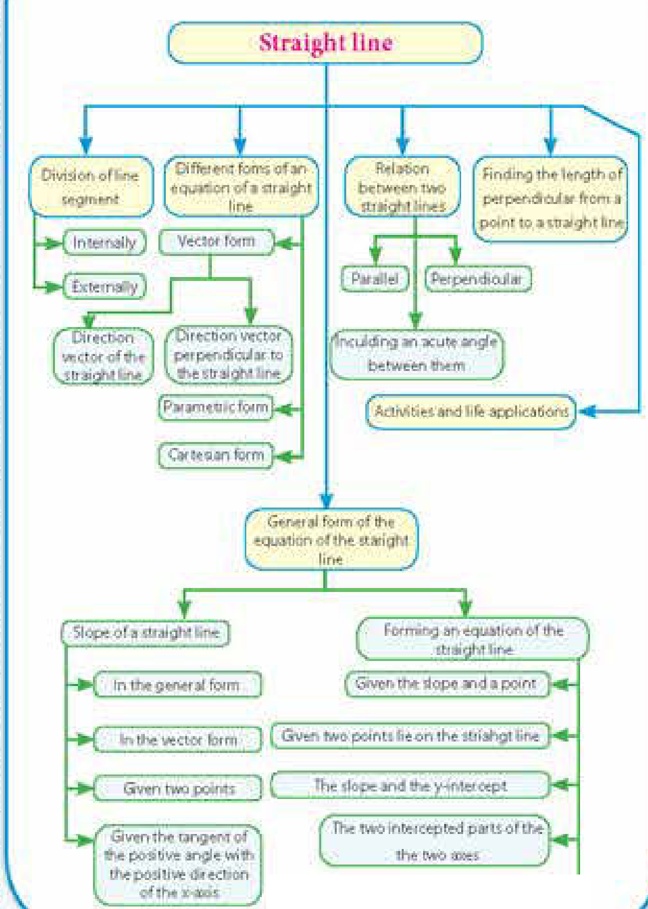
Scientific calculator - Computer - Graphic programs

Brief History

Analytic Geometry is one of the basic branches of mathematics because of its great importance when studying mathematical sciences, physical applications and technical sciences. It helped the study of space and its engineering properties to the modern era and it is related to everything that is new, as it is the key to interpret images in computer science.

Analytic Geometry is considered a gateway to the study of differential geometry (motion engineering) and algebraic geometry where the differential geometry specializes the study of properties of geometric shapes as curves and surfaces by applying differential and integral calculus. Scientists innovated coordinate system consisting of the two orthogonal intersecting axes (x-axis, y-axis) by which we can express each point in the plane by two real number (x, y) using the coordinate system, proved the properties of the Euclidean geometry expressing the straight lines and curves by algebraic equations which are paths to general points moving under conditions of the relation between (x, y), the analytic geometry has facilitated a lot of treatment in various branches of mathematics as it was one of the factors of development and handle them

Chart of the unit



4 - 1 Division of a line segment

You will learn

- Concept of division internally.
- Concept of division externally.
- Finding the ratio of division.

Key-terms

- Internal Division
- External Division
- Ratio of Division

Materials

- Scientific calculator

Think and discuss

You have studied before how to find the coordinates of the midpoint of a line segment. Can you find the coordinates of the division point of a line segment internally or externally given that the ratio of the division?

First: Finding the Coordinates of the point of division of a line segment by a certain ratio:

1- Internal division

If $C \in \overline{AB}$, then point C divides $\overline{AB}$ internally by the ratio $m_2 : m_1$

where $\frac{m_2}{m_1} > 0$ then $\frac{AC}{CB} = \frac{m_2}{m_1}$

and for the two directed segments $\overrightarrow{AC}$, $\overrightarrow{CB}$

The same direction i.e.: $m_1 \times \overrightarrow{AC} = m_2 \times \overrightarrow{CB}$

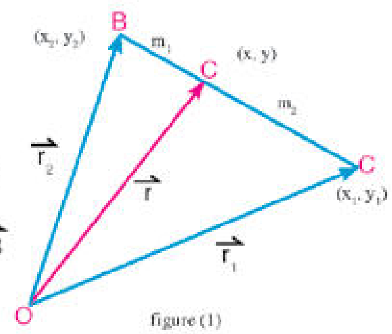


figure (1)

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x, y)$

then $\vec{r_1}$, $\vec{r_2}$, $\vec{r}$ are vectors representing the directed line segments $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ respectively where O is the origin point for the coordinate orthogonal system.

Using subtracting vectors: $m_1 (\overrightarrow{OC} - \overrightarrow{OA}) = m_2 (\overrightarrow{OB} - \overrightarrow{OC})$

$m_1 (\vec{r} - \vec{r_1}) = m_2 (\vec{r_2} - \vec{r})$

By distribution $m_1 \vec{r} - m_1 \vec{r_1} = m_2 \vec{r_2} - m_2 \vec{r}$

$m_1 \vec{r} + m_2 \vec{r} = m_1 \vec{r_1} + m_2 \vec{r_2}$

Then $\vec{r} (m_1 + m_2) = m_1 \vec{r_1} + m_2 \vec{r_2}$

i.e.: $$\vec{r} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2}$$ **which is called the vector form**

Example

1 If A (2, – 1), B (– 3, 4), find the coordinates of point C which divides $\overrightarrow{AB}$ internally by the ratio 3 : 2 in the vector form.

Solution

Let C (x, y)

$\because$ A (2, –1) $\quad \therefore \overrightarrow{r_1} = (2, -1)$, $\because$ B (–3, 4) $\quad \therefore \overrightarrow{r_2} = (-3, 4)$

$m_2 : m_1 = 3 : 2$

$\because \overrightarrow{r} = \dfrac{m_1 \overrightarrow{r_1} + m_2 \overrightarrow{r_2}}{m_1 + m_2}$

$\therefore \overrightarrow{r} = \dfrac{2(2, -1) + 3(-3, 4)}{2 + 3} = \dfrac{(4, -2) + (-9, 12)}{5} = \dfrac{(-5, 10)}{5} = (-1, 2)$

$\therefore$ The coordinates of point C are (–1, 2)

Cartesian form:

$$(x, y) = \frac{m_1(x_1, y_1) + m_2(x_2, y_2)}{m_1 + m_2} = \frac{(m_1 x_1 + m_2 x_2, m_1 y_1 + m_2 y_2)}{m_1 + m_2}$$

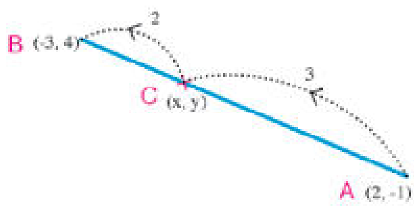
From that we get: $(x , y) = \left(\dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2} , \dfrac{m_1 y_1 + m_2 y_2}{m_1 + m_2}\right)$

Example

2 Solve the previous example using the Cartesian form.

Solution

$(x, y) = \left(\dfrac{2 \times 2 + 3 \times -3}{2 + 3} , \dfrac{2 \times -1 + 3 \times 4}{2 + 3}\right) = (-1, 2)$



Try to solve

1 If A (4, 2), B (8, –6), find the coordinates of point C which divides $\overrightarrow{BA}$ internally by the ration 1 : 3

2- External diviaion

If C $\in \overleftrightarrow{AB}$,C $\notin \overline{BA}$, then C divides $\overrightarrow{AB}$ externally by the ratio $m_2 : m_1$ where $\frac{m_2}{m_1} < 0$ then one of the two values m_1 or m_2 is positive and the other is negative, then the following figure illustrates that there are two probabilities:

C $\in \overrightarrow{AB}$, C $\notin \overline{AB}$

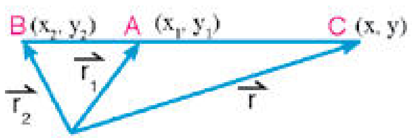


Figure (2)

C $\in \overrightarrow{AB}$,C $\notin \overline{AB}$

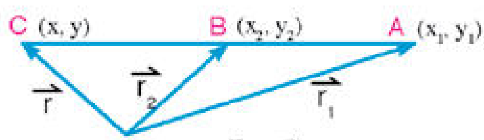
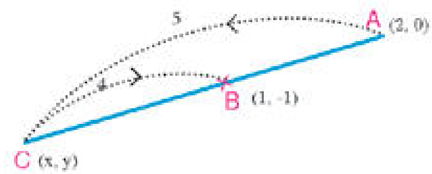


Figure (3)

Example

3 If A (2, 0), B (1, −1), find the coordinates of point C which divides $\overrightarrow{AB}$ externally by the ratio 5 : 4.

Solution



$\because \overrightarrow{r_1} = (2, 0),\ \overrightarrow{r_2} = (1, -1)$

$, m_2 : m_1 = 5 : -4 \quad \therefore \frac{m_2}{m_1} < 0$ negative — by substituting

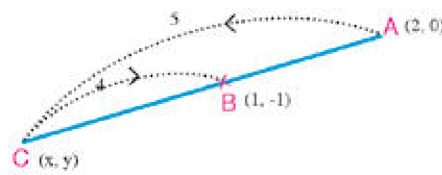
$, \overrightarrow{r} = \frac{m_1 \overrightarrow{r_1} + m_2 \overrightarrow{r_2}}{m_1 + m_2}$ — mathematical formula for the rule

$\therefore \overrightarrow{r} = \frac{-4(2, 0) + 5(1, -1)}{-4 + 5}$ — by distributing

$\overrightarrow{r} = (-8 + 5, 0 - 5) = (-3, -5)$ — by adding and simplifying

$\therefore$ The coordinates of point C are (−3, −5)

Cartesian form:



$(x, y) = \left(\frac{-4 \times 2 + 5 \times 1}{-4 + 5}, \frac{-4 \times 0 + 5 \times -1}{-4 + 5}\right)$

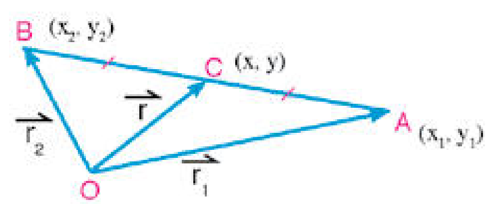
$= (-3, -5)$

Notice that:

If C is the midpoint of $\overrightarrow{AB}$ where A (x_1, y_1), B(x_2, y_2) then: $m_1 = m_2 = m$ then

$\overrightarrow{r} = \frac{\overrightarrow{r_1} + \overrightarrow{r_2}}{2}$ **Vector form**

$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ **Cartesian form**



Try to solve

2 If C (2, 4) is the midpoint of $\overrightarrow{AB}$ where A (x, 4), B (1, y) find the value of x and y

Second : Finding the ratio of Division

If point C divides $\overrightarrow{AB}$ by the ratio $m_2 : m_1$ and:

1- The ratio of division $\frac{m_2}{m_1} > 0$ then the division is internal.

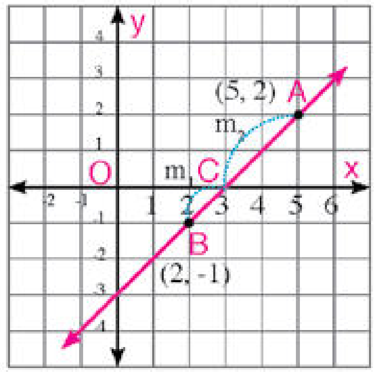
2- The ratio of division $\frac{m_2}{m_1} < 0$ then the division is external.

Example

4 If A (5, 2), B (2, − 1) , find the ratio by which $\overrightarrow{AB}$ is divided by the points of intersection of $\overleftrightarrow{AB}$ with the two axes, showing the type of division in each case, then find the coordinates of the division point.

Solution

First: let the x-axis intersects $\overrightarrow{AB}$ at point C (x, 0)



where $\frac{AC}{CB} = \frac{m_2}{m_1}$ then: $y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$

$\therefore 0 = \frac{m_1 (2) + m_2 (-1)}{m_1 + m_2} = \frac{2m_1 - m_2}{m_1 + m_2}$

$\therefore 2m_1 = m_2$ $\therefore \frac{m_2}{m_1} = \frac{2}{1}$ (ratio of division)

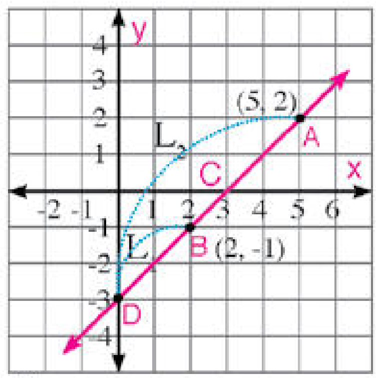
$\because \frac{m_2}{m_1} > 0$

$\therefore$ The division is internal by the ratio 2 : 1

$\therefore$ The coordinates are $C\left(\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, 0\right) = \left(\frac{1 \times 5 + 2 \times 2}{1 + 2}, 0\right)$

$= (3, 0)$

Second: The straight line intersects the y-axis at point D



Let the coordinates of D be (0, y)

where $\frac{AD}{DB} = \frac{m_2}{m_1}$ then $x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

$\therefore 0 = \frac{m_1 \times 5 + m_2 \times 2}{m_1 + m_2}$

$\therefore 2\, m_2 = -5m_1$ $\therefore \frac{m_2}{m_1} = -\frac{5}{2}$ (ratio of division)

$\because \frac{m_2}{m_1} < 0$

$\therefore$ The division is external by the ratio 5 : 2

The coordinates of the point D are $(0, y) = \left(0, \frac{-2 \times 2 + 5 \times -1}{-2 + 5}\right)$

$\therefore (0, -3)$

Think: In the previous example , use the vector form to find the ratio by which $\overrightarrow{AB}$ is divided by the two axes, then find the coordinates of the point of division.

Try to solve

3. If A (– 4, 3), B (8, 6), C ∈ $\overleftrightarrow{AB}$ where C (x, 0), find the ratio by which $\overrightarrow{AB}$ is divided by point C showing the type of division, then find the value of x.

Check your understanding

1. If A (0, – 3), B (3, 6), find the coordinates of point C which divides $\overrightarrow{BA}$ internally by the ratio 1 : 2
2. **Distance:**

   A bus moves from city A to city B where A(5, – 6), B(– 1, 0), it stopped twice during its movement. Find the coordinates of the two points at which the bus has been stopped if:

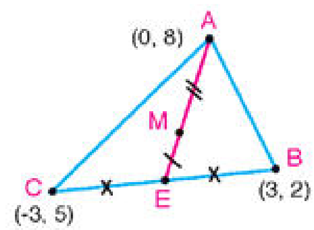
   A It stopped at the middle of the road.

   B It stopped at two thirds of the road from city A.

