Arab Republic of Egypt
Ministry of Education & Technical Education
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رؤية مصر 20/30
EGYPT VISION

MATHEMATICS

Student's Book

For Preparatory Year three

2025-2026

الإدارة المركزية لتطوير المناهج
CACD
Central Administration Of Curriculum Development

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MATHEMATICS

For Preparatory Year three — Student's Book — First Term

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Introduction

Dear students:

It is extremely great pleasure to introduce the mathematics book for third preparatory. We have been specially cautious to make learning mathematics enjoyable and useful since it has many practical applications in real life as well as in other subjects. This gives you a chance to be aware of the importance of learning mathematics, to determine its value and to appreciate mathematicians roles.

This book sheds new lights on the activities as a basic objective. Additionally, we have tried to introduce the subject simply and excitingly to help attaining mathematical knowledge as well as gaining patterns of positive thinking which pave your way to creativity.

This book has been divided into units, each unit contains lessons. Colors and pictures are effectively used to illustrate some mathematical concepts and the properties of figures. Lingual level of previous study has been taken into consideration.

Our great interest here is to help you get the information independently in order to improve your self-study skills.

Calculators and computer sets are used when needed. Exercises, practices, general exams, portfolios, unit test, general tests, and final term tests attached with model answers have been involved to help you review the curriculum completely.

Eventually, we hope getting the right track for the benefits of our students as well as for our dearest Egypt hoping bright future to our dearest students.

Authors

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Statistics

Trigonometry

Unit (4): Trigonometry

Coordinate geometry

Unit 5: Coordinate geometry

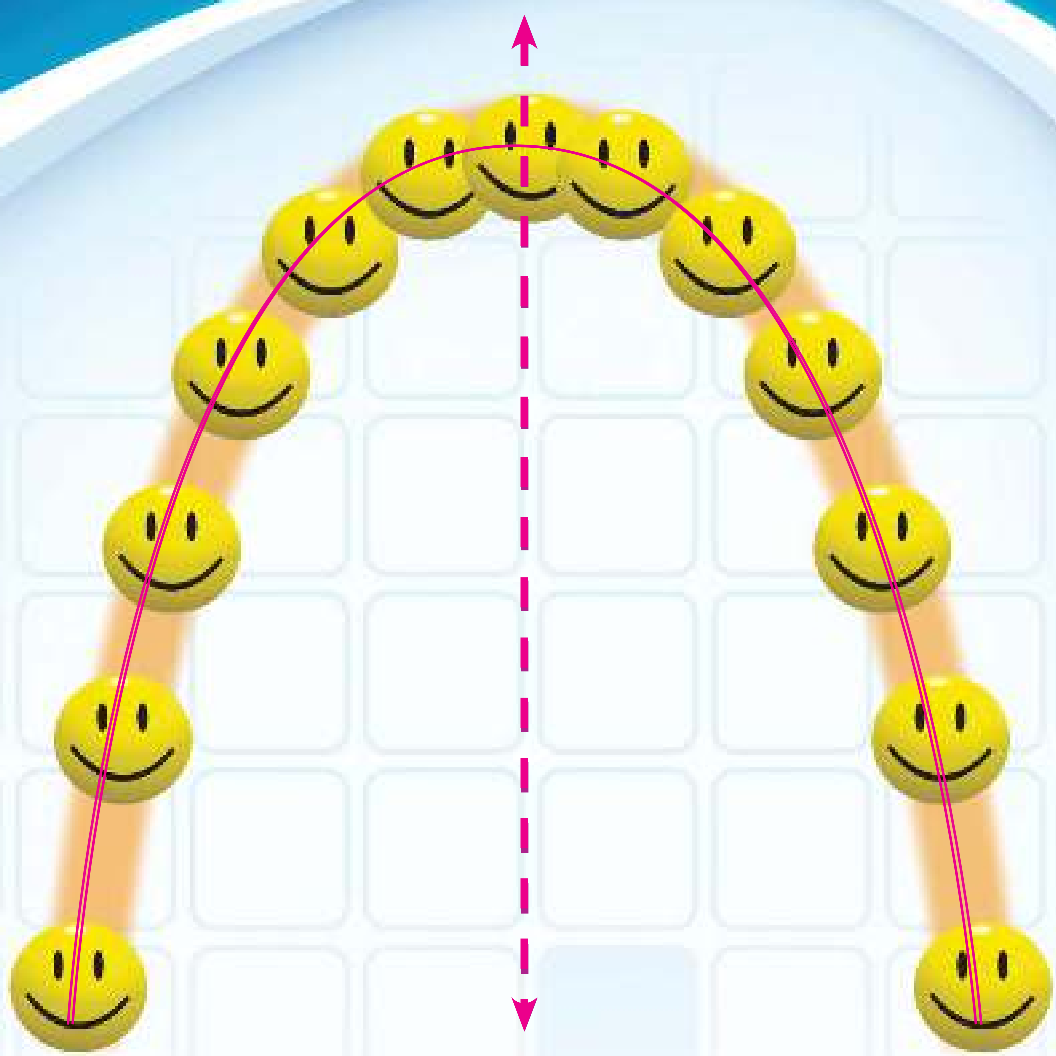
MATHEMATICAL NOTATION

N	The set of natural numbers	$\perp$	Perpendicular to
Z	The set of integers	//	Parallel to
Q	The set of rational numbers	$\overline{AB}$	Straight segment AB
Q`	The set of irrational numbers	$\overrightarrow{AB}$	Ray AB
R	The set of real number	$\overleftrightarrow{AB}$	Straight line AB
$\sqrt{A}$	The Square root of A	m ($\angle$ A)	Measure of angle A
$\sqrt[3]{A}$	The Cube root of A	m ($\overset{\frown}{AB}$)	Measure of arc AB
[a, b]	Closed interval	$\sim$	Similarity
]a, b[	Open interval	$>$	Grater than
[a, b[	Half-open interval	$\geqslant$	Grater than or equal to
]a,b]	Half-open interval	$<$	Less than
[a, ∞ [	Infinite interval	$\leqslant$	Less than or equal to
$\equiv$	Is congruent to	p(e)	Probability of occurring event
n (A)	Number of elements of A	$\overline{x}$	Mean
s	Sample space	σ	Standard deviation
		$\sum$	Sum

Algebra

Unit 1

Relations and Functions

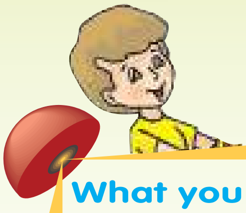


One of the players threw the ball so, it took the direction shown in the figure.

This figure represents one of the functions which you will study and is called " a quadratic function"

1 - 1

Cartesian product



What you'll learn

- ☆ Cartesian product of two non-empty sets.

Key terms

- ☆ Ordered pair.
- ☆ A cartesian product.
- ☆ An arrow diagram.
- ☆ A cartesian diagram.
- ☆ Relation.

Think and Discuss

You have previously studied relation between two variables x, y

1. Find a set of the ordered pairs which satisfy the relation: $y = 2x - 1$ when $x = 0$ and $x = 1$, $x = 2$
2. Represent these ordered pairs graphically in the coordinate plane.
3. Does the ordered pair (3, 5) equal the ordered pair (5, 3)? **(Use the graph)**.

From the previous, we notice:

1. In each ordered pair (a , b), a is called the first projection, and b is called the second projection.
2. Each pair is represented by one and only one point in the coordinate plane.
3. *If* $a \neq b$ *then* $(a, b) \neq (b, a)$, Why?
4. $(a, b) \neq \{a, b\}$.
5. *If* $(a, b) = (x, y)$ *then* $a = x$, $b = y$

Example 1

Find x, y if: $(x - 2, 3) = (5, y + 1)$

Solution

$x - 2 = 5 \quad \therefore x = 7 \quad , \quad 3 = y + 1 \quad \therefore y = 2$

Drill

Find **a and b in each of the following:**

- **A** $(a, b) = (-5, 9)$
- **B** $(a - 2, b + 1) = (2, -3)$
- **C** $(6, b - 3) = (2 - a, -1)$
- **D** $(a - 7, 26) = (-2, b^3 - 1)$

Example 2

If X = {a , b}, Y = { -1 , 0 , 3) then find: $X \times Y$, $Y \times X$, **What do you notice?**

Solution

To find the cartesian product of the set X and Y which is denoted by the symbol $X \times Y$. write the set of all the ordered pairs in which its, first projection is an element of X, and its second projection is an element belongs to Y , and it is written as:

$X \times Y$ = {a , b} × { -1 , 0 , 3} = { (a , -1) , (a , 0), (a , 3) , (b , -1), (b , 0), (b, 3)}

$Y \times X$ = { -1, 0, 3} × {a, b } = { (-1, a), (-1, b), (0, a), (0, b), (3, 0), (3, b)}

So: $X \times Y \neq Y \times X$

We can get $X \times Y$ and $Y \times X$ from the following tables

×		Second projection		
		-1	0	3
First Projection	a	(a, -1)	(a, 0)	(a, 3)
	b	(b, -1)	(b, 0)	(b, 3)

×		Second Projection	
		a	b
First projection	-1	(-1, a)	(-1, b)
	0	(0, a)	(0, b)
	3	(3, a)	(3,b)

Think:

1. When $X \times Y = Y \times X$?
2. Are the number of elements of $X \times Y$ = the number of elements of $Y \times X$?