Exercise (4 - 1)

First : complete each of the following

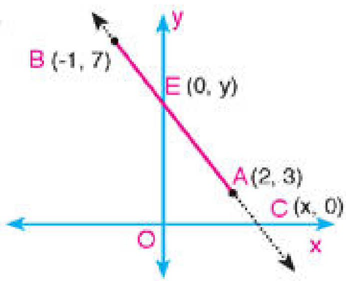
1 **In the figure opposite:** $\overline{AD}$ is a median in $\triangle ABC$, M is the intersection point of the medians, where A(0, 8), B(3, 2), C(−3, 5)



A The coordinates of the point D is (............, ...)

B The coordinates of the point M is (............, ...)

2 **In the figure opposite:** If A (2, 3), B (−1, 7), C and D are points on the coordinate axes



A C divides $\overline{AB}$ .... by the ratio .... : ....

B D divides $\overline{AB}$ .... by the ratio .... : ....

C The coordinates of the point C is (............, ...)

D The coordinates of the point D is (............, ...)

Second : Answer the following questions

3 If A(8, −4), B(−1, 2), find the coordinates of the points which divide $\overrightarrow{AB}$ into 3 equal parts .... , ...

4 If A(3, 1) , B(− 2, 5), find the coordinates of the point C which divides $\overrightarrow{AB}$ internally by the ratio 2 : 3. ....

5 If A(1, 3), B(−4, −2), find the coordinates of the point C where C ∈ $\overrightarrow{AB}$ where 3AC = 2CB

....

6 If A(2, 5) , B(7, −1), find the coordinates of the point C which divides $\overrightarrow{AB}$ externally by the ratio 3 : 2

....

7 If C ∈ $\overrightarrow{BA}$, C ∉ $\overline{AB}$ and A(3, 1), B(4, 2) , AC = 2AB. find the coordinates of the point C. ....

8 If A, B and C are three collinear points where A(2, 5), B(5, 2), C(4, y). Find the ratio by which the point C divides the directed line segment $\overrightarrow{AB}$ showing the type of division, then find the value of y.

.... , ...

4 - 2 Equation of the straight line

You will learn

- Finding the equation of the straight line in terms of a given point and a direction vector.
- Finding the general form of the equation of the straight line
- Finding the equation of the straight line in terms of the intercepted parts from the two axes

Key-terms

- vector direction of Straight line
- Vector equation
- Parametric equation
- Cartisian equation
- General equation

Materials

- Scientific calculator

Think and discuss

You have studied before the general equation of the straight line which is:

$ax + by + c = 0$ where $a, b \neq 0$ and represented it graphically by a straight line.

Which of the following relations represents a straight line?

(A) $3x - 2y = 5$ (B) $y = \sqrt{x} + 1$ (C) $y = 3$

(D) $x - \sqrt{2} = 0$ (E) $y + \frac{1}{x} = 2$ (F) $\frac{x}{3} - \frac{y}{2} = 1$

Notice that the equation $ax + by + c = 0$ where a, b are not equal to zero together is called the general form of the equation of the straight line.

1- If $b = 0$, $a \neq 0$ then: $ax + c = 0$

i.e.: $x = -\frac{c}{a}$ which is an equation of a straight line parallel to the y-axis passes through the point $(-\frac{c}{a}, 0)$

2- If $a = 0$, $b \neq 0$ then: $by + c = 0$

i.e.: $y = -\frac{c}{b}$ is an equation of a straight line parallel to the x-axis and passes through the point $(0, -\frac{c}{b})$

3- If $c = 0$ then: $a x + b y = 0$

which is an equation of a straight line passes through the origin.

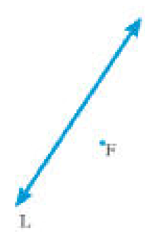
Try to solve

(1) Which of the following straight lines is parallel to the y-axis, which of them is parallel to the x-axis and which of them passes through the origin point, then find the coordinates of the points of intersection with the two axes (if found).

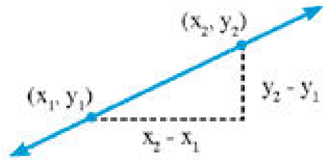
(A) $2x + 3 = 0$ (B) $x + 3y = 0$

(C) $2x + 3y = 12$ (D) $y - 5 = 0$

Critical thinking: If L is a straight line, F is a point in the plane and $F \notin L$. How many straight lines pass through point F and parallel to the straight line L?



Slope of a straight line



You have known before that to determine a straight line completely there are two conditions such as: a given point , and the slope of the line. As you know also the slope of the straight line (m) which passes through the points (x_1, y_1) , (x_2, y_2) equals $\frac{y_2 - y_1}{x_2 - x_1}$

Note (1) **If** $L_1 // L_2$ then $m_1 = m_2$

i.e. if two straight lines are parallel, then they have the same slope and vice versa.

(2) **If** $L_1 \perp L_2$ then $m_1 \times m_2 = -1$

i.e. the product of slopes of two perpendicular straight lines equals –1 and vice versa.

Try to solve

2 Find the slope of the straight line passing through each pair of the following points and show which of these lines are parallel and which are perpendicular:

A (3, 1) , (–2, 5) **B** (4, 0) , (2, –1)

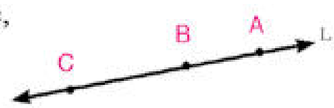
C (7, –1) , (3, –3) **D** (–5, –2) , (–1, 3)

Learn Direction vector of a straight line

Definition **Every non zero vector can be represented by a directed line segment on a straight line is called a direction vector of the straight line L**

If the points A, B, C ∈ L **then** $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ are direction vectors of the straight line.

For example: If $\overrightarrow{u} = (2, 1)$ is a direction vector to a straight line,
then each of the vectors (4, 2), (–2, –1), $(1, \frac{1}{2})$,... is a direction vector to this line.



In general If $\overrightarrow{u} = (a, b)$ is a direction vector to a straight line,

then $K\overrightarrow{u}$ where $K \in R - \{0\}$ is a direction vector to the same straight line. Why?

Try to solve

3 If $\overrightarrow{u} = (2, -3)$ is a direction vector to a straight line, then which of the following is a direction vector to the same straight line?

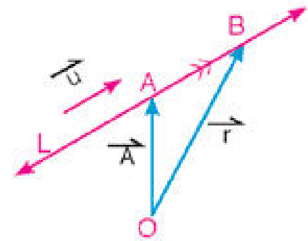
A (–2, 3). **B** (–2, –3).

C (2, 3). **D** (6, –9).

Equaltion of the straight line given a point belonging to it and a direction vector to it

First: Vector form

To determine the equation of the straight line passing through the point A and the vector $\overrightarrow{u}$ is a direction vector to it,



Let point B belong to the straight line L and the vectors.

$\overrightarrow{r}$, $\overrightarrow{A}$ are represented to the two directed line segments $\overrightarrow{OB}$, $\overrightarrow{OA}$ respectively, where O is any point in the plane.

then, there is a number $K \in R - \{0\}$ where $\overrightarrow{AB} = \overrightarrow{r} - \overrightarrow{A} = K\overrightarrow{u}$

Then: $\overrightarrow{r} = \overrightarrow{A} + K\overrightarrow{u}$

This form is called the vector equation of the straight line L which passes through point A and $\overrightarrow{u}$ is its direction vector.

Example

1. Write the vector equation of the straight line which passes through point (2, –3) and its direction vector is (1, 2).

Solution

Let the straight line pass through point A (2, –3) and $\overrightarrow{u} = (1, 2)$

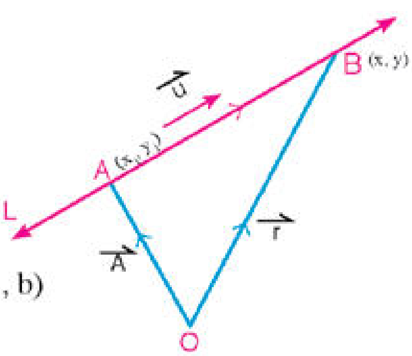
$\because \overrightarrow{r} = \overrightarrow{A} + K\overrightarrow{u}$ vector form of the equation of the straight line.

$\therefore$ The vector equation of the straight line is $\overrightarrow{r} = (2, -3) + K(1, 2)$.

Try to solve

4. Write the vector equation of the straight line which passes through the point (– 4, 3) and the vector (2, 5) is its direction vector.

Second: The parametric equations



The vector equation is $\overrightarrow{r} = \overrightarrow{A} + K\overrightarrow{u}$

If A (x_1, y_1) , B (x, y) are two points in the orthogonal coordinate system, O is the origin point and $\overrightarrow{u} = (a, b)$

then the equation of the straight line is $(x , y) = (x_1 , y_1) + K (a , b)$

Then: $x = x_1 + k a \quad , \quad y = y_1 + kb$

which are the parametric equations of the straight line passing through point (x_1, y_1) and the vector $\overleftarrow{u} = (a, b)$ is its direction vector where. $K \in R - \{0\}$.

Example

(2) Write the parametric equations of the straight line passing through point (4, –3) and its direction vector is (2, 3).

Solution

Let A (4, –3) ∈ L , $\overrightarrow{u}$ = (2, 3)

∴ The vector equation of the line L is (x, y) = (4,– 3) + K(2, 3) vector form

then x = 4 + 2 K , y = –3 + 3 K are the parametric equations

Try to solve

(5) write the parametric equations to the line passing through the point (0, 5) and its direction vector is (–1, 4).

Third : Cartesian Equation

Eliminating K from the parametric equations : $x = x_1 + ka$, $y = y_1 + kb$

We get the equation: $\frac{x - x_1}{a} = \frac{y - y_1}{b}$ i.e.: $\frac{b}{a} = \frac{y - y_1}{x - x_1}$

Put $\frac{b}{a}$ = m (where m in the slope of the line), then the equation becomes in the form: $m = \frac{y - y_1}{x - x_1}$

Example

(3) Find the Cartesian equation of the straight line which passes through the point (3, –4) and its direction vector is (2, –1)

Solution

$m = \frac{-1}{2}$ Slope of the line $m = \frac{b}{a}$

$m = \frac{y - y_1}{x - x_1}$ equation of the line given its slope and a point belonging to it.

$\frac{-1}{2} = \frac{y - (-4)}{x - 3}$ $m = \frac{-1}{2}$, $x_1 = 3$, $y_1 = -4$

$2y + 8 = -x + 3$ Product of means = product of extrems.

$x + 2y + 5 = 0$ general form.

Try to solve

(6) Find the cartesian equation of the straight line passing through the point (3, –4) and makes 45° with the positive direction of the x –axis.

Critical thinking: Find the vector equation and cartesian equation to the straight line passing through the point (x_1, y_1) and its direction vector $\overrightarrow{u}$ = (a,b) in each of the following cases:

First: if the line is parallel to the y–axis.

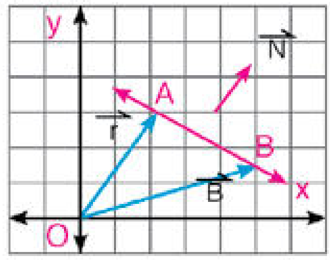
Second: if the line is parallel to the x–axis.

Third: if the line passes through the origin.

Add to your knowledge

Direction vector to the line passing through the origin point and the point (x_1, y_1) is $\overrightarrow{u} = (x_1, y_1)$ and its slope is $\frac{y_1}{x_1}$

Learn The perependicular direction vector of a straight line



If $\vec{u} = (a, b)$ is a direction vector of a straight line **then** which of the family vectors are in the form K (b, –a) where $K \in R - \{0\}$ is a perpendicular direction vector to the vector $\vec{u}$.

Conversely if $\vec{N} = (a, b)$ is perpendicular to the straight line , then which of the family vectors are in the form K (b, –a) where $K \in R - \{0\}$ is a direction vector of the line.

For example: If $\vec{u} = (3, 2)$ is a direction vector of the straight line , then its perpendicular direction vector is (–2, 3), (2, –3), (–4, 6), ...

Try to solve

7) If $\vec{u} = (\frac{1}{2}, 1)$ is a direction vector to the line, then all the following vectors are perpendicular to the line except the vector:

A $(1, -\frac{1}{2})$ B (2, –1) C $(-1, -\frac{1}{2})$ D (4, –2)

Example

4) If the straight line passing through the point A (– 3, 5) and the vector (– 1, 2) is perpendicular to it, then find :

A The vector equation of the straight line.

B The Cartesian equation of the straight line.

Solution

A ∵ The line passes through the point A (–3, 5) and is perpendicular to the vector (–1, 2).

∴ The direction vector of the line is $\vec{u} = (2, 1)$

∵ The vector equation of the line is: $\vec{r} = \vec{A} + K\vec{u}$

∴ $\vec{r} = (-3, 5) + K(2, 1)$

B ∵ The equation of the line whose slope "m" and passes through the point (x_1, y_1) is:
$m = \frac{y - y_1}{x - x_1}$

$\therefore \frac{1}{2} = \frac{y - 5}{x + 3}$ $\therefore x + 3 = 2y - 10$

Then $x - 2y + 13 = 0$ is the Cartesian equation of the line.

Think: Find the Cartesian equation of the same straight line by eliminating k from the two parametric equations.

Try to solve

8) If the straight line passes through the point A (2, – 3) is perpendicular to the vector $\vec{u} = (-1, 2)$ then find:

A The vector equation of the line. B The parametric equations of the line.

C The Cartesian equation of the line.

Learn: The Equation of the straight line in terms of the two intercepted parts from the two axes

We know that the equation of the line whose slope (m) and intersects a part from the y-axis of length b is $y = mx + c$

From the figure opposite , we get

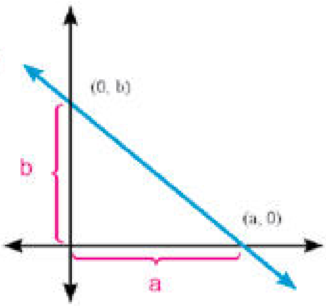
The slope of the line passing through the points (a, 0), (0, b) is: $m = \frac{-b}{a}$ (why?)

$\frac{y - y_1}{x - x_1} = m$ — equation of a line given the slope and a point

$\frac{y - 0}{x - a} = -\frac{b}{a}$ — substituting the coordinates of the intersection points

$ay = -bx + ab$ — product of means = product of extrems

$bx + ay = ab$ — Divide both sides by a b



$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

Example

5) Find the intercepted parts from the two axes by the straight line whose equation is : $3x + 4y - 12 = 0$

Solution

Put the equation in the form $\frac{x}{a} + \frac{y}{b} = 1$

$\therefore \frac{x}{4} + \frac{y}{3} = 1$ (Why?)

$\therefore$ The lengths of the intercepted parts from the x-axis and from the y-axis are 4 and 3 respectively

Try to solve

9) Find the intercepted parts from the two axes by the straight line whose equation is: $5x - 3y = 15$

Check your understanding

Find the general equation of the straight line in the following cases:

A) It intersects the two axes at the points (3, 0), (0, – 4).

B) It passes through the point (3, 1) and is parallel to the straight line whose equation $2x - 3y + 7 = 0$

C) It passes through the point (0, – 1) and its direction vector is (2, -3)

Exercise (4 - 2)

First: Complete each of the following

1. If the straight line passes through the points (3, 0) , (0, 2) is parallel to the straight line whose equation $y = ax - 3$, then a equals ....

2. The vector equation of the straight line passes through the point (3, 5) and parallel to the x-axis is ....

3. The cartesian equation of the straight line passes through the point (–2, 7) and parallel to the y-axis is ....

4. The vector equation of the straight line passes through the origin point and the point (1, 2) is ....

5. The equation of the straight line which makes 45° with the positive direction of the x-axis and cuts 5 units from the positive part of the y-axis is ....

6. The cartesian equation of the striaight line which cuts the positive parts of the x-axis and the y-axis with magnitudes 2, 3 respectively is ....

7. Area of the triangle enclosed by the x-axis and the y - axis and the striaight line $2x + 3y = 6$ equals ....

Second : Answer the following questions

8. If A (3, –2), B (5, 6), C (1, –2) Find the slope of each of the following straight lines: $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$ ...., ..., ...

9. If the equations of the straight lines L_1 , L_2 are $2x - 3y + a = 0$ and $3x + by - 6 = 0$ respectively , Find:
   - **A** The slope of the straight line L_1 ....
   - **B** The value of b which makes the two lines L_1, L_2 parallel equals ....
   - **C** The value of b which makes the two lines L_1, L_2 perpendicular equals..........
   - **D** If the straight line passes through the point (1, 3), then find the value of a. ....

10. If the straight line whose equation $ax - 4y + 5 = 0$ makes an angle whose tangent is 0.75 with the positive direction of the x-axis, then find the value of a . ...

11. Find the vector equation of the straight line whose slope $\frac{1}{3}$ and passes through the point (2, −1). ....

12. Find the parametric equations of the straifht line which makes 45°with the positive direction of the x-axis and passes through the point (3, −5). ....

13. Find the vector equation of the straight line which passes through the points (2, −3), (5, 1)

....

14. Find the general equation of the straight line which passes through the points (5, 0), (0, −7)

....

15. If A(0, 2), B(2, 1), C(−2, 3) are three points in the plane, find the vector equation of the straight line $\overleftrightarrow{AB}$, then prove that the points A, B and C are collinear .

....

16. If A(5, −6), B(3, 7), C(1, −3), find the equation of the straight line which passes through the point A and bisects $\overline{BC}$.

....

17. Find the cartesian equation of the straight line which passing through the point (3, −5) and parallel to the straight line whose equation is $x + 2y - 7 = 0$

....

18. Find the vector equation of the straight line which passing through the point (5, 7) and is perpendicular to the straight line whose equation is $\vec{r} = (3, 0) + K(4, 3)$

....

19. **Geometry:** $\overline{AB}$ is a diameter in the circle M, if B(− 7 , 11) , M(− 2 , 3), find the equation of the tangent to the circle at the point A.

....

20. **Geometry:** If the straight line whose equation $3x + 4y - 12 = 0$ intersects the x-axis and the y-axis at A and B respectively. Find:

   A. Area of △ OAB where O is the origin point. ....

   B. The equation of the straight line which is perpendicular to $\overline{AB}$ and passes through its midpoint. ....

4 - 3 Measure of the angle between two straight lines

You will learn

- Finding the measure of the acute angle between two straight lines.

Key-terms

- Anagle between two straight lines

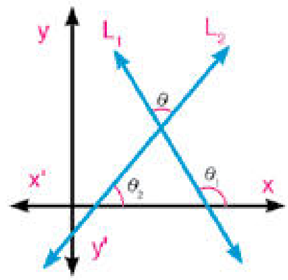
Materials

- Scientific calculator

Learn: Measure of the acute angle between two straight lines

If θ is the measure of the acute angle between the two straight lines L_1, L_2 whose slopes m_1, m_2 respectively then:

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \text{ where } m_1 m_2 \neq -1$$



Example

1. Find the measure of the acute angle between the two straight lines whose equations are

$3x - 4y - 11 = 0$, $x + 7y + 5 = 0$

Solution

We find the slope of each straight line:

$m_1 = \frac{-3}{-4} = \frac{3}{4}$ slope of the first line

$m_2 = \frac{-1}{7}$ slope of the second line

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ Formula

$\tan\theta = \left|\frac{\frac{3}{4} - (-\frac{1}{7})}{1 + \frac{3}{4}(-\frac{1}{7})}\right|$ substituting the values of m_1, m_2

$= \left|\frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{28}}\right| = \left|\frac{\frac{21+4}{28}}{\frac{28-3}{28}}\right| = 1$

$\theta = 45°$

Remember

Slope of the straight line whose equation $ax + by + c = 0$ equals $\frac{-a}{b}$

Oral exercises:

Mention the relation between the two straight lines L_1, L_2 in the following cases:

A. If the tangent of the angle between them equals zero.

B. If the tangent of the angle between them is undefined.

C. If the slope of the first is m_1 and the slope of the second is m_2, mention the relation between m_1, m_2 in A and B.

Try to solve

1. Find the measure of the acute angle between each of the following pairs of straight lines:

   A. $\vec{r} = (0, -2) + K(3, -1)$, $\vec{r} = (0, 5) + K(1, 2)$.

   B. $x + 2y + 3 = 0$, $x - 3y + 1 = 0$

   C. $2y = 3$, $2x + y = 4$

Example

2. **Geometry:** ABC is a triangle in which A (0, 5), B (2, –1) and C (6, 3). Prove that the triangle is isosceles, then find the measure of angle A.

Solution

The distance between two points $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ formula

$AB = \sqrt{(0-2)^2 + (5-(-1))^2} = 2\sqrt{10}$

$AC = \sqrt{(0-6)^2 + (5-3)^2} = 2\sqrt{10}$

$BC = \sqrt{(2-6)^2 + (-1-3)^2} = 4\sqrt{2}$

The triangle is isosceles because $AB = AC$

We notice that $(BC)^2 < (AB)^2 + (AC)^2$

i.e. $\angle A$ is acute

$m_1 = \frac{5-(-1)}{0-2} = -3$ Slope of $\overleftrightarrow{AB}$

$m_2 = \frac{5-3}{0-6} = -\frac{1}{3}$ Slope of $\overleftrightarrow{AC}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ Formula

$\tan A = \left|\frac{-3-(-\frac{1}{3})}{1+(-3)(-\frac{1}{3})}\right| = \frac{4}{3}$ Substituting the values of m_1, m_2

$m(\angle A) = 53°\ 7'\ 49''$ Use the calculator

Note

When using the formula of the angle between two lines to find the measure of the interior angle of a triangle, you have to find first the type of this angle (acute- right - obtuse)

Check your understanding

1. Find the measure of the acute angle included between the two straight lines $\vec{r} = (2, 0) + K(-2, 1)$, $\vec{r} = (-3, 1) + K(6, 3)$.

2. Find the measure of the acute angle included between the straight line $x - 2y + 3 = 0$ and the straight line passing through the points (4, –1), (2, 1).

3. A B C is a triangle in which A (0, 2), B(3, 1), C(–2, –1). Find the measure of angle A

Exercise (4 - 3)

First : Complete each of the following

1. Measure of the angle between the straight lines whose slopes 2 , $-\frac{1}{2}$ equals ....

2. Measure of the angle between the straight lines whose equation x = 3 , y = 4 equals ....

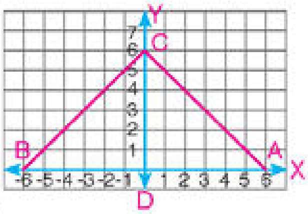
3. Measure of the acute angle between the straight line whose equation $\vec{r} = (2, 2) + t(1,1)$ and the straight line whose equation x = 0 equals ....

4. If the two straight lines whose equations $ax + 3y - 7 = 0$ and $2x - 3y + 5 = 0$ are parallel, then a equals ....

5. If the two straight lines whose equations $ax + 7y - 9 = 0$ and $7x - 2y + 12 = 0$ are perpendicular then a equals ....

Second : Activity

The figure opposite shows a triangular piece of land, its vertices are A (6, 0) , B (-6, 0) , C (0, 6). Complete each of the following:



6. Measure of the acute angle between $\overleftrightarrow{AC}$ and the x-axis equals ....

7. Measure of the angle between the straight lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ equals ....

8. The vector equation of the straight line $\overleftrightarrow{AC}$ is ....

9. The vector equation of the straight line $\overleftrightarrow{BC}$ is ....

10. The cartesian equation of the straight line which passes through the point C and parallel to $\overleftrightarrow{AB}$ is ....

11. Area of the triangle ABC equals ....

Third : Choose the correct answer from the given answers

12. Measure of the acute angle between the straight line which passes through the points (0, 1) , (-1, 0) and the positive direction of the x-axis equals: ....

    A zero° B 45° C 60° D 90°

13. Measure of the acute angle between the straight line whose equation $\vec{r} = (0, 3) + t(1, 1)$ and the straight line whose equation x = 0 equals: ....

    A 30° B 45° C 60° D 90°

14. Measure of the acute angle between the straight lines whose equation $\sqrt{3}\ x - y = 4$ and $y = 3$ equals ....

   (A) $30°$ (B) $45°$ (C) $60°$ (D) $90°$

15. The straight line perperpendicular to the straight line whose equation $\overrightarrow{r} = (0, 5) + t(\sqrt{3}\ , 1)$ and makes an angle of measure .... with the positive direction of the x-axis

   (A) $30°$ (B) $60°$ (C) $120°$ (D) $150°$

Fourth : Answer the following questions

16. Find the measure of the acute angle between each of the following pairs of the straight lines whose equations are:

   (A) $\overrightarrow{r} = (5, 0)\ ,\ x - y + 4 = 0$ ....

   (B) $\overrightarrow{r} = (0, 1) + t(1, 1)\ ,\ 2x - y - 3 = 0$ ....

   (C) $y - \sqrt{3}\ x - 5 = 0\ ,\ x - \sqrt{3}\ y - 6 = 0$ ....

17. If θ is the measure of the acute angle between the straight lines whose equations $x - 6y + 6 = 0\ ,\ ax - 2y + 4 = 0$ and $\tan\theta = \frac{3}{4}$, then find the value of a. ....

18. If L_1: $ax - 3y + 7 = 0\ ,\ L_2 : 4x + 6y - 5 = 0\ ,\ L_3 : \frac{x}{3} - \frac{y}{2} = 3$, then find the value of a which makes:

   (A) $L_1 \mathbin{//} L_3$ ....

   (B) $L_1 \perp L_2$ ....

19. If the measure of the acute angle between the straight lines whose equations $x + ky - 8 = 0\ ,\ 2x - y - 5 = 0$ equals $\frac{\pi}{4}$, then find the value of K. ....

4 - 4 The length of the perpendicular from a point to a straight line

You will learn

- Finding the length of a perpendicular from a given point to a straight line.

Key-terms

- Perpendicular
- Straight Line

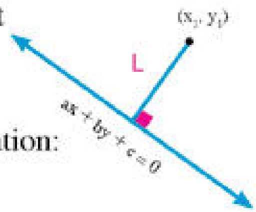
Materials

- Scientific calculator

Learn

Finding the length of the perpendicular from a point to a straight line

If the point (x_1, y_1) does not belong to the straight line whose equation is $ax + by + c = 0$ then length of perpendicular (L) drawn from this point to the straight line is determined by the relation:



$$L = \frac{|a x_1 + b y_1 + c|}{\sqrt{a^2 + b^2}}$$

Example

1. Find the length of the perpendicular from the point (4, –5) to the straight line $\overrightarrow{r} = (0, 2) + K(4, 3)$.

Solution

Let $(x, y) = (0, 2) + K (4, 3)$

$\therefore\ x = 4K \ , \quad y = 2 + 3K$ (parametric equations to the vector equation)

$\frac{x}{4} = \frac{y - 2}{3}$ by eliminating K

$3x = 4y - 8$ Product of means = product of exterems

$3x - 4y + 8 = 0$ Cartesian equation

$L = \frac{|a x_1 + b y_1 + c|}{\sqrt{a^2 + b^2}}$ Formula

Substituting: $a = 3$, $b = -4$, $c = 8$, $x_1 = 4$, $y_1 = -5$

$$L = \frac{|3 \times 4 - 4 \times - 5 + 8|}{\sqrt{3^2 + 4^2}}$$

$$= \frac{|12 + 20 + 8|}{\sqrt{9 + 16}} = \frac{|40|}{\sqrt{25}} = \frac{40}{5} = 8 \text{ unit of length}$$

Try to solve

1. Find the length of a perpendicular drawn from the point (2, –5) to the straight line:

$\overrightarrow{r} = (-1, 0) + K(12, 5)$.

2 **Oral exercises:** Write the length of the perpendicular from the point A to the straight line M in each of the following cases:

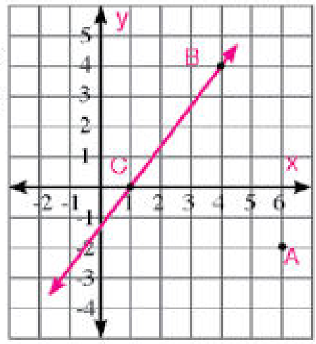
A A (0 , 0) , M : ax + by + c = 0

B A (x_1, y_1) , M : y = 0

C A (x_1, y_1) , M : x = 0

Example

2 **In the figure opposite**: Find the length of the perpendicular drawn from the point A (6, –2) to the straight line passing through the points B (4, 4), C (1, 0), then find the area of the triangle ABC.



Solution

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Formula

$\because$ C (1, 0) , B (4, 4)

$\therefore m = \frac{4 - 0}{4 - 1} = \frac{4}{3}$ Substituting the point (4, 4), (1, 0)

$m = \frac{y - y_1}{x - x_1}$ equation of the line given the slope and a point belonging to it

$\frac{4}{3} = \frac{y - 0}{x - 1}$ substituting $m = \frac{4}{3}$

Then: $4x - 3y - 4 = 0$ Cartesian equation

$L = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ formula

length of the perpendicular from the point A (6, –2) to the line : 4x – 3y – 4 = 0

is: $L = \frac{|4 \times 6 - 3 \times -2 - 4|}{\sqrt{4^2 + 3^2}} = \frac{|24 + 6 - 4|}{\sqrt{25}} = \frac{26}{5} = 5\frac{1}{5}$ unit of length

Consider $\overline{BC}$ is the base of the triangle ABC

$\because BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ formula

$= \sqrt{(4 - 1)^2 + (4 - 0)^2} = 5$ units substituting the points (4, 4), (1, 0)

Area of the triangle ABC = $\frac{1}{2}$ length of base × height formula

$= \frac{1}{2} \times 5 \times \frac{26}{5} = 13$ square unit

Try to solve

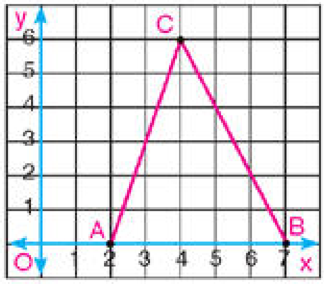
3 Find the length of the perpendicular drawn from the point (5, 2) to the straight line which passes through the points (0, –3), (4, 0)

Check your understanding

1 **Roads** Two adjacent roads, the path of the first road is represented by the equation 3x – 4y – 7 = 0 and the path of the second road is represented by the equation 3x – 4y + 11 = 0.
Prove that the two roads are parallel, then find the shortest distance between them.