We notice that :

1. *If* X and Y are two finite and non empty sets then :

   $X \times Y = \{(a , b) : a \in X , b \in Y\}$

2. $X \times Y \neq Y \times X$ *where* : $X \neq Y$

   $n (X \times Y) = n (Y \times X) = n (X) \times n (Y)$

   where n denotes the number of set elements .

3. *If* $(k, m) \in X \times Y$ *then* $k \in X$, $m \in Y$

4. *If* X is a non-empty set,

   then: $X \times X = \{(a , b) : a , b \in X\}$

   and written as X^2 and it is read as (X two).

Example 3

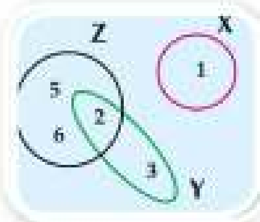
If X = {1}, Y = {2, 3}, Z = {2, 5, 6} represent the sets of X , Y , Z with venn diagram then find:

First : A $X \times Y$ B $Y \times Z$ C $X \times Z$ D Y^2

Second: $(X \times Y) \cup (Y \times Z)$ **Third:** $X \times (Y \cap Z)$

Fourth: $(X \times Y) \cap (X \times Z)$ **Fifth:** $(Z - Y) \times (X \cup Y)$

Solution



First :

A $X \times Y = \{1\} \times \{2, 3\} = \{(1, 2), (1, 3)\}$

B $Y \times Z = \{2, 3\} \times \{2, 5, 6\}$

$= \{(2, 2), (2, 5), (2, 6), (3, 2), (3, 5), (3, 6)\}$.

C $X \times Z = \{1\} \times \{2, 5, 6\} = \{(1, 2), (1, 5), (1, 6)\}$

D $Y^2 = Y \times Y = \{2, 3\} \times \{2, 3\}$

$= \{(2, 2), (2, 3), (3, 2), (3, 3)\}$

Second : $(X \times Y) \cup (Y \times Z) = \{(1, 2), (1, 3), (2, 2), (2, 5), (2, 6), (3, 2), (3, 5), (3, 6)\}$

Third : $X \times (Y \cap Z) = \{1\} \times \{2\} = \{(1, 2)\}$

Fourth : $(X \times Y) \cap (X \times Z) = \{(1, 2), (1, 3)\} \cap \{(1, 2), (1, 5), (1, 6)\} = \{(1, 2)\}$.

Fifth : $Z - Y = \{5, 6\}$ $\therefore (Z - Y) \times (X \cup Y) =$ .... Complete

Drill

If X = {2 , - 1}, Y = {4 , 0} , Z = {4 , 5 , -2} Find

A $X \times Y$ B $Y \times Z$ C X^2

D $n(X \times Z)$ E $n(Y^2)$ F $n(Z^2)$

The representation of the cartesian product:

Example 4

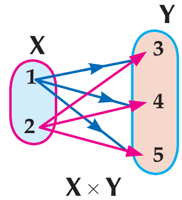
1 If X = {1, 2}, Y {3, 4, 5} *Find:* $X \times Y$, and represent it:

First: by the arrow diagram. **Second: by the cartesian diagram.**

Solution

$X \times Y = \{1, 2\} \times \{3, 4, 5\} = \{ (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$

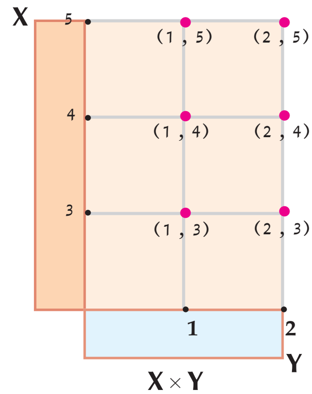
Where the cartesian product of X × Y is represented by an arrow diagram, or a grahpical net, as follows:



First: An arrow diagram

Draw an arrow from each element that represents the first projection (The elements of set of X) to each element that represents the second projection (The elements of set of Y)

i . e: The arrow diagram of the cartesian product represents each ordered pair by an arrow that starts from its first projection and ends at the second projection.



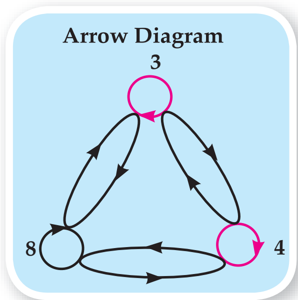
Second: Cartesian diagram (the perpendicular graphical net.

On a perpendicular graph net, the elements of set X is represented horizontally and the elements of set Y vertically. The intersection points of the horizontal and vertical lines represent the ordered pairs of the elements of the cartesian product X x Y.

Example 5

If X = {3, 4 , 8} then find, X × X and represent it with an arrow diagram.

Solution



$X \times X = \{3, 4, 8\} \times \{3, 4, 8\}$

$=\{ (3, 3), (3, 4), (3, 8), (4, 3), (4, 4), (4, 8), (8, 3), (8, 4), (8, 8)\}$.

Notice in the figure: the ordered pairs are represented by arrows, and the ordered pairs in which the first projection is equal to the second projection as: (3 , 3) , (4, 4), (8, 8) are represented by a buttonhole to show that the arrow comes from a point and ends in the same point.

Notice that: n (X) = 3, then n (X × X) = 3 × 3 = 9

In this case, the cartesian product X × X can be represented graphically by 9 points where each point represents an ordered pair. But if X is an infinite set، then the number of elements of X×X is infinite.

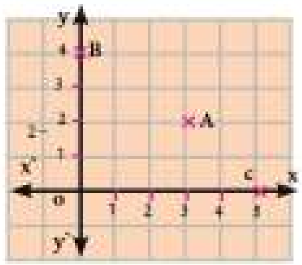
Think: How can you represent the cartesian product of each of the following?

N × N , Z × Z , Q × Q and R × R.

The cartesiem product of the infinite sets and its graphical representation:

First: **To represent the cartesian product of $N \times N = \{(x, y) : x \in N, y \in N\}$**

1. Draw two perpendicular straight lines, one of them is $\overleftrightarrow{xx`}$ horizontialy and the other $\overleftrightarrow{yy`}$ vertically and are intersected at point o.
2. Represent the natural numbers N on each of the horizontal and vertical straight lines starting with the origin point 0 which represents the number zero.



3. Draw vertical straight lines and horizontal straight lines from the points which represent the natural numbers, you will get the opposite figure, and thus, the points of intersection of the set of these straight lines are represented by the perpendicular graphical net of the cartesian product of $N \times N$.

Notice that: Each point of this net represents one the ordered pairs in the cartesian product of $N \times N$.

For Example: point A represents the ordered pair (3 , 2) and point B represents (0, 4).

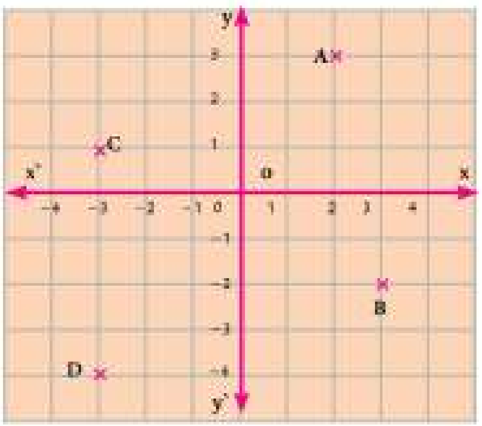
Complete: point C represents the ordered pair (...... , ...), point O represents the ordered pair (...... , ...).