Exercise (4 - 4)

First : Complete each of the following:



1 **The figure opposite shows** karim's house A (2, 0) and the school B (7, 0) and the mosque C (4, 6): Complete each of the following:

A The equation of $\overleftrightarrow{AB}$ is ....

B The length of $\overline{AB}$ equals ....

C Shortest distacnce between the Mosque C and the road from the house to the school equals ....

D Measure of the acute angle between the straight lines $\overleftrightarrow{AC}$ and Y = 0 equals ....

E Area of (△ ABC) equals ....

Second : Multiple choice

2 Length of perpendicular from the point (– 3 , 5) on the y-axis equals ....

A 2 **B** 3 **C** 5 **D** 8

3 The distance between the straight lines whose equations y – 3 = 0 , y + 2 = 0 equals ....

A 1 **B** 2 **C** 3 **D** 5

4 Length of perpendicular from the point (1 ,1) to the straight line whose equation x + y = 0 equals ....

A 1 **B** $\sqrt{2}$ **C** 2 **D** $2\sqrt{2}$

5 If the length of perpendicular drawn fron (3 , 1) to the straight line whose equation 3x – 4y + c = 0 equals 2 unit of length, then C equals ....

A Zero **B** 3 **C** 5 **D** 7

6 Find the length of the perpendicular drawn from **A** to the straight line L in exercises **A** – **D**

A A (0 , 0) , L: $\vec{r}$ = (0 , 5) + t (3 , 4) ....

B A (2 , – 4) , L: 12x + 5y – 43 = 0 ....

C $A(5, 2)$, $L: 8x + 15y - 19 = 0$ ....

D $A(-2, -1)$, $L: \overrightarrow{r} = (0, -7) + t(1, 2)$ ....

7 Find the length of the radius of the circle whose centre $(-2, 5)$, and touches the straight line whose equation $3x + 4y + 1 = 0$

....

8 Find the distance between $(1, 5)$ and the straight line passing through the points $(5, -3)$, $(1, 0)$

....

9 Prove that the straight lines whose equations $3x - 4y - 12 = 0$ and $6x - 8y + 21 = 0$ are parallel , then find the distance between them.

....

....

10 If $A(4, 3)$, $B(-2, 5)$, $C(-1, -2)$ are vertices of the triangle ABC , Draw $\overline{BD} \perp \overline{AC}$.

A Prove that $\triangle ABC$ is an isosceles triangle

B Find the equation of $\overline{BD}$

C Find the length of $\overline{BD}$

....

....

11 ABCD is a parallelogram, if $A(-3, 2)$, $B(2, 3)$, $C(5, 7)$. Find the coordinates of the vertex D, then find the area of the parallelogram .

....

....

12 **Geometry:** Acircle of centre the origin point in which two chords whose equations $4x - 3y + 10 = 0$, $5x - 12y + 26 = 0$. Prove that the two chords are equal in length.

....

General Exercises

First : Complete each of the following:

1. The slope of the line whose equation $2x - 3y + 5 = 0$ equals ....

2. The direction vector perpendicular to the line whose equation $\overrightarrow{r} = (2, -1) + t(3, -5)$ is ..

3. The equation of the line which makes an angle of measure 135° with the positive direction of the x–axis and passes through the point (4, 0) is ....

4. The line whose equation $3x + 4y - 12 = 0$ intersects the two axes at the points ....

5. Measure of the acute angle between the line passes through (0, 2), (–2, 0) and the line whose equation $y = 0$ equals ....

Second; Multiple choice:

6. Measure of the acute angle between the two lines whose equations: $x - 3y + 5 = 0$, $x + 2y - 7 = 0$ equals ....
   - **A** 15°
   - **B** 30°
   - **C** 45°
   - **D** 60°

7. The equation of the line which passes through the point (2, –3) and parallel to the x–axis is: ....
   - **A** $x + 3 = 0$
   - **B** $y + 3 = 0$
   - **C** $x - 2 = 0$
   - **D** $y - 3 = 0$

8. Length of perpendicular drawn from the origin to the line whose equation $3x - 4y - 15 = 0$ equals ....
   - **A** 3
   - **B** 4
   - **C** 5
   - **D** 15

9. All the following equations represent the equation of the line which passes through the points (3, 0), (0, 2) except the equation: ....
   - **A** $\overrightarrow{r} = (3, 0) + t\,(3, -2)$
   - **B** $\overrightarrow{r} = (0, 2) + t\,(3, -2)$
   - **C** $\overrightarrow{r} = (3, 0) + t\,(2, 3)$
   - **D** $\overrightarrow{r} = (0, 2) + t\,(-6, 4)$

10. Length of perpendicular drawn from the point (0, –5) to the straight line whose equation $x + 7 = 0$ equals ....
    - **A** 2
    - **B** 5
    - **C** 7
    - **D** 12

Third:

11. If A(5, 2), B(–3, 1), find the ratio by which $\overrightarrow{AB}$ is divided by the x–axis, showing the type of division. ....

12. Find the length of perpendicular drawn from A(2, 5) to the line whose eqaution $\overrightarrow{r} = (1, -2) + t\,(3, 4)$. ....

13. Prove that the triangle ABC is right at B where A(5, 2), B(2, –2), C(–2, 1) then calculate its area ....

Unit 5

Trigonometry

Unit objectives

By the end of this unit, the student should be able to:

- Deduce the basic relations among trigonometric functions .
- Prove that the validity of the identities on trigonometric functions .
- Solve simple trigonometric equations in the general form in the interval $[0, 2\pi[$
- Recognize the general solution for the trigonometric equation.
- Solve the right angled triangle.
- Solve applications that involve angles of elevation and depression.
- Recognize the circular sector and how to find its area.
- Recognize the circular segment and how to find its area.
- Find the area of the triangle, the area of the quadrilateral and the area of the regular polygon.
- Solve miscellaneous exercises on trigonometry.
- Use the information technology to recognize the various applications of the basic concepts of trigonometry.
- Model some physical and biological phenomena which are represented by trigonometric functions.
- Use activities for computer programs

Key - Terms

- Trigonometric identities
- Trigonometric equation
- Angle of elevation
- Angle of depression
- Circular sector
- Circular Segment

Lessons of the Unit

Lesson (5 - 1): Trigonometric Identities.
Lesson (5 - 2): Solving Trigonometric Equations.
Lesson (5 - 3): Solving the Right Angled Triangle.
Lesson (5 - 4): Angles of Elevation and Angles of Depression.
Lesson (5 - 5): Circular Sector
Lesson (5 - 6): Circular Segment.
Lesson (5 - 7): Areas.

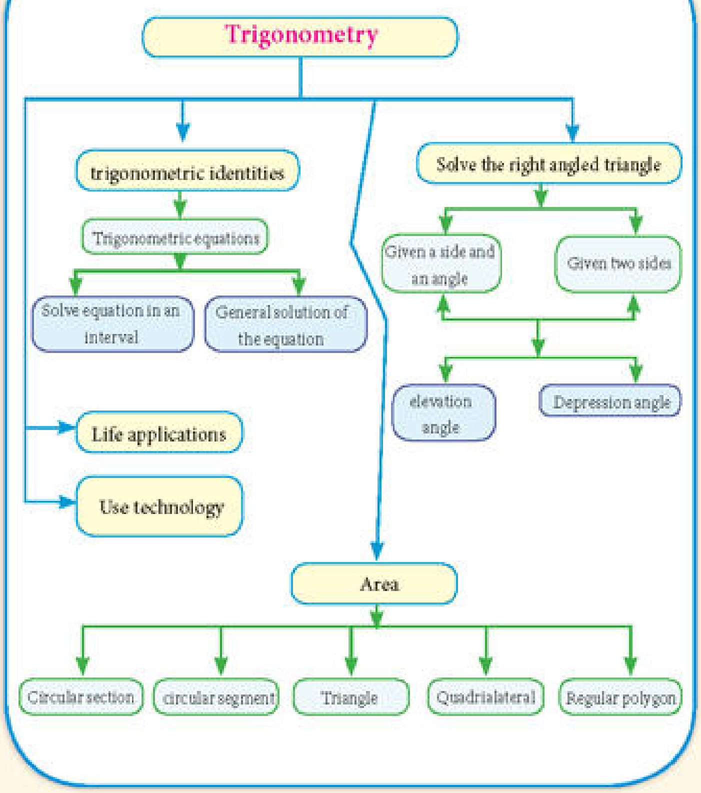
Materials

Scientific calculator – squared paper – computer connected with internet – Graphic programs

Brief History

Trigonometry is a branch of mathematics. It is clear from its name that it deals with the special calculations of angles and sides of the triangle. Some historians say that the mathematician Nosir Aldin Altusi is the first to separate trigonometry and astronomy. Historians say that Talis (600 BC) used trigonometry when he could measure the height of the pyramid by comparing between the length of the shadow of a vertical stick and the length of his shadow at the same moment.. Trigonometry has had a share of the interests of the Arabs. The terminology (the tangent) was described by the Arab scientist Abu al waffa albozgany in the tenth AD century, and this terminology was taken from the shadows of objects which formed as a result of the path of light emitted by the sun in straight lines. Arabs have several additions in the plane and spherical (according to the surface of the sphere) trigonometry, Westerners took from Arabs important information and they added to them a lot until the trigonometry became included in several mathematical researches. Its applications became in various scientific and practical aspects and has contributed in pushing forward the wheel of progress and civilization.

Chart of the unit



5 - 1 Trigonometric Identities

You will learn

- Concept of trigonometric identity.
- Simplify the trigonometric expressions.
- Prove the validity of trigonometric identity.

Key-terms

- Equation
- Identity

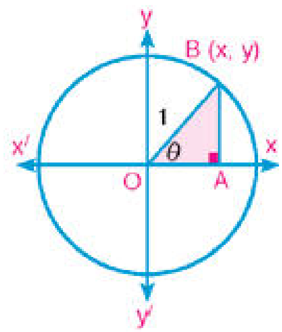
Materials

- Scientific calculator

Basic Relations Among Trigonometric Functions

Think and discuss

You have studied in the first term some properties of trigonometric functions and its graphic. In this unit you will use the trigonometric identities, to simplify the expressions and solve trigonometric equation.



You have studied the unit circle, and known that the directed angle $\angle AOB$ is in the standard position and its terminal side $\overrightarrow{OB}$ intersects the unit circle at the point B(x, y) where m($\angle$AOB) = θ, B(cosθ, sinθ). Is it possible to deduce some basic relations among the trigonometric functions?

Learn

Trigonometric Identities and Equations

The identity: is an equality, and it is true for all real values of the variable , which each of the two sides of the equality is known.

For example: $\sin(\frac{\pi}{2} - \theta) = \cos\theta$ is a true inequality for all real values of θ.

The equation: is a true equality for some real numbers which satisfy this equality, and it is not true for some others which are not satisfy it.

For example: $\sin\theta = \frac{1}{2}$, $\theta \in [0, 2\pi[$

We get that: the value of θ which satisfy this equation and belong to the interval $[0, 2\pi[$ are $\frac{\pi}{6}, \frac{5\pi}{6}$ only.

Try to solve

1. Which of the following relations represents an equation and which of them represents an identity.

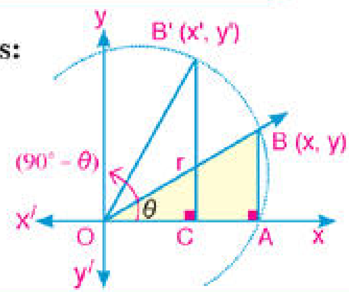
   A) $\cos\theta = \frac{\sqrt{3}}{2}$
   B) $\tan(\frac{3\pi}{2} + \theta) = -\cot\theta$
   C) $\cot\theta = -\frac{1}{\sqrt{3}}$
   D) $\sin(\pi - \theta) = \sin\theta$

Basic Trigonometric Identities

1- You have studied the basic trigonometric functions and their reciprocals and known that:

$$\sin\theta = \frac{1}{\csc\theta}, \quad \cos\theta = \frac{1}{\sec\theta}, \quad \tan\theta = \frac{1}{\cot\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

2- Trigonometric function of two complementary angles:



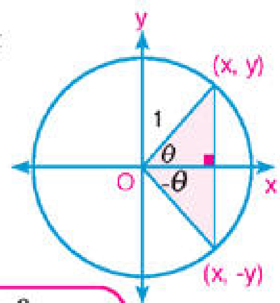
$$\sin(\frac{\pi}{2} - \theta) = \cos\theta, \quad \cos(\frac{\pi}{2} - \theta) = \sin\theta$$
$$\tan(\frac{\pi}{2} - \theta) = \cot\theta, \quad \csc(\frac{\pi}{2} - \theta) = \sec\theta$$
$$\sec(\frac{\pi}{2} - \theta) = \csc\theta, \quad \cot(\frac{\pi}{2} - \theta) = \tan\theta$$

From congruency of the triangles: OAB, B'CO **We find that:** $y' = x$, $x' = y$

3- Identity the two angles $\theta, -\theta$:

We notice that from the figure opposite:

- $x = \cos\theta$, $x = \cos(-\theta)$
- $y = \sin\theta$, $-y = \sin(-\theta)$



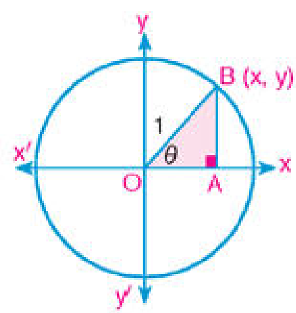
Add to your Knowledge

Identities of angles $\theta, -\theta$ **are called identities of even and odd functions, you will study in the second form secondary.**

thus:

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta$$
$$\csc(-\theta) = -\csc\theta, \quad \sec(-\theta) = \sec\theta$$
$$\tan(-\theta) = -\tan\theta, \quad \cot(-\theta) = -\cot\theta$$

4- Pythagorian identities:



We know that from the unit circle:

$x^2 + y^2 = 1$ ① **Substituting** $x = \cos\theta$, $y = \sin\theta$

then: $\cos^2\theta + \sin^2\theta = 1$

Divide both sides of the relation ① by x^2 then:

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{1}{x^2}$$

i.e.: $1 + \tan^2\theta = \sec^2\theta$

Divide both sides of the relation ① by y^2 then:

$$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{1}{y^2}$$

i.e.: $1 + \cot^2\theta = \csc^2\theta$

5- **Expressing $\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$, in terms of $\sin\theta$, $\cos\theta$:**

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} \quad , \quad \therefore \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Simplifying the trigonometric expressions :

We mean by simplifying the trigonometric expressions is to put it in simplest form, by using the basic trigonometric identities.