Second: **To represent the cartesien product of $Z \times Z = \{(x, y): x \in Z, y \in Z\}$.**

We represent the set of integers on each of the two horizontal and vertical straight lines where the point (O) represents the ordered pair (0, 0).

Thus, each point of the net points represents one of the pairs in the cartesian product $Z \times Z$

This net is known as the coordinat plane of $Z \times Z$.



Drill

Identify the ordered pairs which represented by the points A , B , C and D in the previous graphical net.

Third: **To represent the cartesian product $Q \times Q = \{(x, y) : x \in Q, y \in Q\}$**

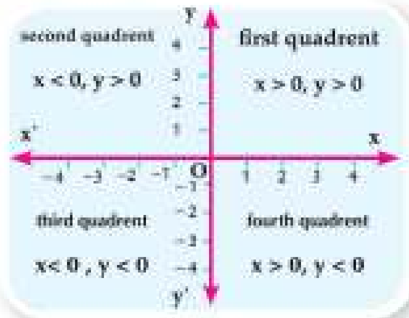
Draw a perpendicular graphical net and represent the set of rational numbers Q on the two horizontal and vertical straight lines, then identify the points: A $(3, \frac{5}{2})$, B $(-\frac{3}{2}, 4)$, C $(-3, -\frac{3}{2})$ and D $(\frac{5}{2}, -\frac{3}{2})$

Fourth: Representing the cartesian product RXR = {(x, y) : x ∈ R, y ∈ R}

the set of real numbers can be represented on each of the two horizontal and vertical straight lines, and point O represents the ordered pair (o , o).

The horizontal straight line $\overleftrightarrow{xx'}$ is called the x - axis, and the vertical straight line $\overleftrightarrow{yy'}$ is called the y - axis.

Then, the net is divided into four parts (quadrants) as in the opposite figure:

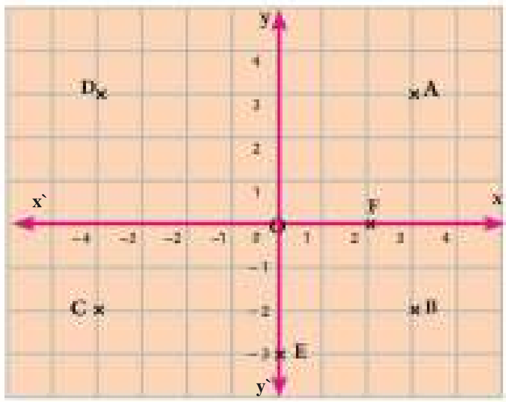


Example 6

Draw a perpendicular square net of the cartesian product R × R, then tell the quadrant or the axis where each of the following points is located:

A (3, 3), B (3, -2), C (-4, -2), D (-4, 3), E (0, - 3),F (2, 0)

Solution



A	(3, 3)	is located in the first quadrant.
B	(3, -2)	is located in the fourth quadrant.
C	(- 4, - 2)	is the located in the third qudrant
D	(- 4, 3)	is the located in the second qudrant
E	(0, - 3)	is the locate on y - axis
F	(2, 0)	is the located on x - axis.

Drill

If X = [- 2 , 3] find the location which represents X × X.

Show which of the following points belongs to the cartesian product of X × X

A (1, 2), B (3, - 1), C (- 1, 4) and D (-2, 0)

For More Exercises, go to MOE website

1 - 2

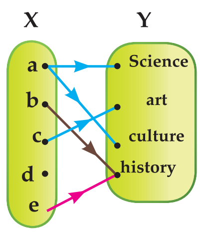
Relations

Think and Discuss

In the festival "Reading for All", five students represent the set of X = {a , b, c , d , e} went to the school library to read some books which are represented by the set Y = [science, literature, culture and history] student A read a book in science and a book in culture, student b read a book in histroy, student c read a literary book, pupil e read a book of the historical books, but student d didn't read any of these books.

1. Write the previous statements in the form of ordered pairs from X to Y.
2. Represent a set of the ordered pairs in the form of an arrow diagram.

Notice that: The expression " read" connects some of the elements of the set X with the elements of set Y, and it determines a relation between X and Y which is denoted by the symbol R. This relation can be represented by an arrow diagram - as shown in the opposite figure , where we draw an arrow beginning from the student and ending at the type of books he reads.



We can also express the relation from X to Y by the net of the following ordered pairs:

{(a, Science), (a, Culture), (b, Histrory), (c, Literature), (e, History)}.

This set of ordered pairs are called the relation R.

Think:Is R a subset from the cartesian product X × Z?

Example 1

If X = {-1, 1, 2}, Y = {2, 4, 6, 8}, and R is a relation from X to Y where a R b means: «b = 2 a + 4», for each a ∈ X, b ∈ Y

Write and represent R once in an arrow diagram and another by a cartesian diagram.

What you'll learn

- ☆ A relation of set of X to the set of Y.
- ☆ A relation from a set on it self.

Key terms

- ☆ Relation.

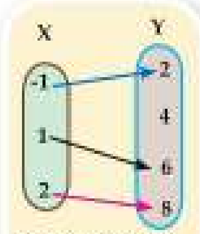
Solution

When: A = -1 $\therefore B = 2 \times (-1) + 4 = 2$

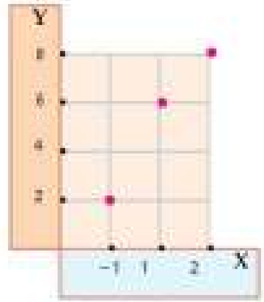
When: A = 1 $\therefore B = 2 \times 1 + 4 = 6$

When: A = 2 $\therefore B = 2 \times 2 + 4 = 8$

$\therefore R = \{(-1, 2), (1, 6), (2, 8)\}$



Arrow diagram



Graph digram

From the previous, we deduce that

1. The relation from X to Y where X, Y are two non-empty sets is a relation, connecting some or all the elements of X with some or all the elements of Y.
2. $X \times Y$ is the set of ordered pairs where the first projection in each ordered pair belongs to X and the second projection belongs to Y.
3. If R is a relation from X to Y , then $R \subset X \times Y$.

The relation from a set to itself

If R is a relation from a set X to X (itself) then R is called a relation on X and $R \subset X \times X$.

Example 2

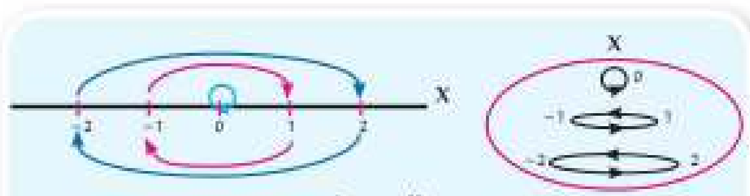
If X = {-2, -1, 0, 1, 2} and R is a given relation on X where a R b means:

«The number a is the additive inverse of the number b» for each of a , b $\in$ X

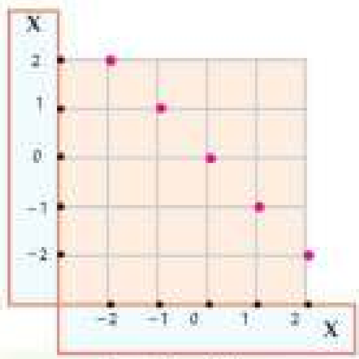
Write the relation R and represent it by an arrow diagram and also by, cartesian diagram.

Solution

$R = \{ (-2, 2), (-1, 1), (0, 0), (1, -1), (2, -2)\}$



Arrow diagram



Cartesian Diagram

Drill

If X = {1, 2, 3}, Y = {12, 21, 47, 52}, and R is the relation from X to Y where a R b means : (a is a digit from the digits of b), for each $a \in X$, $b \in Y$

First: Write R and represent it by an arrow diagram and also, by a cartasian diagram.

Second: Show which of the following relations are correct and why?

1 R 52   2 R 21   3 R 47

For More Exercises, go to MOE website

1 - 3 Functions (Mapping)

What you'll learn

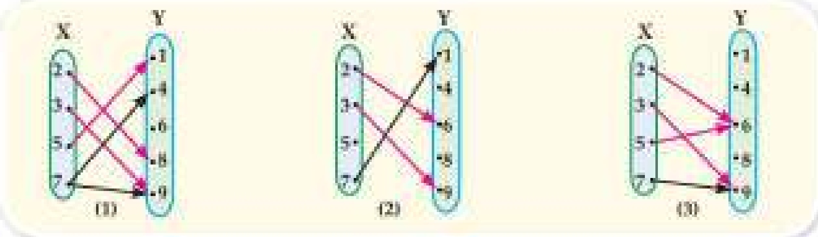
- ☆ Concept of the function.
- ☆ Symbolical expression of the function.

Key terms

- ☆ Functions.
- ☆ Domain
- ☆ Codomain
- ☆ Range

Think and Discuss

The following figures represent three relations from X to Y.



1. Write each relation and represent it by a cartesian diagram.
2. Which of these relations satifies the following condition: each element of X is connected to only one element of Y.

Definition:

A relation from X to Y is said to be a function if:

Each of the elements of X appears only once as a first projection in one of the ordered pairs of the relation.

The Symbolic representation of the function:

1. The function is denoted by one of the following symbols: f or m or Q or... and the function f from the set X to the set Y.

   is written mathimatecally as:

   $f : X \rightarrow Y$ and is read as: «f is a fuction from X to Y».

Notes:

1. If f is a function from X to itself, we say that f is a function on X.
2. If the ordered pair (x, y) belongs to the function, then the element y is called the image of the element x by the function f, and we express it by one of the following two forms:

   $f : x \mapsto y$ is read as : the function: f maps x to y

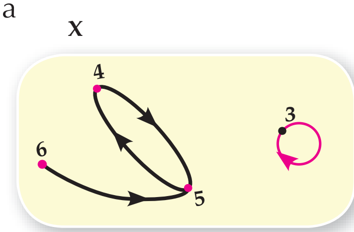
   Or $f(x) = y$ it is read as: f is a function where f (x) = y

Example 1

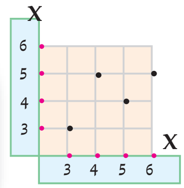
If f is a function on X where: X = {3, 4, 5, 6} and f (3) = 3, f (4) = 5, f (5) = 4, f (6) = 5.
Represent f by an arrow diagram and also, by a cartesian diagram.

Solution

f = { (3, 3), (4, 5), (5, 4), (6, 5)}



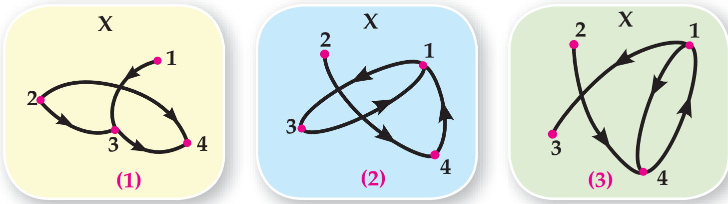
Arrow digaram



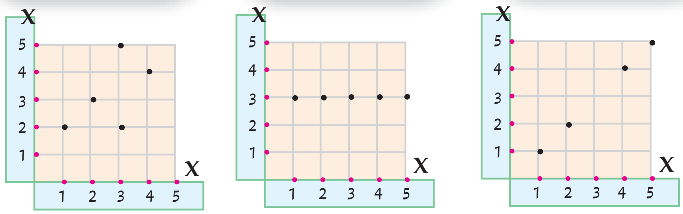
Cartesian diagram

Drill

1. If X = {1, 2, 3, 4} which of the following arrow diagrams represent a function on the set X?



2. Which of the following cartesian diagrams represent a fuction from X to X.

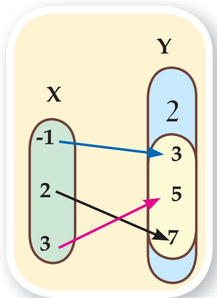


(1) (2) (3)

Think: **Is every relation a function? Explain your answer and give examples.**

The Domain, the codomain and the range

If f is a function from X to Y.
i. e: f : X → Y , then
The set X is called the domain of the function f.
The set Y is called the codomain of the function f.
The set of images of the elements of the domain of X by the function f, is called the range of the function.



For example: If f : X → Y.

, X = { - 1, 2, 3}, Y = {2, 3, 5, 7}, f = { (-1, 3), (3, 5), (2, 7)} then:

1. The domain of the function f is the set X = { - 1, 2, 3}
2. The codomain of the function f is the set Y = {2, 3, 5, 7}
3. The range of the function f is the set of the images of the elements of X by the function f and equal to { 3, 5, 7}.

Note that: The range is a subset of the codomain of the function.

Example 2

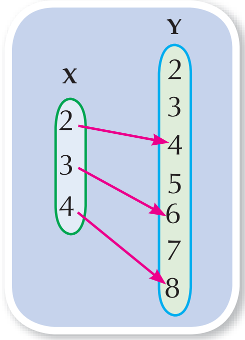
If X = {2, 3, 4}, Y = {y : y ∈ N, 2 ≤ y < 9} where N is the set of natural numbers, and R is a relation from X to Y where a R b means: « $a = \frac{1}{2}b$» for each of a ∈ X, b ∈ Y, write R and represent it by an arrow diagram show that R is a function from X to Y and find its range.

Solution

Y = {2, 3, 4, 5, 6, 7, 8} , R = {(2, 4), (3, 6), (4, 8)}

R is a function because every element of the X has only one arrow coming out to one element of Y.

The function range = {4, 6, 8}



For More Exercises, go to MOE website

1 - 4

Polynomial functions

Think and Discuss

In the functions f: R → R , $f_1(x) = 5$

f: R → R , $f_2(x) = 3x - 8$

f: R → R , $f_3(x) = 4x^2 - 5x + 8$

We notice that:

1. The domain and the codomain of the function is the set of the real numbers R.
2. The rule of function (image of x) is a term or an algebraic expression.
3. What the power of the variable x in the previous functions?

Definition

The function f : R → R where:

$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ where $a_0, a_1, a_2, \dots a_n \in R$
$n \in N$, $a_n \neq 0$, is called a polynomial of degree n.

And thus: the degree of the polynomial is the highest power of the variable in the function rule.

Drill

1. Which of the following functions represents polynomial:

   A $f_1(x) = x^3 + x^2 + 3$ B $F_2(x) = x^3 + \frac{1}{x} + 7$

   C $f_3(x) = x^2 + \sqrt{x} + 8$ D $F_4(x) = x(x + \frac{1}{x} - 2)$

2. If f : R → R then mention the degree of the function in the following:

   A $f(x) = 3 - 2x$ B $f(x) = x^2 - (x^2 - 3)$

   C $f(x) = x(x - 2x^2)$ D $f(x) = x^2 (x - 3)^2$

What you'll learn

- ☆ The linear function and its graphical representation.

Key terms

- ☆ Polynomial function.
- ☆ Linear function.
- ☆ quadratic Function
- ☆ The graphical representation of function.

Example 1

If f (x) = x^2 - x + 3 then find: f (-2) , f (0) , f ($\sqrt{3}$)

Solution

$\because$ f (x) = x^2 - x + 3 $\therefore$ f (-2) = $(-2)^2$ - (-2) + 3 = 4 + 2 + 3 = 9

f (0) = 3 , f ($\sqrt{3}$) = $(\sqrt{3})^2 - \sqrt{3} + 3 = 6 - \sqrt{3}$

Drill

If f(x) = x^2 - 3 x , g (x) = x - 3

A Find f ($\sqrt{2}$) + 3 g ($\sqrt{2}$)

B Prove that f (3) = g (3) = 0

Linear function

Definition

The function f: R → R where f(x) = a x + b , a, b $\in$ R, a $\neq$ 0 this function is called a linear function or a function of the first degree.

The graphical representation of the linear fuction:

Example 2

Represent graphically the function f: R → R, f(x) = 2 x - 3

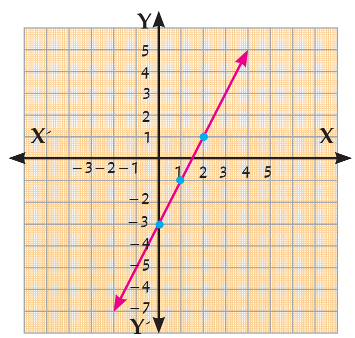
Solution

$\because$ f (x) = 2 x - 3

$\therefore$ f (0) = 0 - 3 = -3, f (1) = 2 - 3 = -1 , f (2) = 4 - 3 = 1

You can put these ordered pairs in a table as the following:

x	0	1	2
y = f(x)	-3	-1	1



The ordered pairs of the cartesian product of R × R is represented on the square net.