Example

Notice that

$\sin\theta \times \sin\theta = (\sin\theta)^2 = \sin^2\theta$

1 **Write the simplest form:** $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$

Solution

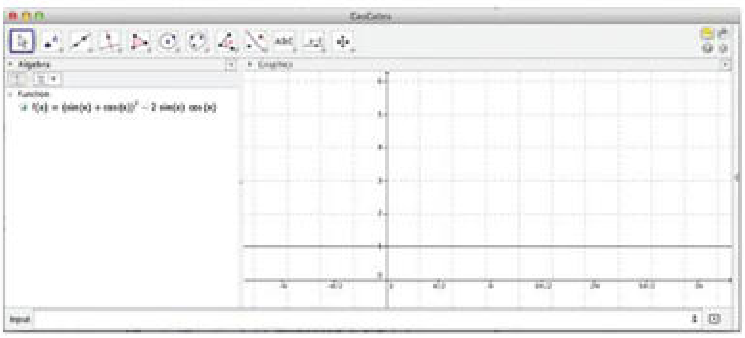
A The expression $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$

$= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 2\sin\theta\cos\theta$ **remove the parentheses**

$= \sin^2\theta + \cos^2\theta$ **Simplify**

$= 1$ **apply phythagorian identity:**

You can check the result by using one of the graphic programs shown below:



2 Write in the simplest form: $\frac{1 + \tan^2\theta}{1 + \cot^2\theta}$

Solution

the expression : $\frac{1 + \tan^2\theta}{1 + \cot^2\theta}$

$= \frac{\sec^2\theta}{\csc^2\theta}$ **apply pythagorian identity**

$= \frac{1}{\cos^2\theta} \div \frac{1}{\sin^2\theta}$

$= \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

Try to solve

2 Put each of the following expressions in the simplest form, then check the result:

A $\frac{1}{\cos^2\theta} - \frac{1}{\cot^2\theta}$ **B** $\cos(\frac{\pi}{2} - \theta)\sec(\frac{\pi}{2} - \theta)$ **C** $\frac{\sin(\frac{\pi}{2} - \theta)}{\cos(2\pi - \theta)}$

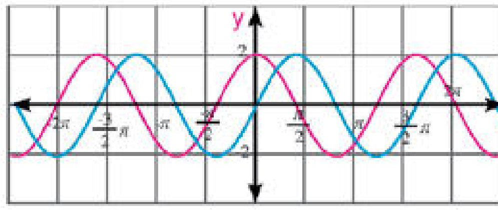
Trigonometric Identities

To prove the validity of the trigonometric identity, we prove that the two functions determining its both sides are equal

To verify that the statement : $\cos 2\theta = 2\sin\theta\cos\theta$ is not true

We draw the graph of each of the two functions:

$f(x) = \cos 2\theta$, $g(x) = 2\sin\theta\cos\theta$

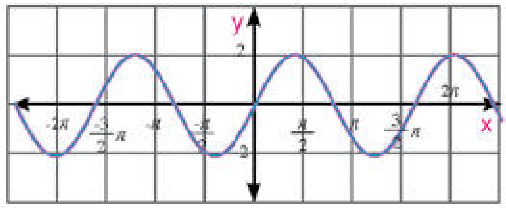


Refer to the graph opposite

We get two functions that are not congruent:

i.e. $f(x) \neq g(x)$, thus this relation is not identity.

We can check algebraically by putting $\theta = 0$, then $f(0) = 1$, $g(0) = 0$ then the two functions are not equal.



While in the equality : $\sin 2\theta = 2\sin\theta\cos\theta$

by putting $f(x) = \sin 2\theta$, $g(x) = 2\sin\theta\cos\theta$

We find that the graphic representation to the figure is congruent to the curve of the two functions: **i.e.** $f(x) = g(x)$

Thus this equality is identity.

Remember

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta = 1 - \sin^2\theta$

Example

3 **Prove the validity of the identity:** $\frac{\cos^2\theta}{1 - \sin\theta} = 1 + \sin\theta$

Solution

L.H.S $= \frac{\cos^2\theta}{1 - \sin\theta} = \frac{1 - \sin^2\theta}{1 - \sin\theta}$

$= \frac{(1 + \sin\theta)(1 - \sin\theta)}{1 - \sin\theta} = 1 + \sin\theta =$ R.H.S

Example

4 **Prove the validity of the identity :** $\tan\theta + \cot\theta = \sec\theta\csc\theta$

Solution

L.H.S $= \tan\theta + \cot\theta$

$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$

$= \frac{1}{\cos\theta\sin\theta}$

$= \sec\theta\csc\theta =$ R.H.S

Try to solve

3 **Prove the validity of the identity:** $\frac{(1-\sin^2\theta)(1-\cos^2\theta)}{\tan^2\theta} = \cos^4\theta$

Example

5 **Prove the validity of the identity:** $\frac{1-\cot^2\theta}{1+\cot^2\theta} = 2\sin^2\theta - 1$

Solution

L.H.S $= \frac{1-\cot^2\theta}{1+\cot^2\theta}$

$= \frac{1-\cot^2\theta}{\csc^2\theta} = \frac{1-\frac{\cos^2\theta}{\sin^2\theta}}{\frac{1}{\sin^2\theta}}$ change to sin θ, cos θ

$= \frac{1-\frac{\cos^2\theta}{\sin^2\theta}}{\frac{1}{\sin^2\theta}} \times \frac{\sin^2\theta}{\sin^2\theta} = \sin^2\theta - \cos^2\theta$

$= \sin^2\theta - (1 - \sin^2\theta)$

$= 2\sin^2\theta - 1 =$ R.H.S

Think: Are there other solutions for the example?

Try to solve

4 Prove the validity of each of the following identities:

A $\frac{1+\cot^2\theta}{1+\tan^2\theta} = \cot^2\theta$ B $\frac{\tan\theta + \cot\theta}{\sec\theta\csc\theta} = 1$

C $(\sec\theta - \tan^2\theta) = \frac{1-\sin\theta}{1+\sin\theta}$

Check your understanding

1 **Discover the wrong answer:**

$\sin^2\theta + \cos^2\theta$ equals:

A 1 B $2\cos^2\theta - 1$ C $1 - 2\sin^2\theta$ D $1 + 2\sin\theta\cos\theta$

2 **Prove the validity of the following identities:**

A $\frac{\sin\theta\cos\theta}{\tan\theta} + \frac{\tan\theta}{\sec\theta\csc\theta} = 1$ B $\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} + \frac{\sin^3\theta - \cos^3\theta}{\sin\theta - \cos\theta} = 2$

Exercise (5 - 1)

First: Multiple choice

1. $\frac{\tan\theta \cot\theta}{\sec\theta}$ equals: .... in the simplest form:

   (A) $\sin\theta$ (B) $\cos\theta$ (C) $\sec\theta$ (D) $\csc\theta$

2. $\sin\theta \cos\theta \tan\theta$ equals .... in the simplest form

   (A) $\sin^2\theta$ (B) $\cos^2\theta$ (C) $\tan^2\theta$ (D) $1 - \sin^2\theta$

3. $\sin(90^\circ - \theta)\csc(90^\circ - \theta)$ equals : .... in the simplest form

   (A) 1 (B) $\sin^2\theta$ (C) $\cos^2\theta$ (D) $\sin\theta\cos\theta$

4. $\frac{1-\cos^2\beta}{\sin^2\beta - 1}$ equals : .... in the simplest form

   (A) $-\tan^2\beta$ (B) $-\cot^2\beta$ (C) $\tan^2\beta$ (D) $\cot^2\beta$

Second: Answer the following questions

5. Prove the validity of the following identities:

   (A) $\tan\mu + \cot\mu = \sec\mu \csc\mu$

   (B) $\csc\alpha - \sin\alpha = \cos\alpha \cot\alpha$

   (C) $\cot^2\mu - \cos^2\mu = \cot^2\mu \cos^2\mu$

   (D) $\tan^2\alpha - \sin^2\alpha = \tan^2\alpha \sin^2\alpha$

   (E) $\sin^2\alpha + \tan^2\alpha \sin^2\alpha = \tan^2\alpha$

   (F) $\sin(90^\circ - \mu)\cos\mu = 1 - \sin^2\mu$

6. Prove the validity of the following identities:

   (A) $\frac{\csc\theta}{\cos\theta}(1 - \sin^2\theta) = \cot\theta$

   (B) $\frac{1}{\sin^2(90^\circ - \theta)} - \tan^2\theta = 1$

   (C) $\frac{1}{1+\tan^2\alpha} - \frac{1}{1+\tan^2\beta} = \cos^2\alpha - \cos^2\beta$

   (D) $\frac{1+\tan^2\theta}{\sec^4\theta} = 1 - \sin^2\theta$

   (E) $(\sec\phi - \tan\phi)^2 = \frac{1-\sin\phi}{1+\sin\phi}$

   (F) $\frac{1}{1+\cot\theta} = \frac{\tan\theta}{1+\tan\theta}$

   (G) $\frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos^2\theta + \cos\theta\sin^2\theta} = \csc\theta - \sec\theta$

Solving Trigonometric Equations

5 - 2

Solving trigonometric equation by real solutions

You will learn

- Finding the general solution for the trigonometric equations
- Solving equations in the interval $[0, 2\pi[$

Key-terms

- Trigonometric equation
- General solution

Materials

- Scientific calculator
- Graphic calculator

Think and discuss

We have previously studied before solving linear and quadratic equations (algebraically and graphically) In this lesson, we will solve the trigonometric equations using the basic identities. Is there a similarity between solving algebraic equations and solving trigonometric equations?

Group work

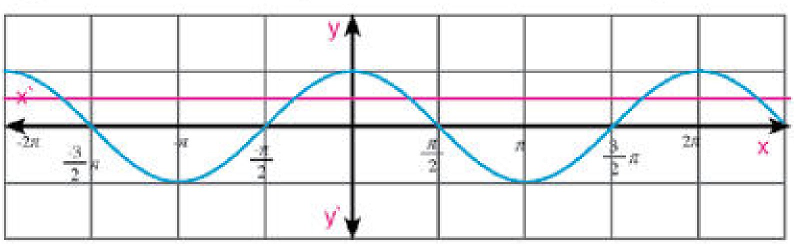
Work with your classmate to draw the trigonometric function $y = \cos\theta$ and the function $y = \frac{1}{2}$ and notice their common points of intersection.

1- Draw the cirve of the function $y_1 = \cos\theta$, $y_2 = \frac{1}{2}$ and notice their common points of intersection.

2- How many solutions for the equation $\cos\theta = \frac{1}{2}$ in the interval $[0, 2\pi[$?

3- Are there other solutions for the equation $\cos\theta = \frac{1}{2}$ in the graph?

The following graph represents the solution of the equation $\cos\theta = \frac{1}{2}$ we get the equation has two solutions $\frac{\pi}{3}, \frac{5\pi}{3}$ when $\theta \in [0, 2\pi[$, by adding 2π or -2π we will get other solutions for the equation.



General solution of the trigonometric equations

Example

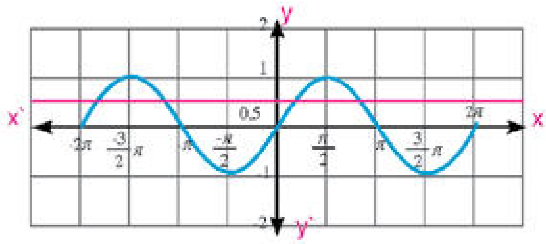
1 **Find the general solution of each of the following equations :**

A $\sin\theta = \frac{1}{2}$ **B** $\cos\theta = \frac{\sqrt{2}}{2}$ **C** $\tan\theta = \sqrt{3}$

Solution

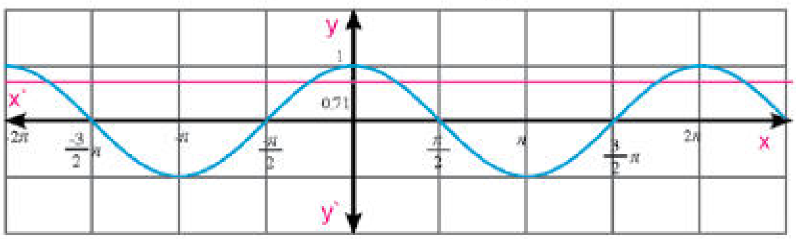
A $\because \sin\theta = \frac{1}{2}$ $\therefore \theta = \frac{\pi}{6}$

i.e. the general solution for the equation is $\frac{\pi}{6} + 2n\pi$ or $-\frac{\pi}{6} + \pi$ or $-\frac{\pi}{6} + \pi + 2n\pi$, $n \in Z$



B $\because \cos\theta = \frac{\sqrt{2}}{2}$ $\therefore \theta = \frac{\pi}{4}$

i.e. the general solution of the equation is $2n\pi \pm \frac{\pi}{4}$, $n \in Z$



C $\because \tan\theta = \sqrt{3}$ $\therefore \theta = \frac{\pi}{3}$

i.e. the general solution of the equation is $\frac{\pi}{3} + n\pi$, $n \in Z$

Try to solve

1 **Find the general solution of each of the following equations:**

A $\sin\theta = \frac{\sqrt{3}}{2}$ **B** $2\cos\theta = 1$ **C** $\tan\theta = \frac{\sqrt{3}}{2}$

Example

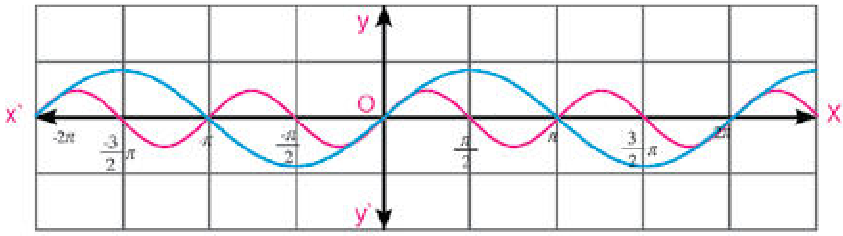
2 **Find the general solution of the equation: $\sin\theta\cos\theta = \frac{\sqrt{3}}{2}\sin\theta$**

Solution

$\sin\theta\cos\theta - \frac{\sqrt{3}}{2}\sin\theta = 0$ $\sin\theta(\cos\theta - \frac{\sqrt{3}}{2}) = 0$

Zero factor property	$\sin\theta = 0$	or	$\cos\theta - \frac{\sqrt{3}}{2} = 0$
			$\cos\theta = \frac{\sqrt{3}}{2}$
	$\theta = 0$,		$\theta = \frac{\pi}{6}$
	$\theta = n\pi$, $n \in Z$		$\theta = \pm\frac{\pi}{6} + 2n\pi$, $n \in Z$

The following graph represents a part of the solution of the equation.



Critical thinking: Is it necessary that all trigonometric functions have real solutions? Explain this by giving examples.