Remarks:

1. It is enough to find two ordered pairs belonging to the function, it is prefered to find third ordered pairs to check the graph.
2. If $f : R \rightarrow R$, $f(x) = a x$, where $a \neq 0$ then it represents graphically by a straight line passing through the origin (0, 0)

Drill

Reprsent graphically each of the following functions:

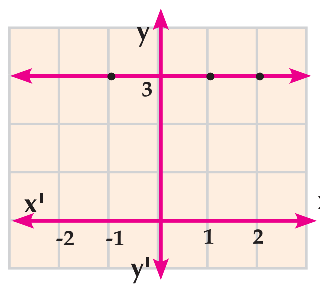
1. $f: f(x) = x + 2$
2. $g : g(x) = 3x$
3. $\ell : \ell(x) = -2x$

Special case: If $f : R \rightarrow R$, $f(x) = b$ where $b \in R$ then f is called a constant function.

For example: $f(x) = 3$

and it is written as $y = 3$

x	-1	1	2
y = f(x)	3	3	3



It is represented by a straight line paralled to the x-axis.

Drill

Represent the following functions graphically:

1. $f(x) = 5$
2. $f(x) = -4$
3. $f(x) = 0$
4. $f(x) = 2\frac{1}{2}$

The quadratic function

The function $f : R \rightarrow R$ where $f(x) = a x^2 + b x + c$, a, b, c are real numbers, $a \neq 0$ is called a quadratic function and it is a function of second degree.

The graphical representation of the quadratic function.

Example 3

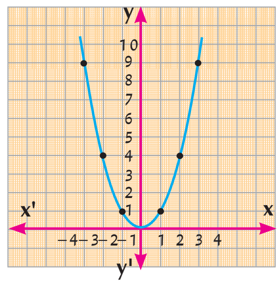
Represent graphically the quadratic function f, where $f(x) = x^2$, $x \in R$ consider $x \in [-3, 3]$

Solution

Identify some of the ordered pairs (x , f (x)) which belong to the function f where $x \in R$ and that the interval is [- 3 , 3] gives some possible values the variable x.
f(- 3) = 9 ,f (- 2) = 4, f (- 1) = 1, f (0) = 0, f (1) = 1, f (2) = 4, f (3) = 9

Put these ordered pairs in a table as follows:

x	3	2	1	0	-1	-2	-3
y = f(x)	9	4	1	0	1	4	9



Identify in the cartesian plane the points which represent these ordered pairs, then draw a curve passing through these points.

Notice that:

1. The curve of the function f is symmetrical about the y-axis and the equation of the symmetrical axis is x = 0
2. The coordinate of the vertex of the curve is (0, 0), and the minimum value of the function = 0

Generylly:

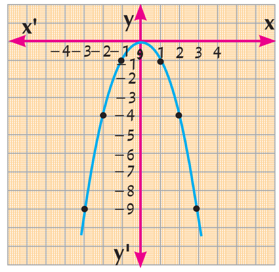
The function $f(x) = ax^2 + bx + c$, where a,b,c are real numbers, $a \neq 0$ has the following properties:

1. The coordinates of the vertex of the curve = $(\frac{-b}{2a}, f(\frac{-b}{2a}))$
2. The curve of the function is open upwards U when the coefficient of x^2 is positive ($a > 0$) and in this case the function has a minimum value equals $f(\frac{-b}{2a})$
3. The curve of the function is open downward ∩ when the cofficient of x^2 is negative ($a < 0$) and in this case the function has a maximum value equals $f(\frac{-b}{2a})$
4. The curve of the function is symmetric about the vertical line which passes through the vertex point of the curve and the equation of this line is:
$X = \frac{-b}{2a}$ and this line is called the axis of symmetry of the function.

Example 4

Represent graphically the quadratic function f where:

$f(x) = -x^2$, $x \in R$ where $x \in [-3, 3]$



Solution

Repeat the previous solution steps:

x	3	2	1	0	-1	-2	-3
y = f(x)	-9	-4	-1	0	-1	-4	-9

From the previous drawing, we notice:

1. The curve of the function f is symmetrical about the y-axis, thus, the equation of the symmetrical axis is x = 0
2. The coordinate of the vertex of the curve is (0, 0) and the maximum value of the function = 0

Algebra

Unit 2: Ratio, proportion, Direct Variation and Inverse Variation

Do you Know ?

The weight of a human body on the surface of the moon equals $\frac{1}{6}$ of the weight on the surface of Earth.

Imagine you are going to a trip on the moon: What will your weight be?

2 - 1 Ratio

What you'll learn

- ☆ Ratio.
- ☆ Properties of ratio.

Key Terms

- ☆ Antecedent.
- ☆ Consequent.
- ☆ The two terms of the ratio.

Think and Discuss

We have learned in the previous phases the subject of ratio and that ratio is: a comparison between two quantaties.

for example: If there are 4 boys and 3 girls so the ratio between the number of boys to the number of girls can be written as 4 to 3 or $\frac{4}{3}$ Generally, if a and b are two real numbers

Then, the ratio between the two numbers a and b

Can be written as a to b or a:b or $\frac{a}{b}$.

a will be called an antecedant and b is consequent and a and b together are the two terms of ratio.

Complete and answer the questions:

1. Is the ratio changed if each of its two terms is multiplied in a fixed amount not equalling to zero?

   $\frac{3}{5} \overset{?}{=} \frac{3 \times ....}{5 \times ....}$

2. Is the ratio changed if you add a real number to each of its two terms?

   $\frac{2}{3} \overset{?}{=} \frac{2 + ...}{3 + ...}$

3. If $\frac{a}{b} = \frac{3}{5}$, Is a = 3, b = 5 for the values of a and b?

Example

Find the number which if added to the two terms of ratio 7 : 11 it will be 2 : 3

Solution

Consider the number is x.

$\therefore \frac{x+7}{x+11} = \frac{2}{3}$ $\quad\quad$ $\therefore 3(x+7) = 2(x+11)$

$\therefore 3x + 21 = 2x + 22$ $\quad\quad$ $\therefore 3x - 2x = 22 - 21$

$\therefore x = 1$

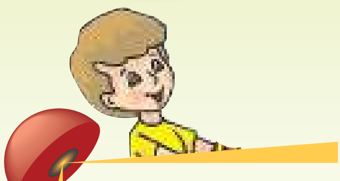
Drill

Find the positive number which if we add its square to each of the two terms of ratio 5 : 11 it becomes 3 : 5.

For More Exercises, go to MOE website

2 - 2

Proportion



What you'll learn

- Proportion
- Properties of proportion
- Continued properties

Key Terms

- Proportion
- First proportional
- Second proportional
- Third proportional
- Fourth proportional
- Extremes
- Means

If $\frac{a}{b} = \frac{c}{d}$ then it's said that a, b, c and d are in proportion.

If a, b, c and d are in proportion, **then** $\frac{a}{b} = \frac{c}{d}$

Definition:

The proportion is the equality of two ratios or more.

In ratio $\frac{a}{b} = \frac{c}{d}$

So, ***a*** is called the **first proportional**, ***b*** is called the **second proportional**, ***c*** is called the **third proportional**, and ***d*** is called the **fourth proportional**.

a and d are called extremes, b and c are called means

The properties of proportion

first: if $\frac{a}{b} = \frac{c}{d}$ ***then:***

1. $a = m\,c$, $b = m\,d$ **where** $m \in R^*$
2. $a\,d = b\,c$ **(product of the extremes equals product of the means)**
3. $\frac{a}{c} = \frac{b}{d}$

Check the previous properties by giving numerical examples of your own

Second: If: $ad = bc$ ***then :*** $\frac{a}{b} = \frac{c}{d}$

$\frac{a}{c} = \frac{b}{d}$

Check the proporties in the following numeric example:

You know that: $4 \times 8 = 2 \times 16$

then: $\frac{4}{2} = \frac{.....}{.....}$, $\frac{4}{16} = \frac{.....}{.....}$

Example 1

If $\frac{x}{y} = \frac{2}{3}$ find the value of the ratio: $\frac{3x + 2y}{6y - x}$

Solution

Consider x = 2m, y = 3m (where m constant ≠ zero)

$\therefore \frac{3x+2y}{6y-x} = \frac{3 \times 2m + 2 \times 3m}{6 \times 3m - 2m} = \frac{12m}{16m} = \frac{3}{4}$

Another Solution

Divide the numerator and denominator on y, then substitute for the value of $\frac{x}{y}$

$\therefore$ The expression $= \frac{3 \times \frac{x}{y} + 2}{6 - \frac{x}{y}} = \frac{3 \times \frac{2}{3} + 2}{6 - \frac{2}{3}}$ ⟶ *Complete* $= \frac{\dots}{\dots} = \frac{\dots}{\dots}$

Example 2

Find the fourth proportional for the numbers 4, 12, 16

Solution

Consider the fourth proportional to be x

$\frac{4}{12} = \frac{16}{x}$

$\therefore 4 \times x = 12 \times 16$ *[product of the extremes = product of the means]*

$\therefore x = \frac{12 \times 16}{4} = 48$ $\therefore$ The fourth proportional = 48

Example 3

Find the number that if added to the numbers 3, 5, 8 and 12 it becomes proportional .

Solution

Consider the number is x i.e. 3 + x, 5 + x, 8 + x, 12 + x **are in proportional**

$\therefore \frac{3+x}{5+x} = \frac{8+x}{12+x}$ $\therefore (5 + x)(8 + x) = (3 + x)(12 + x)$

$\therefore 40 + 13x + x^2 = 36 + 15x + x^2$ $\therefore 15x - 13x = 40 - 36$

$\therefore 2x = 4$ $\therefore x = 2$

Drill

1. A. Find the second proportional of the numbers 2, ... , 4, 6

   B. Find the third proportional of the numbers 8, 6, ... , 12

2. If $\frac{a}{b} = \frac{3}{5}$ find the value of $7a + 9b : 4a + 2b$

Third: If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots$, $m_1, m_2, m_3, \ldots \in R^*$

then: $\frac{a m_1 + c m_2 + e m_3 + \ldots}{b m_1 + d m_2 + f m_3 + \ldots}$ = one of the ratios

For example: If: $\frac{a}{2} = \frac{b}{3} = \frac{c}{4}$ multiply the first two terms of the first ratio by 2, multiply the two terms of the second ratio by -5 and multiplying the two terms of the third ratio by 3, then

$\frac{2a - 5b + 3c}{2 \times 2 - 3 \times 5 + 3 \times 4}$ = one of these ratios

i.e.: $2a - 5b + 3c$ = one of these ratios

Example 4

If: a, b, c and d are proportional quantities, ***then prove that :*** $\frac{3a - 2c}{5a + 3c} = \frac{3b - 2d}{5b + 3d}$

Solution

$\because$ If a , b , c and d are proprotional quantities $\quad \therefore \frac{a}{b} = \frac{c}{d}$

Multiply the first two means by five and the second means by 3, then the sum of antecedents and the sum of consequents = one of these ratios .

$\therefore \frac{5a + 3c}{5b + 3d}$ = one of these ratios **(1)**

Multiply the two terms of ratio by 3 and the second by -2 then the sum of antecendents : the sum of consequents = one of these ratios .

$\therefore \frac{3a - 2c}{3b - 2d}$ = one of these ratios **(2)**

from (1), (2) $\therefore \frac{5a + 3c}{5b + 3d} = \frac{3a - 2c}{3b - 2d}$ $\quad \therefore \frac{3a - 2c}{5a + 3c} = \frac{3b - 2d}{5b + 3d}$ **(Q.E.D)**

Another Solution

Consider $\frac{a}{b} = \frac{c}{d} = m$ where m is a constant expression

$a = b\,m$, $c = d\,m$ and substitute in both sides.

Drill

If $\frac{a}{b} = \frac{c}{d}$ *prove that :*

First: $\frac{a+b}{b} = \frac{c+d}{d}$ **Second:** $= \frac{a-b}{b} = \frac{c-d}{d}$

Hint: Consider $\frac{a}{b} = \frac{c}{d} = m$ where m is a constant expression ≠ zero and complete or in any other way.

Continued proportional

2, 6, 18 are three numbers. Compare between the proportions $\frac{2}{6}, \frac{6}{18}$

1. Is there a relation between $(6)^2$ and the product of 2×18?
2. If you replace the number 6 with (-6) is there a relation between $(-6)^2$ and the product of 2×18?

Definition:

The quantities a, b and c are said to be in continued proportional if:
$\frac{a}{b} = \frac{b}{c}$ **a is called the first proportional , b is called the middle proportional, and c is called the third proportional, where :** $b^2 = ac$ **or** $b = \pm\sqrt{ac}$

Example 5

Find the middle proportional between 3, 27

Solution

The middle proportional $= \pm\sqrt{3\times27} = \pm 9$

Example 6

If b is a middle proportional between a and c, prove that : $\frac{a^2 + b^2}{b^2 + c^2} = \frac{a}{c}$

Solution

b is middle proportional between a and c *i.e.* a, b, c in continued proportional

Consider $\frac{a}{b} = \frac{b}{c} = m$ $\therefore b = c\,m$

$a = b\,m = c\,m \times m = c\,m^2$

L.H.S $= \frac{a^2 + b^2}{b^2 + c^2} = \frac{c^2 m^4 + c^2 m^2}{c^2 m^2 + c^2}$

$= \frac{c^2 m^2 (m^2+1)}{c^2 (m^2 +1)} = m^2$ **(1)**

R.H.S $= \frac{a}{c} = \frac{c\,m^2}{c} = m^2$ **(2)**

From **(1)**, **(2)** we get $\frac{a^2 + b^2}{b^2 + c^2} = \frac{a}{c}$

Another Solution

Consider : $\frac{a}{b} = \frac{b}{c} = m$ $\therefore \frac{a^2}{b^2} = \frac{b^2}{c^2} = m^2$

From the first ratio and the second ratio $m^2 = \frac{a^2 + b^2}{b^2 + c^2}$ = L.H.S

$m^2 = \frac{a}{b} \times \frac{b}{c} = \frac{a}{c}$ = R.H.S

From **(1)**, **(2)** $\therefore \frac{a^2 + b^2}{b^2 + c^2} = \frac{a}{c}$

Drill

If a , b , c and d are in continued proportional . Prove that ; $\frac{a - 2b}{b - 2c} = \frac{3b + 4c}{3c + 4d}$

Hint Let $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = m$

then $c = dm$, $b = dm^2$, $a = dm^3$ complete

For More Exercises, go to MOE website

2 - 3

Direct Variation and Inverse Variation

What you'll learn

- ☆ Direct variation
- ☆ Inverse variation
- ☆ Difference between direct variation and inverse variation.

Key Terms

- ☆ Variation
- ☆ Direct variation
- ☆ Inverse variation

First: Direct variation

Think and Discuss (1)

A car moves at a uniform velocity (V) 15 m/sec. If the covered distance (d) in meter in a time (t) per second to give the relation: $d = v\,t$.

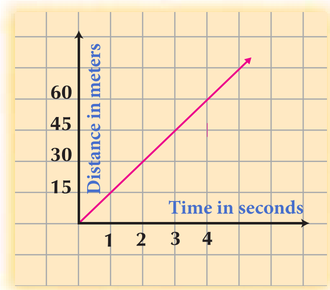
t	1	2	3	4
d	15	30	45	60

A Represent the relation between d and t graphically.

B Does the graphical representation pass through the origin point (0, 0)?

C Find $\frac{d}{t}$ in each case, what do you notice?

We notice from the above :



$\frac{d}{t}$ equals a constant expression which is 15

i.e.: $d = 15\,n$ and is said to be directly due to n and written symbolically $d \propto n$.

Definition:

y is said to be varies directly with x and is written as $y \propto x$ and written $y = m\,x$ (where m constant $\neq 0$). If the variable x takes the two values x_1, x_2 and the variable y takes the two variables y_1, y_2 respectively , then: $\frac{y_1}{y_2} = \frac{x_1}{x_2}$

From the previous, we conclude:

1. The previous relation is a linear relation between x and y and the two variables x and y, and is represented by a straight line passing through the origin point.
2. If $y \propto x$ then $y = m\,x$

   and if $y = m\,x$ then $y \propto x$.