Try to solve

2 Find the general solution of each of the following equations:

A $\cos^2\theta - \cos\theta = 0$ **B** $2\sin^2\theta = \sin\theta$ **C** $\sqrt{2}\sin\theta\cos\theta - \sin\theta = 0$

Solve the trigonometric equations in the interval $[0, 2\pi[$

Example

3 Solve the equation: $\sin\theta\cos\theta - \frac{1}{2}\cos\theta = 0$ where $0^\circ < \theta < 180^\circ$

Solution

$\cos\theta(\sin\theta - \frac{1}{2}) = 0$ **by factorization**

$\cos 0 = \theta$ or $\sin\theta = \frac{1}{2}$

$\theta = 90^\circ$ or $\theta = 30^\circ$ or 150°

Solution of the equation is: 30° or 90° or 150°

Try to solve

3 If $0^\circ < \theta \leqslant 360^\circ$ Find the solution set of each of the following equations:

A $2\sin\theta\cos\theta + 3\cos\theta = 0$ **B** $4\sin^2\theta - 3\sin\theta\cos\theta = 0$

Check your understanding

1 Find the general solution of each of the following equations in radian:

A $\tan\theta = 1$ **B** $\cos\theta = \sin 2\theta$ **C** $2\sin\theta - \sqrt{3} = 0$

Exercise (5 - 2)

First : Complete each of the following

1. The general solution of the equation $\cos\theta = 1$ for all values of θ is ....
2. The general solution of the equation $\sin\theta = 1$ where $\theta \in [\pi, 2\pi[$ is ....
3. The general solution of the equation $\sin\theta = \cos\theta$ for all values of θ is ....
4. The solution set of the equation $\cot\theta = \sqrt{3}$ where $\theta \in [\pi, 2\pi[$ is ....

Second : Multiple choice

5. If $0^\circ \leqslant \theta < 360^\circ$, $\sin\theta + 1 = 0$, then θ equals ....
   A 0° B 90° C 180° D 270°
6. If $0^\circ \leqslant \theta < 360^\circ$, $\cos\theta + 1 = 0$, then θ equals ....
   A 90° B 180° C 270° D 360°
7. If $0^\circ \leqslant \theta < 180^\circ$, $\sqrt{3}\tan\theta - 1 = 0$, then θ equals ....
   A 30° B 60° C 120° D 150°
8. If $180^\circ \leqslant \theta < 360^\circ$, $2\cos\theta + 1 = 0$, then θ equals ....
   A 210° B 240° C 300° D 330°

Third: Answer the following questions

9. Find the general solution for each of the following equations.
   A $\sin\theta = \frac{1}{2}$ ....
   B $2\cos\theta - \sqrt{3} = 0$ ....
   C $\sqrt{3}\tan\theta - 1 = 0$ ....
10. Solve each of the following equations in the interval $[0, \frac{3\pi}{2}[$:
   A $\tan^2\theta - \tan\theta = 0$ ....
   B $2\sin\theta\cos\theta - \cos\theta = 0$ ....
   C $2\sin^2\theta - 3\sin\theta - 2 = 0$ ....

5 - 3 Solving the Right Angled Triangle

You will learn

- Solving the right angled triangle given lengths of two sides.
- Solving the right angled triangle given length of a side and measure of its acute angles.

Key-terms

- Solution of a tringle

Materials

- Scientific calcualtor

Think and discuss

We know that the triangle has six elements (three sides and three angles). The solution of the triangle means finding the measures of its six elements. To solve the right angled triangle, we must be given the length of two sides or the length of one of its sides and the measure of one of its acute angles.

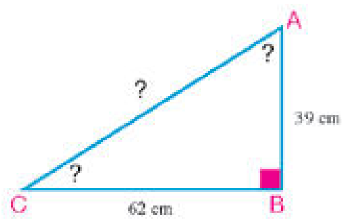
Solving the right angled triangle given the lengths of two sides:

Example

1 **Solve the triangle ABC right angled at B, in which AB = 39 cm, BC = 62cm**

Solution

First: **we find m (∠ C):**



$\because \tan C = \frac{AB}{BC}$

$\therefore \tan C = \frac{39}{62} \simeq 0.6290322581$

Use the calculator:

$m(\angle C) = 32°10'17''$

→ 3 9 ÷ 6 2 = Shift Tan⁻¹ Ans = °'''

We find m(∠ A):

$m(\angle A) = 90° - 32° 10'17'' = 57° 49'43''$

Use the calculator:

→ 9 0 – 3 2 °''' 1 0 °''' 1 7 °''' = °'''

Second: **we find length of:** $\overline{AC}$

$\because \sin C = \frac{AB}{AC}$ $\because \sin 32°10'17'' = \frac{39}{AC}$

→ 3 9 (÷ sin 3 2 °''' 1 0 °''' 1 7 °''') =

then $AC = \frac{39}{\sin 32^\circ 10' 17''} \simeq 73.24581124$ cm

Think

- Are there other trigonometric functions you can use to find the length of $\overline{AC}$? Mention these functions if they exist .
- Can you use the pythagorian theorem to find the length of $\overline{AC}$? Write the steps of the solutions if possible .
- Which do you prefer: the use of the pythagorian theorem to find the length of $\overline{AC}$ or the use of one of the trigonometric functions? Why?

Try to solve

1. Solve the triangle ABC, right angled at B in each of the following cases :

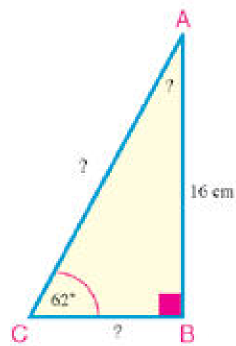
   A CB = 8 cm , BC = 12 cm   **B** BC = 5 cm , AC = 13 cm

Solving the right angled triangle given the length of its sides and the measure of one of its acute angles

Example

2. Solve the triangle ABC right angled at B, where m (∠ C) = 62°, AB = 16 cm, approximating the result to the nearest hundredth.

Solution



We find m(∠ A):

m(∠ A) = 90° – 62° = 28°

we find the length $\overline{BC}$:

∵ $\tan C = \frac{AB}{BC}$ **i.e.:** $\tan 62^\circ = \frac{16}{BC}$ then

$BC \times \tan 62^\circ = 16$

$BC = \frac{16}{\tan 62^\circ} = 8.507350907 \simeq 8.51$ cm

We find the length of $\overline{AC}$:

∵ $\sin C = \frac{AB}{AC}$ **i.e.:** $\sin 62^\circ = \frac{16}{AC}$

$AC \times \sin 62^\circ = 16$

$AC = \frac{16}{\sin 62^\circ} = 18.12112081 \simeq 18.12$ cm

Try to solve

2. Solve the triangle ABC, right angled at B in each of the following cases:

   A AB = 8 cm , m(∠ C) = 34°   **B** AC = 26 cm , m(∠ A) = 53° 12'

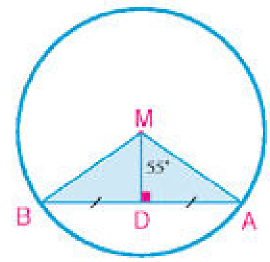
Critical thinking:

Can you solve the right angled triangle given the measures of its acute angles? Explain your answer.

Example

3 **Geometry:** A circle of radius 7 cm, a chord was drawn in it opposite to a central angle of measure 110°. Calculate the length of this chord to the nearest thousandth.

Solution



In the figure opposite: We draw $\overline{MD} \perp \overline{AB}$

From the properties of the circle: D is the midpoint of $\overline{AB}$

m(∠ AMD) = 110° ÷ 2 = 55°

We find the length of $\overline{AD}$ **in the right triangle** ADM:

sin (AMD) $= \frac{AD}{AM}$ **from the definition of "sine function"**

i.e.: sin 55° $= \frac{AD}{7}$

product of means = product of extremes :

AD = 7 × sin 55° ≃ 5.73406431 cm

Finding the length of $\overline{AB}$**:** A B = 2 × AD

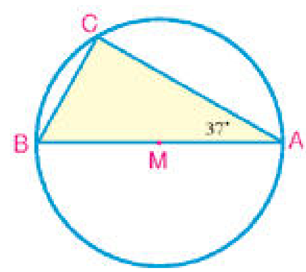
i.e.: A B = 2 × 5.73406431 = 11.46812862 ≃ 11.468 cm

Try to solve

3 **Geometry:** **The figure opposite shows the circle of centre M,** $\overline{AB}$ **is a diameter in it ,**

if AC **= 12 cm,** m(∠ A) **= 37° find the length of the radius of the circle.**

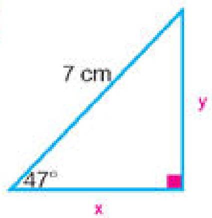
to the nearest hundredth.



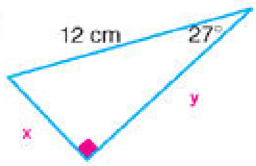
Exercise (5 - 3)

1 Find the value of x, y in each of the following figures

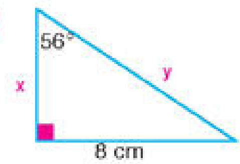
A



B

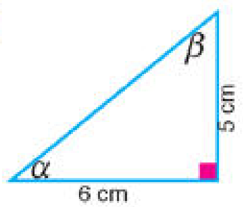


C

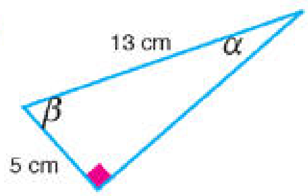


2 Find the value of each of the angles α, β in degree measure in each of the following figures:

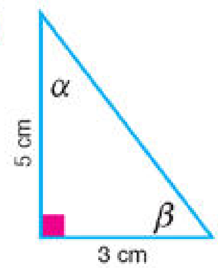
A



B



C



3 Solve the triangle ABC right at B approximating the angles to the nearest degree and the length to the nearest cm where:

A AB = 4 cm, BC = 6 cm

B AB = 12.5 cm, BC = 17.6 cm

4. Solve the triangle ABC right at B approximating the angles to the nearest thousandth in radian measure, and the length to the nearest thousandth cm where:

A m ($\angle$A) = 0.925^{rad} . BC = 8 cm **B** m ($\angle$A) = 1.169^{rad} . AB = 18 cm

5. ABC is a triangle, draw $\overrightarrow{AB} \perp \overline{BC}$, if AD = 6 cm, m($\angle$B) = 52° , m ($\angle$C) = 28°, find the length of $\overline{BC}$ to the nearest centimetre.

6. **Geometry:** Apiece of land is in the shape of a rhombus ABCD of side length 12 metres, m ($\angle$ABC) = 100°. find the length of each of its diagonals $\overline{AC}$ and $\overline{BD}$ to the nearest metre.

7. **Geometry:** ABCD is an isosceles trapezium in which $\overline{AD}$ // $\overline{BC}$, AB = CD = 5cm, AD = 4cm, BC = 10cm. Find the measure of each of its four angles.

Angles of Elevation and Angles of Depression

5 - 4

You will learn

- Concept of angles of elevation and depression.
- Using the right angled triangle to solve problems including angles of elevation and depression.

Key-terms

- Angle of Elevation
- Angle of Depression

Materials

- Scientific calcualor

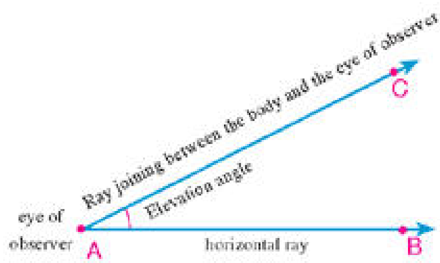
Think and discuss

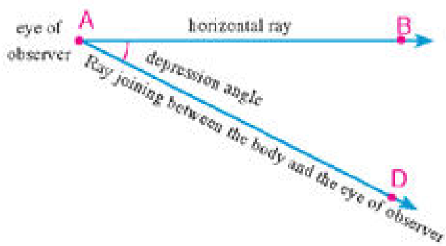
Can you find the height of a minaret from the ground when you are at a given distance from it without measuring the actual length of this minaret?

Learn

Angles of Elevation and Angles of Depression

1- If a person A observed point C above his horizontal sight $\overrightarrow{AB}$ then the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is called elevation angle of C of the horizontal level of the sight of person A.

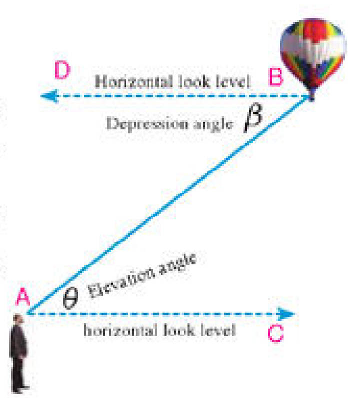




2- If a person A observed point D down his horizontal sight $\overrightarrow{AB}$ then the angle between $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is called the depression angle of D of the horizontal level of the sight of person A.

3- In the figure opposite:

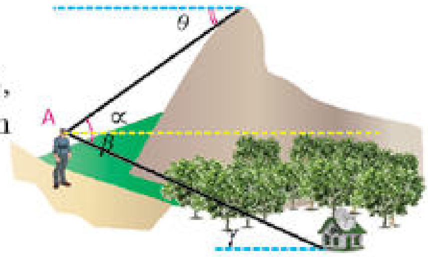
- $\angle$ CAB is the elevation angle of the balloon with respect to the person at A.
- $\angle$ DBA is the depression angle of the person at A with respect to the balloon, in this case , then : $\beta = \theta$



Try to solve

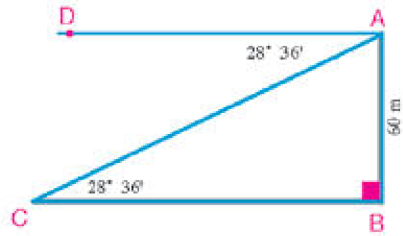
1 **In the figure opposite**

First: determine the type of each angle (γ), (β), (θ), (α) in terms of being an elevation angle or a depression angle with respect to the observer at A.
Second: Write the pairs of equal angles.



Example

1 From the top of a tower 60 metres high, the angle of depression of a body located in a horizontal level which passes through the base of the tower equals 28° 36'. Find how far was the body from the base of the tower to the nearest metres.



Solution

Let A be the top of the tower $\overline{AB}$
then $\angle$ DAC is the depression angle of the body
then: m($\angle$C) = m($\angle$DAC)

Definition of tangent function : $\tan C = \frac{AB}{BC}$

Substituting AB = 60: $\tan 28^\circ\ 36' = \frac{60}{BC}$

$BC \times \tan 28^\circ\ 36' = 60$

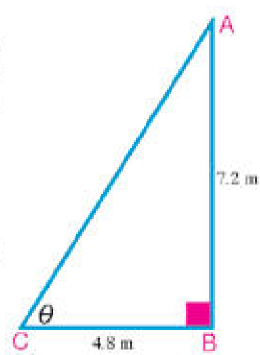
$BC = \frac{60}{\tan 28^\circ\ 36'} = 125.22966 \simeq 125$ metres

Try to solve

2 A person observed the top of a hill 2.56 km from the point on the ground. He found its depression angle was 63°. Find the distance between the top and the observer to the nearest metre.