Example 1

If $y \propto x$ then $y = 14$ when $x = 42$, ***then find***

first: the relation between x and y **second:** find the value of y when $x = 60$

Solution

First: $\because y \propto x$ $\therefore y = m\,x$ (where m constant $\neq 0$)

subtitute for the values of x and y in the relation

$\therefore 14 = 42 \times m$ $\therefore m = \frac{14}{42} = \frac{1}{3}$ $\therefore$ the relation is: $y = \frac{1}{3}\,x$

Second: when $x = 60$ $\therefore y = \frac{1}{3} \times 60 = 20$

notice : You can find the relation $\frac{y_1}{y_2} = \frac{x_1}{x_2}$ to find the value of y in the second requirement

Second: Inverse variation

If the area of the rectangle m and one of both dimensions x and the other dimension y, then:

A. ***Write*** the relation between m, x and y.

B. If the area of the rectangle is constant and equal to 30 cm^2 ***complete*** the following table:

x	3	5	6	10
y	………	………	………	………

C. ***Find*** x y in each case . What do you notice?

From the previous , we notice that:

$x\,y = 30$ ***i.e.:*** $y = \frac{30}{x}$ ***i.e.*** y inversely changes with Y and Written symbolically $y \propto \frac{1}{x}$

Similiarly: $x = \frac{30}{y}$ ***i.e.:*** x inverserly changes with Y and Written symbolically $x \propto \frac{1}{y}$

Definition:

y is said to be changed inversely with x and written $y \propto \frac{1}{x}$ if $x\,y = m$ (where m constant $\neq 0$)

and if the variable x takes the two values x_1, x_2 accordingly, the variable y takes the two values y_1, y_2 respectively: $\frac{y_1}{y_2} = \frac{x_2}{x_1}$

From the previous, we conclude that :

1. The previous relation is not a linear relation between the two variables x and y and is not represented by a straight line .
2. If y inversely changes with x ***then:*** $y = \frac{m}{x}$ (where m constant $\neq 0$)
   and if $y = \frac{m}{x}$ then $y \propto \frac{1}{x}$.

Example 2

If $y \propto \frac{1}{x}$ and y = 3 when x = 2

first: ***find*** the relation between x and y. **second:** ***find*** the value of y when x = 1.5.

Solution

$\because y \propto \frac{1}{x}$ $\therefore y = \frac{m}{x}$ (where m constant $\neq 0$)

substitute for the two values of x and y in the relation

$\therefore 3 = \frac{m}{2}$ $\therefore m = 2 \times 3 = 6$

$\therefore$ the relation is : $y = \frac{6}{x}$

when x = 1.5 $\therefore y = \frac{6}{1.5} = 4$

Note: you can find the value of y from the relation $\frac{y_1}{y_2} = \frac{x_2}{x_1}$

Drill

Show **which of the following tables represents the direct variation and which represents the inverse variation and which does not represent the direct variation or inverse variation while mentioning the reason in each case:**

x	y
3	20
5	12
4	15
6	10

x	y
2	9
4	18
12	54
16	72

x	y
5	9
10	18
15	27
25	45

x	y
3	6
-2	-9
-18	1
9	-2

Example 3

Connecting with Physics : If the relation between velocity (v) in (m/sec) and time t (sec) is v = 9.8 t

First: *determine* the kind of variation between v and t .

Second: **A** *Find* the values of v when t = 2 seconds , t = 4 seconds

B *Find* the value of t when v = 24.5 m/sec

Solution

First: $\because$ v = constant × t *i.e.* $v \propto t$ *i.e.* v directly changes with **t**.

Second: **A** when t = 2 then v = 9.8 × 2 = 19.6 m/s

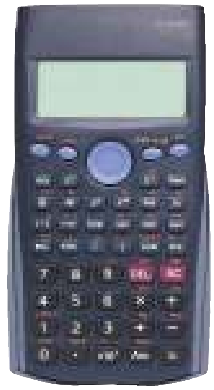
when t = 4 then v = 9.8 × 4 = 39.2 m/s

B When V = 24.5 then 24.5 = 9.8 × t $\therefore t = \frac{24.5}{9.8}$ = 2.5 seconds.

Example 4

Connecting with Geometry: If the height of a right constant cylinder (constant volume) is (h) varies inversely as the square of its radius length r. If the (h) is = 27 cm, when the radius = 10.5 cm, Find (h) when r = 15.75 cm.

Solution :



$\because v \propto \frac{1}{r^2}$ $\therefore v = m \times \frac{1}{r^2}$ (Where m constant $\neq 0$)

$v = 27$ when $r = 10.5$

$\therefore 27 = m \times \frac{1}{(10.5)^2}$ $\therefore m = 27 \times (10.5)^2$ **(1)**

Substitute $\therefore v = 27 \times (10.5)^2 \times \frac{1}{r^2}$ from **(1)**

when $r = 15.75$ cm $\therefore v = 27 \times (10.5)^2 \times \frac{1}{(15.75)^2} = 12$ cm

Use the calculator to find the last step as follows:

27	×	10,5	x^2	-	10,75	x^2	=

For More Exercises, go to MOE website

Statistics

Unit 3: Statistics

Ice Cream stores produce different kinds of ice cream. The manager conducted a survey on the favorite ice cream the consumers prefer.

Statistics helps you select the sample representing the consumers.

3 - 1

Collecting Data

What you'll learn

- Resources of collecting data
- Methods of collecting data
- How to select a sample
- Types of samples

Key terms

- Primary resources
- Secondary resources
- Method of mass population
- Method of sample
- Biased choice
- Random choice sample
- Random sample
- Layer sample

Think and Discuss

The method of collecting data is considered one of the most important phases that statistical research mainly depends on. Collecting data in such scientific methods will lead to get accurate outcomes when doing operations of statistical inferrence and proper decision making.

1. What are the resources of collecting data?
2. How is the method of collecting data identified?

Resources of collecting data

1 *Primary resources (Field resources):*

These are the resources which we originally get data through interviewing or questionaires (survey). This type is distinguished by accuracy. However, it needs time and efforts beside it is highly expensive to conduct such a type.

2 *Secondary resources (historical resources):*

We can get our data from authorities and agancies formally work such as central agency for mobilization and statistics , internet and media This type is a good type of resources such that it saves time and money.

The method of collecting data

The method of collecting data is determined according to the aim and the size of the statistical society under study.

For example: The students of a school represent a statistical society whose value is the student .

First : *Method of mass population :*

It means to collect the data related to the phenomenon of the statistical society. It's used to include all the society such as the population. This type is including all the values and it's unbiased in addition the outcomes are so accurate.

The disadvantages of such a method are ; it needs long time and great efforts. Further more, it costs much money.

Second: ***Mehods of samples:***

It mainly depends upon selecting a sample from the statistical society that it represents.

We conduct reseaches on the sample. The outcomes we get are generalized on the whole society.

Advantages of using methods of samples:

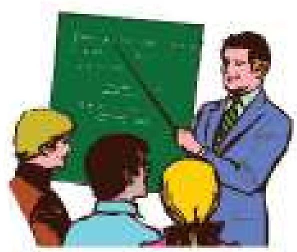
1. It saves time, efforts and money.
2. The only way to collect data about gigantic societies (like fish).
3. The only method to study some limited societies such as:
   - A. Check the patient blood by getting a sample (checking the whole blood leads to death).
   - B. Check the production of a factory producting electric lamps to determine the validity of the lamp. (Know for how long the lamp can be used before getting burned).

Some of the disadvantages of the sample methods are : the outcomes of such type are not accurate if the selected sample doesn't represent all the society well in such a case the sample is **called biased**.

How we select samples and the conditions must be found in getting a sample:

First: the biased selction (samples are not randomly selected)

It means that we select the sample in a way to satisfy the objectives of the research. This is called as the sample delibrate. For example, when we want to know how the students understood a lesson in mathematics we must analyze the outcomes of the test by considering the outcomes of a group of students studied the same topic without the other students this is not a random selection.



Second: Random seclecion (random samples)

It means to select a sample such that the chance of getting any value from the society is equal.

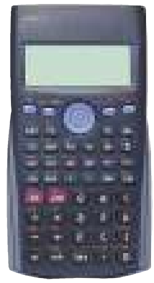
Of the most important types of the random samples :

1. Simple random sample:
   Is the simplest type of samples and it can be get from the homogeneous socities where their selection is related to the size and number of units in the society.

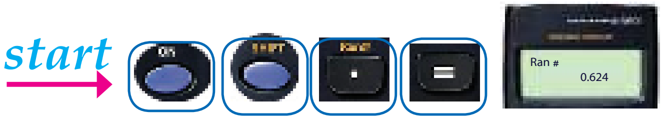
A. **If the size of the society is small:**

When we choose 5 students of a 40-student class, then we can prepare a card for each student on which their names or numbers are written, where all the cards are identical, put them back again in the box and draw a card from the box randomly and return the ball back again. Repeat this experiment till you get the sample needed.