Example

2 A light pole of height 7.2 metres gives a shade on the ground of length 4.8 metres. Find in radian the measure of the elevation angle of the sun at that moment.



Solution

Let A be the top of the light pole $\overline{AB}$, BC be the length of the shade of the pole, and θ be the elevation angle of the sun

$\because \tan C = \frac{AB}{BC}$ $\therefore \tan\theta = \frac{7.2}{4.8} = 1.5$

$m(\angle\theta) = 56^\circ 18' 36''$

$\therefore$ **the eleveation angle of the sun in radian** $= 56^\circ 18' 36'' \times \frac{\pi}{180^\circ} \simeq 0.9827937232^{rad}$

Note:

It is possible to use the calculator to find θ in radian directly without finding it in degree as follows:

1- Turn the calculator on the radian system (Radian): (4: Rad) 4 Mode Shift ←

2- Enter data (Data): ($\tan^{-1}$) tan Shift 5 . 1

3- Recall outputs (call outputs): =

R MATH
0.982793732

Try to solve

3 From the top of a rock 180 metres high from sea level, the depression angle of a boat 300 metres apart from the base of the rock was measured. What is the radian measure of the depression angle?

Exercise (5 - 4)

1. The length of the thread of a kite is 42 metres, if the angle which the thread makes with the horizontal ground equals 63°, Find to the nearest metre the height of the kite from the surface of the ground. ....

2. A man found that the angle of elevation of the top of a minaret 42m distance from its base was 52°. What is the height of the minaret to the nearest metre? ....

3. A mountain of height 1820 metres, It is observed from its top that the measure of depression angle of a point on the ground was 68°. What is the distance between the point and the observer to the nearest metre? ....

4. The upper end of a ladder rests on a vertical wall, it is 3.8m from the surface of the ground, the lower end rests on a horizontal ground. If the angle of inclination of the ladder to the ground is 64°. Find to the nearest hundredth each of:
   - **A** The distance between the lower end and the wall
   - **B** The length of the ladder

   ....

   ....

5. From the top surface of a house 8 metres high, a person found that the elevation angle of the top of an opposite building was of measure 63°, and observed the depression angle of its base, it was 28°. Find the height of the building to the nearest metre.

   ....

5 - 5 Circular Sector

You will learn

- Concept of the circular sector
- Finding the area of the circular sector

Key-terms

- Circular Sector

Materials

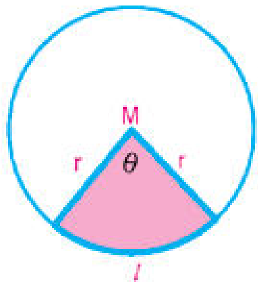
- Scientific Calculator

Think and discuss

The circular sector:

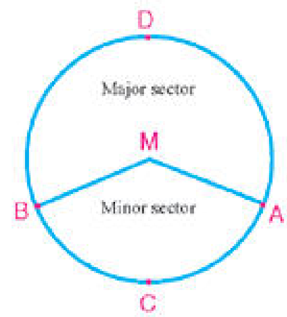
You have previously studied the relation between the length of the arc (l) from a circle of radius (r) and measure of the central angle opposite to this arc (θ) and known that: $l = \theta^{rad} \times r$.

Can you find the area of this shaded part of the circle in the figure opposite?



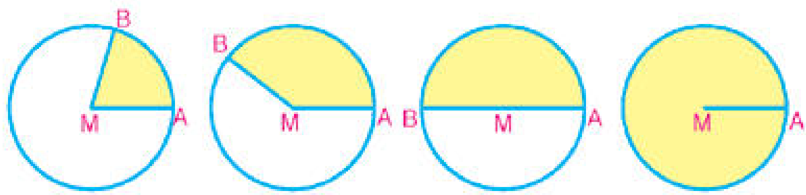
The circular sector: is a part of the surface of the circle bounded by two radii and an arc .

In the figure opposite, $\overline{MA}$ and $\overline{MB}$ divide the circle into two circular sectors, the minor sector MACB and the major sector MADB. $\angle$ AMB is called the angle of the minor sector and the reflex angle, $\angle$ AMB is called the angle of the major sector.



Area of the Circular sector

Activity:



The figure shown above represents a number of congruent circles:

1- Are the increase of Areas of the circular sectors caused by the increase of the radius length of the circle?

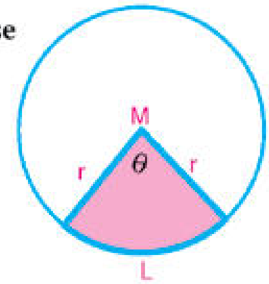
2- Are the increase of Areas of the circular sectors caused by the increase in measure of the angle of the circular sector?

3- If it continues to increase in measure of the angle of the sector until the terminal side $\overrightarrow{MB}$ congruent to the initial side $\overrightarrow{MA}$, then what do you expect to be the area of the sector?

Learn

First Area of the circular sector given measure of its central angle and the length of the radius

Area of the sector represents a part of the area of a circle whose central angle equals 2π.



From the previous activity, we deduce that:

$$\frac{\text{Area of the sector}}{\text{Area of the circle}} = \frac{\theta^{rad}}{2\pi}$$

i.e. area of the sector $= \frac{\theta^{rad}}{2\pi} \times$ area of the circle

$$= \frac{\theta^{rad}}{2\pi} \times \pi\, r^2 = \frac{1}{2} r^2\, \theta^{rad}$$

Area of the circular sector $= \frac{1}{2} r^2\, \theta^{rad}$ (where θ is the angle of the sector, r is the radius of the circle)

Critical thinking: Do you consider the circle a circular sector? Explain.

Example

(1) Find the area of the circular sector in which the length of the radius of its circle is 10cm and the measure of its angle is 1.2^{rad}

Solution

Formula: Area of the circular sector $= \frac{1}{2} r^2\, \theta^{rad}$

Substituting $r = 10$, $\theta^{rad} = 1.2^{rad}$: $= \frac{1}{2}(10)^2 \times 1.2 = 60\ cm^2$

Try to solve

(1) Area of a circular sector is $270\ cm^2$ and the length of the radius of its circle equals 15 cm, find in radian the measure of its angle.

Second: Area of the circular sector given its degree angle:

$\because \frac{\text{Area of the sector}}{\text{area of its circle}} = \frac{\frac{1}{2} r^2 \times \theta^{rad}}{\pi\, r^2}$

but $\frac{\theta^{rad}}{2\pi} = \frac{x^\circ}{360^\circ}$

$\therefore$ Area of the sector $= \frac{x^\circ}{360^\circ} \times$ area of the circle

> **Remember**
> Relation between the degree measure and the radian measure is:
> $\frac{\theta^{rad}}{\pi} = \frac{x^\circ}{180^\circ}$

Example

(2) A circular sector in which the length of the radius of its circle equals 16cm, and the measure of its angle equals 120°, find its area to the nearest square centimetre .

Solution

Formula: area of the sector $= \frac{x^\circ}{360^\circ} \times \pi\, r^2$

Substituting $r = 16$, $x^\circ = 120^\circ$: $= \frac{120^\circ}{360^\circ} \times \pi\,(16)^2 \simeq 268\ cm^2$

Try to solve

2. A circular sector in which the measure of its angle equals 60° and the length of the radius of its circle equals 12 cm. Find its area to the nearest tenth.

Third: Area of the circular sector given the length of its arc

You know that: Area of the circular sector $= \frac{1}{2} r^2 \theta^{rad}$

$$= \frac{1}{2} r^2 \times \frac{l}{r} = \frac{1}{2} l \, r$$

(by substituting : $\theta^{rad} = \frac{l}{r}$)

Remember

Length of the arc which is opposite to a central angle of measure θ in a circle of radius r is determined by the relation:

$l = \theta^{rad} \times r$

Example

3. Find the area of the circular sector whose perimeter equals 28cm, and the length of the radius of its circle equals 8cm.

Solution

Perimeter of the sector $= 2r + l$:	i.e. $2r + l = 28$
Substituting r = 8 cm:	$2 \times 8 + l = 28$
Simplify:	$l = 28 - 16 = 12$ cm

Formula: Area of the sector $= \frac{1}{2} l \, r$

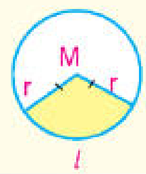
Substituting : $l = 12$ cm, $r = 8$ cm:

Area of the sector $= \frac{1}{2} \times 12 \times 8 = 48 \text{ cm}^2$

Add to your Knowledge

Perimeter of the sector whose length of the arc l and length of the radius of its circle r is determined by the relation:

perimeter of the sector $= 2r + l$

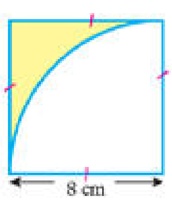


Try to solve

3. Geography: If you know that the equator is a circle of radius 6380 km, find the distance between two cities on the equator, if the arc between them is opposite to an angle of measure 30° at the centre of the Earth.

Check your understanding

1. Find in terms of π the area of the shaded part in each of the following figures:

A


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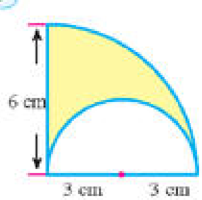

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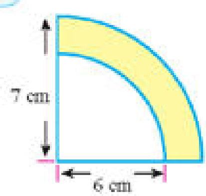


D


Exercise (5 - 5)

First : Complete each of the following

1. Area of the circular sector, in which $l = 6$ cm, $r = 4$ cm equals ....
2. Area of the circular sector, in which $r = 4$ cm, and its perimeter 20 cm equals .... cm^2.
3. Perimeter of the circular sector in which , its area 24 cm^2, length of its arc 8 cm equals ....

Second : Multiple choice

1. Area of the circular sector in which , measure of its angle 1.2^{rad} and length of the radius of its circle 4 cm equals ....
   - A) 4.8 cm^2
   - B) 9.6 cm^2
   - C) 12.8 cm^2
   - D) 19.6 cm^2
2. Perimeter of the circular sector in which length of its arc 4 cm and length of the diameter of its circle 10 cm equals ....
   - A) 14 cm
   - B) 20 cm
   - C) 30 cm
   - D) 40 cm
3. Area of the circular sector in which, measure of its angle 120°, length of the radius of its circle 3 cm equals ....
   - A) 3π cm^2
   - B) 6π cm^2
   - C) 9π cm^2
   - D) 12π cm^2
4. Area of the circular sector in which, its perimeter 12 cm, length of its arc 6 cm equals ....
   - A) 6 cm^2
   - B) 9 cm^2
   - C) 12 cm^2
   - D) 18 cm^2
5. If the area of the circular sector equals 110 cm^2, measure of its angle equals 2.2^{rad}, then length of the radius of its circle equals: ....
   - A) 2 cm
   - B) 5 cm
   - C) 10 cm
   - D) 20 cm

Third : Answer the following questions

1. Find the area of the circular sector in which, length of the diameter of its circle is 20 cm and measure of its angle is 120°. ....
2. Find the area of the circular sector in which length of its arc is 16cm, and length of the radius of its circle is 9cm. ....
3. Find the area of the circular sector in which , length of its arc is 7cm and its perimeter equals 25cm. ....
4. **Agriculture:** Abasin flowers is in the shape of a circular sector, its area equals 48 m^2 , length of its arc equals 6m. Find its perimeter and the length of the radius of its circle.
   ....
5. The perimeter of a circular sector equals 24 cm and length of its arc equals 10 cm. Find the area of its surface.
   ....

Circular Segment

5 - 6

You will learn

- The circular segment.
- Finding the area of the circular segment.

Key-terms

- Circular Segment

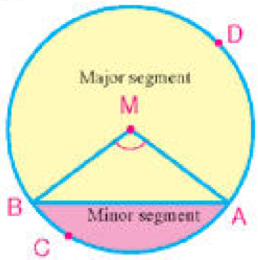
Materials

- Scientific Calculator

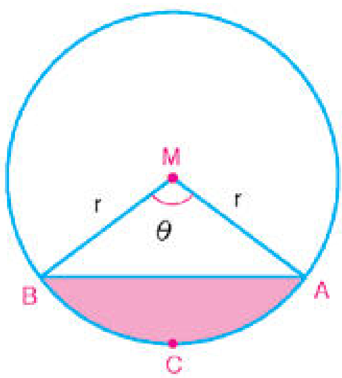
Learn The circular segment

The circular segment is a part of the surface of the circle bounded by an arc and a chord passing by the ends of this arc.

The chord $\overline{AB}$ divides the circle into two circular segments: **The minor segment** ACB and the major segment ADB, ∠AMB is called angle of the minor segment, while the reflex angle ∠AMB is called the angle of the major segment.



Finding the area of the circular segment:

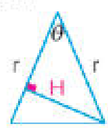


Remember

Area of the triangle $= \frac{1}{2}$ r× h where:

$\sin\theta = \frac{h}{r}$

$h = r \sin\theta$



Area of the triangle =

$\frac{1}{2} \times r \times r \sin\theta$

Area of the minor segment ACB

$= $ area of the minor sector MAB − area of the triangle MAB

$= \frac{1}{2} r^2 \theta^{rad} - \frac{1}{2} \times r \times r \sin\theta$

Area of the circular segment $= \frac{1}{2} r^2(\theta^{rad} - \sin\theta)$

Where r is the length of the radius of its circle , θ is the measure of the angle of the segment.

Think: Can you find the area of the major segment given the area of the minor segment? Explain.

Example

1. Find the area of the circular segment whose length of the radius of its circle equals 8cm, and the measure of its angle equals 150°.

Solution

$\theta^{rad} = 150^\circ \times \frac{\pi}{180^\circ} \simeq \frac{5\pi}{6}$

$\sin\theta = \sin 150^\circ$

Area of the circular segment $= \frac{1}{2} r^2 (\theta^{rad} - \sin\theta)$

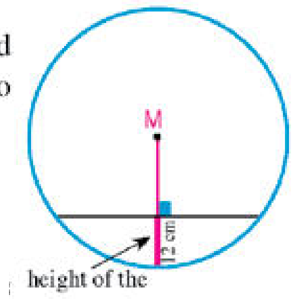
Area of the circular segment $= \frac{1}{2} \times 64 \left(\frac{5\pi}{6} - \sin 150^\circ\right) \simeq 67.7758 \text{ cm}^2$

Try to solve

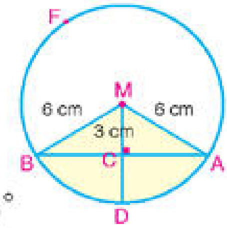
1. Find the area of the circular segment whose length of the radius of its circle equals 10cm and the measure of its circle equals 2.2^{rad} approximating the result the nearest hundredth.

Try to solve

2. Find the area of the major segment in which the length of its chord equals 12 cm and its height equals 2 cm approximating the result to the nearest square centimetre.



Exercise (5 - 6)



1 **In the figure drawn**:

M is a circle of radius 6 cm, $\overline{MC} \perp \overline{AB}$, M C = 3 cm complete:

A The height of the circular segment ADB = ... cm

B The height of the major circular segment AFB = ... cm

C Measure of the angle of the minor circular segment ADB = ...°

D Measure of the angle of the major circular segment AFB = ...°

E Area of the triangle MAB = ... cm^2.

F Area of the circular sector M A D B in terms of π equals .... cm^2.

G Area of the minor circular segment in terms of π equals .... cm^2.

2 **Find the area of the circular segment in which,**

A Length of the radius of its circle is 12 cm and measure of its angle equals 1.4^{rad}. ....

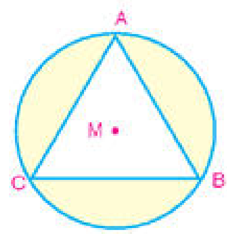
B The length of the radius of its circle equals 8 cm, and measure of its angle equals 135°.

....

C The length of the radius of its circle equals 14 cm and the length of its arc equals 22 cm.

....

3 **In the figure drawn:**



ABC is an equilateral triangle drawn in the circle M in which, the length of its radius equals 8 cm. find the area of each shaded circular segments. ....

4 Find the area of the major circular segment in which the length of its chord equals the length of the radius of its circle equals 12 cm.

....

5 - 7 Areas

You will learn
- Area of the triangle.
- Area of the quadrilateral .
- Area of the regular polygon

Key-terms
- regular polygon

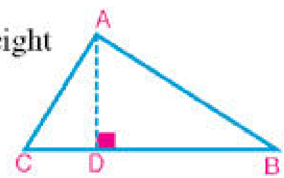
Materials
- Scientific calculator

Think and discuss

The Area of a Triangle

You have previously studied the area of the triangle , and known that its area is determined as follows:

Area of the triangle = $\frac{1}{2}$ length of the base × height



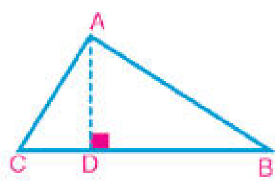
In the figure opposite:

Area of the triangle = $\frac{1}{2}$ BC × AD

Think: Can you apply this relation on the right angled triangle and the obtuse angled triangle?

The Area of a triangle in terms of the lengths of two sides and the included angle

Learn



From the figure opposite:

$\sin B = \frac{AD}{AB}$ **i.e.:** AD = AB sin B

From the area of the triangle:

Area of the triangle $= \frac{1}{2}$ BC × AD

$= \frac{1}{2} \times$ BC × AB sin B

Oral exercises: Find the area of the triangle in terms of each of:

A CA, CB, ∠C

B AB, AC, ∠ A

In general:

> Area of the triangle = half the product of the lengths of two sides × sine the included angle between them.

Example

(1) Find the area of the triangle ABC in which AB = 9 cm, AC = 12 cm, m(∠A) = 48° approximating the result to the nearest hundredth.

Solution

Area of the triangle A B C = $\frac{1}{2} \times AB \times AC \sin A$

Substituting AB = 9 cm , AC = 12 cm, m(∠A) = 48°

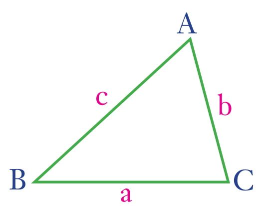
Area of the triangle ABC = $\frac{1}{2} \times 9 \times 12 \times \sin 48 \simeq 40.13$ cm^2

→ 1 ÷ 2 × 9 × 1 2 × Sin 4 8 =

Try to solve

(1) Find the area of the triangle ABC in which BC = 16 cm, BA = 22 cm , m(∠B) = 63 ° approximating the result to the nearest thousandth.

Finding the surface area of the triangle in terms of its side lengths (Heron's formula)



The surface area of the triangle whose side lengths are a, b and c is:

$\Delta = \sqrt{p(p-a)(p-b)(p-c)}$ where P is half of the triangle perimeter

Example

(2) Find the surface area of the triangle whose side lengths are 6, 8 and 10 centimetres using Heron's formula

Solution

$\because$ 2P = 6 + 8 + 10 = 24 cm P = 12 cm

P - a= 12 - 6 = 6cm , p - b = 12 - 8 = 4 cm , p - c= 12 - 10 = 2 cm

$\therefore$ Area of $\Delta = \sqrt{p(p-a)(p-a)(p-c)}$

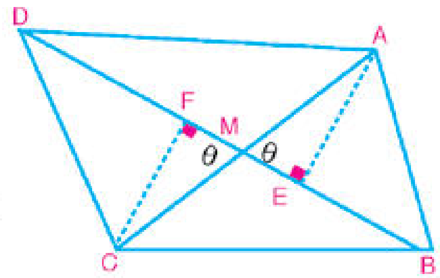
$= \sqrt{12 \times 6 \times 4 \times 2} = 24$ cm^2

Try to solve

(2) Find the surface area of the triangle A B C in which:

a= 5cm , b = 12 cm , c = 13cm using Heron's formula.

The Area of a Convex Quadrilateral



In the figure opposite:

ABCD is a quadrilateral in which $\overline{AC} \cap \overline{BD} = \{M\}$

$\overline{AE} \perp \overline{BD}$, $\overline{CF} \perp \overline{BD}$, θ is the included angle between the two diagonals.

Area of the quadrilateral = area △ABD + △CBD

$= \frac{1}{2} BD \times AE + \frac{1}{2} BD \times CF$

$= \frac{1}{2} BD (AE + CF) \qquad = \frac{1}{2} BD (AM \sin\theta + CM \sin\theta)$

$= \frac{1}{2} BD \times \sin\theta (AM + CM) = \frac{1}{2} BD \times AC \times \sin\theta$

In general : Area of the quadrilateral in terms of the lengths of its diagonals and the included angle between them is:

Area of the quadrilateral = $\frac{1}{2}$ product of the lengths of its diagonals × sine the included angle between them

Think: Does the area of the quadrilateral change if we replace the angle θ by its complementary angle? Explain your answer.

Example

3 Find the area of the quadrilateral in which the lengths of its diagonals are 12 cm, 16 cm and the measure of the included angle between them is 68° approximating the result to the nearest square centimetre.

Solution

Formula of the area is:

Area of the quadrilateral = $\frac{1}{2}$ product of the lengths of its diagonals × sine the included angle between them

∴ Area of the quadrilateral $= \frac{1}{2} \times 12 \times 16 \times \sin 68° \simeq 89 \text{ cm}^2$

Try to solve

3 Find the area of the quadrilateral in which the lengths of its diagonals are 32 cm, 46 cm and the measure of the included angle between them is 122° approximating the result to the nearest tenth.

4 **Critical thinking:** Calculate using the previous formula area of each of the following:

A. A square of diagonal length is 10 cm

B. A rhombus of diagonals lengths are 8 cm, 12 cm. What do you notice?

The area of a regular polygon

Figure (1) represents a regular polygon, in which **n** is the number of its sides, and **x** is the length of its side.

Figure (2) represents one of the triangle which is taken from figure (1).

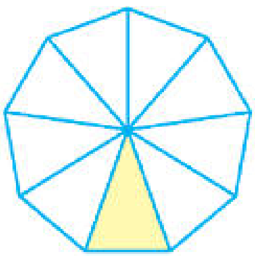


figure (1)

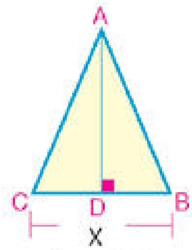


figure (2)

$\because m(\angle BAC) = \frac{2\pi}{n}$ (why)?

$\therefore \cot \frac{\pi}{n} = \frac{AD}{BD}$ **i.e.** $AD = BD \times \cot \frac{\pi}{n}$

$AD = \frac{1}{2} x \cot \frac{\pi}{n}$ (where x is the length of its polygon)

Area of the triangle $= \frac{1}{2} BC \times AD = \frac{1}{2} x \times \frac{1}{2} x \cot \frac{\pi}{n}$

$= \frac{1}{4} x^2 \times \cot \frac{\pi}{n}$

> Area of the polygon whose number of sides is n , and x is the length of its side $= \frac{1}{4} n x^2 \times \cot \frac{\pi}{n}$

Example

4 Find the area of the regular octagon in which the length of its side equals 6 cm approximating the result to the nearest hundredth.

Solution

Formula Area of the regular polygon $= \frac{1}{4} n x^2 \times \cot \frac{\pi}{n}$

Substituting n = 8, x = 6 cm:

The area $= \frac{1}{4} \times 8 \times (6)^2 \times \cot \frac{180^\circ}{8}$

$= 72 \times \frac{1}{\tan 22.5^\circ} \simeq 173.8 \text{ cm}^2$

Oral exercises:

Use the previous formula to find the area of each of the following:

1- Equilateral triangle 2- square 3- regular hexagon

Try to solve

4 Find the area of the regular pentagon in which the length of its side is 16 cm approximating the results of the nearest thousandth.

C A rhombus in which AB = 8cm, and measure of the included angle between two adjacent sides in it equals 58

4 Find the area of each of the following regular polygons approximating the result to the nearest tenth

A A regular pentagon of side length equals 16cm.

B A regular hexagon of side length equals 12cm.

5 **Constructions:** the figure opposite represents a set of bicycles lead to the entrance of the residential compound is in the shape of an isosceles trapezium, its larger base is down and its length equals 7 metres, its smaller base is up and its length equals 3 metres, each leg inclines by an angle of measure 75 to the larger base. Find:

3 metres
75° 75°
7 metres

A Length of its base at the middle.

B Length of each of its legs to the nearest tenth .

C Area of the trapezium to the nearest metre.

6 **Basing decorations:** Basin is designed to fish decoration , its base is in the shape of a regular pentagon, the length of its diagonal equals 72 cm. Find to the nearest square centimetre the area of its base.

7 **Folowers:** Karim designs a Garden to his house, and hope to determine a special part for flowers, is in the form of a regular hexagon of area 54 $\sqrt{3}$ m^2. Find the length of its side.

Exercise (5 - 7)

1 Find the area of the triangle ABC in each of the following cases:

A A B = 6cm, B C = 8cm, m(∠ B) = 90°

B A C = 12cm, length of perpendicular drawn from B to $\overline{AC}$ equals 7 cm.

C A B = 16cm, B C = 20 cm, m(∠ B) = 46°

D Construct A B C = 1...

2 **Find the area of the figure ABCD in each of the following cases:**

A Parallelogram in which A B = 8cm, B C = 11cm, m(∠ B) = 60°

B Trapezium in which, length of its parallel bases $\overline{AD}$ and $\overline{BC}$ are equals to 7cm , 11cm respectively, the length of the perpendicular drawn from D to $\overline{BC}$ equals 6cm.

3 Find the area of each of the following regular polygons approximating the result to the nearest tenth

A A regular pentagon of side length equals 16cm.

B ...

Find the area of the triangle ABC in the following cases

4 a = 15 cm , b = 12 cm , c = 9 cm

5 a = 16 cm , b = 18 cm , c = 24 cm

6 a = 32 cm , b = 36 cm , c = 30 cm

General Exercises

1 Simplify each of the following:

A $\sin(-\theta)\csc(\theta)$ ....

B $(\sin\theta+\cos\theta)^2-\frac{2}{\sec\theta\csc\theta}$ ....

2 Prove the validity of the following identities:

A $\cos(180°+\theta)\csc(90°+\theta)=-1$

B $\sec\theta-\sin\theta\tan\theta=\cos\theta$

C $\tan^2\theta+\cot^2\theta-(\sec^2\theta+\csc^2\theta)=-2$

D $(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2=2$

3 Find the solution set of each of the following equation in $[0, 2\pi[$

A $2\sin\theta - 1 = 0$ ....

B $\sqrt{3}\cot\theta = 1$ ....

C $2\cos^2\theta+\cos\theta=0$ ....

4 Find the general solution of each of the following equations :

A $\tan\theta - \sqrt{3} = 0$ ....

B $\cos 5\theta = \sin 4\theta$ ....

C $\sec 4\theta = \csc 2\theta$ ....

5 Solve the triangle A B C right at B in which:

A $m(\angle A) = 43°$, B C = 12 cm

B A B = 12.8 cm , B C = 19.2 cm

....

First : Multiple choice:

6 $\frac{1-\sin^2\theta}{1-\cos^2\theta}$ equals .... in the simplest form:

A -1 **B** 1 **C** $\tan^2\theta$ **D** $\cot^2\theta$

7 The solution set of the equation $\sqrt{3}\tan\theta = 1$ where $0° < \theta < 270°$ is: ....

A 30° **B** 150° **C** 210° **D** 240°

8 If the area of a circular sector equals 48cm^2 and the length of its arc equals 12cm, then the length of the radius of its circle equals: ....

A 4cm **B** 8cm **C** 12cm **D** 16cm

9 Area of DABC in which a = 6 cm , b = 9 cm , c= 13 cm equals .... cm^2 (ta the nearest tenth)

A 15.7 **B** 21.3 **C** 27.3 **D** 35.3

10 Find in the form of a rational number the value of each of:

A $\sin\theta$ if $\cos\theta = \frac{4}{5}$, $0° < \theta < 90°$ ....

B $\tan\theta$ if $\csc\theta = \frac{13}{5}$, $90° < \theta < 180°$ ....

11 From the top of a rock 40 metres high , two ships were observed in one ray on the sea with the rock and were measured to be 35° 12' and 53° 6'. Find the distance between the two ships.

12 A circle M of radius length 7.5 cm, $\overline{MA}$, $\overline{MB}$ are radii where A B = 12cm . Find the area of the minor circular sector M A B to the nearest square centimetres .

13 Find the area of the circular segment in which the length of the radius of its circle equals 10cm and the length of its arc equals 26.19 cm.

14 ABC is an equilateral triangle of side length 24 cm , A circle is drawn passing through its vertices. Find the length of the radius of the circle , then Find the area of the minor circular segment in which its chord is $\overline{BC}$.

15 Find the area of the quadrilateral A B C D in which AC = 14 cm , BD = 18 cm and measure of the angle between its diagonals is 78° approximating the result to the nearest hundredth.

16 Find the area of a regular polygon of 8–sides of side length 10cm approximating the result to the nearest tenth.

17 From a point 8 metres apart of the base of a tree, it was found that measure of elevation angle of the top of the tree 22°, find the height of the tree to the nearest hundredth.

18 A circular sector of area 270cm^2, the length of the radius of its circle equals 15cm, find the length of the arc of the sector and measure of its central angle in radian measure.

19 Find the area of the major circular segment in which the length of the radius of its circle equals 5cm, and the length of its chord equals 8cm.

20 **Analytic geometry:**

First: Find the area of the figure opposite.

Second: Find to the nearest tenth the area of each of the following figures:

A an isosceles triangle, the length of one of its legs equals 12cm and the measure of the included angle between them equals 64°.

B A regular figure of 12 sides and the length of its side equals 19cm.