B If the size of the society is big:



suppose we want to select the sample (5 students) from all the students whose numbers 800. The process of selection will be difficult to be done. So, we number the students from 1 to 800, then use the calculator or excel program to give 10 random digits in the field from 0.000 to 0.999 and take out the decimal point to make the field from zero to 999 you can take out the decimal digits which are more than 800 as follows:



Repeat pressing on [=] the appearence of numbers will be successive.

2 5 digits unrepreated are enough to give the digits of the sample for the students.

Layer random sample:

When the society needed to be examined is heterogeneous or made up of qualitative sets that are differenct in characteristics, the society is divided into homogeneous sets according to the characteristics forming it. Each set is called a layer and the researcher selects a random sample which each layer is represented according to its size in the society, such as a sample is called the layer sample .

For example: when we want to study an educational level of a society of 400 persons where the ratio of males to females is 3:2 and we want to select a sample of 50 persons, we must select 30 persons from the male layer and 20 persons from the female layer randomly.

Dispersion

3 - 2

Think and Discuss

You have previously learned the central tendancy (mean - domain - mode) and you used them to calculate a set of data to identify one value describing the trend of these data in centeralization around this value.

If the weekly wages in pounds of two sets of workers A and B in a factory are as follows:

Set A: 170, 180, 180, 230, 240
Set B: 50, 180, 180, 190, 400

1. ***Find*** the mean to the wages of the two sets A and B.
2. ***Compare*** the wages of the two sets A and B. **What do you deduce?**

You know that The mean = $\dfrac{\text{Total of these values}}{\text{Their number}}$

then:

the mean of wages for set A = $\dfrac{170 + 180 + 180 + 230 + 240}{5}$

$= \dfrac{1000}{5}$ = LE 200

The mean of wages of set B = $\dfrac{50 + 180 + 180 + 190 + 400}{5}$

$= \dfrac{1000}{5}$ = LE 200

Compare the wages of the two sets A and B to find :

1. The mean of wages for set A = the mean of wages of set B

= LE 200

2. The median of wages = the mode wage = LE 180 for each set A and B

What you'll learn

- ☆ Dispersions (Range- standard deviation)

Key term

- ☆ Central tendency
- ☆ Mean
- ☆ Dispersion
- ☆ Range
- ☆ Standard deviation

We notice that :

(1) The wages of the two sets are different but both have the same measures of central tendency.

(2) The wages of set A are close so the values are included between 170 and 240 pounds where the wages of set B are divergent so the values are included between 50 and 400 pounds.

i.e. **The wages of set B is more divergent than the wages of set A.**

So When we compare two sets, we must consider the dispersion of the values of both sets and being divergent from each other .

Dispersion: **to any set of values means divergent or the differences between its values. The dispersion is small if the difference between the values are little whereas th dispersion is great if the difference between the values are very big (if the difference between the values are great). When the dispersion is zero, then all the values are equal.**

i.e. the dispersion is a measure that express how much the sets are homogenous

From the previous, we deduce:

To compare two sets of data or more, we must have a measure to the central tendency and another for dispersion for each set.

Dispersions measurements

1 *Range: (The simplest measure of dispersions)*

It is the diference between the greatest value and the smallest value in the set.
Compare the two sets above :

First set: 51, 53, 55, 57, 58, 60

Second set : 42, 45, 47, 49, 52, 92

We find that the range of the first set = 60 - 51 = 9

the range of the second set = 92 - 42 = 50

So the second set is more divergent than the first set

Notice that :

(1) The range is the simplest and easiest method of measuring dispersion.

(2) The range is influenced greatly by the outlier. it is clear that the values of the second set disparses in a range of 50 when we remove the last value (92) from and the range = 52 - 42 = 10 or $\frac{1}{5}$ of the previous range .

(3) Since the range doesn't influence by any value in the set except the greatest and smallest values, it doesn't give a clear picture to the dispersion of the set.

2 *Standard deviation :*

Is the commonest measure of dispersions and the most accurate (under certain conditions) which is the positive square root to the average of **squares deviations of values from the mean.**

i.e.:

$$\text{Standard deviation } \sigma = \sqrt{\frac{\sum (x - \overline{x})^2}{n}}$$

where σ denotes to: (sigma) to tell the standard deviation to the society of data.

$\overline{x}$ (x Bar) denotes the mean of the values of society.

n denotes the number of values .

$\sum$ denotes addition.

First : *calculating the standard deviation to a set of data :*

Example

Calculate the standard deviation for the values : 12, 13, 16, 18, 21

Solution

To calculate the standard deviation , form the table opposite the mean of a set of values

$$\overline{x} = \frac{\text{Total of these values}}{\text{Their numbers}}$$

$$\therefore \overline{x} = \frac{\sum x}{n}$$

$$\overline{x} = \frac{12 + 13 + 16 + 18 + 21}{5} = \frac{80}{5} = 16$$

$$\because \text{ The standard deviation } \sigma = \sqrt{\frac{\sum (x - \overline{x})^2}{n}}$$

	x	$x - \overline{x}$	$(x - \overline{x})^2$
	12	12 - 16 = -4	16
	13	13 - 16 = -3	9
	16	16 - 16 = 0	zero
	18	18 - 16 = 2	4
	21	21 - 16 = 5	25
Sum	80		54

$$\therefore \text{ The standard deviation } \sigma = \sqrt{\frac{54}{5}} = \sqrt{10.8} = \simeq 3.286$$

Second: ***Calculating the standard deviation to a frequency distribution :***

For any frequency distribution :

the standard deviation $\sigma = \sqrt{\dfrac{\sum (x-\bar{x})^2 k}{\sum k}}$

where : **x** represents the value or the center of the set ,

k represents the frequency of the value or the set

$\sum k$ is the total of frequency , $\bar{x}$ is the mean $= \dfrac{\sum x\,k}{\sum K}$

Example

The following are the frequency distribution for a number of defective units which found in 100 boxes of manufactured units :

Number of defective units	zero	1	2	3	4	5
Number of boxes	3	16	17	25	20	19

Find the standard deviation to the defective units .

Solution

Consider the number of defetive units **(x)** and the number of the corresponding boxes **(k)** to calculate the standard deviation to the defective units form the following table **:**

The mean $\bar{x}$

$= \dfrac{\sum x \times k}{\sum k} = \dfrac{300}{100} = 3$

The standard variation σ

$= \sqrt{\dfrac{\sum (x-\bar{x})^2 k}{\sum k}}$

$= \sqrt{\dfrac{204}{100}} \simeq 1.428$ units

Number of defective units	Number of boxes k	$x \times k$	$x - \bar{x}$	$(x - \bar{x})^2$	$(x - \bar{x})^2 k$
zero	3	zero	-3	9	27
1	16	16	-2	4	64
2	17	34	-1	1	17
3	25	75	zero	zero	zero
4	20	80	1	1	20
5	19	95	2	4	76
Total	100	300			204

Drill

The following frequency distribution shows the goals scored in a number of football matches:

Number of goals	Zero	1	2	3	4	5	6
Number of matches	1	4	6	9	5	3	2

Find the standard deviation for the numbers of goals.

Example

The following frequency distribution shows the marks of 40 students in an exam:

Sets	0-	4-	8-	12-	16-20	Total
Frequency	2	5	8	15	10	40

Find the standard deviation for this distribution.

Solution

1. Find the centers of sets x

   Then: The center of the first set $= \frac{0+4}{2} = 2$

   The center of the second set $= \frac{4+8}{2} = 6$

   and then record them in the third column.

2. Multiply the centers of sets × its corresponding frequencies: i.e. **x** × **k** and record in the fourth column. Then find the mean $\overline{x} = \frac{\sum x\,k}{\sum k}$

3. Find the deviation of the center of each set (x) from the mean i.e. find $(x - \overline{x})$

4. Find squares of deviations of the center of each set from the mean: i.e $(x - \overline{x})^2$

5. Find the product of the square deviation of the center of each set from the mean × frequency of this set; i.e. $(x - \overline{x})^2 \times k$

6. Calculate the standard deviation $\sigma = \sqrt{\frac{\sum (x-\overline{x})^2\,k}{\sum k}}$

Sets	Frequency (k)	Center of sets (x)	$x \times k$	$x - \bar{x}$	$(x - \bar{x})^2$	$(x - \bar{x})^2 k$
0-	2	2	4	- 10.6	112.36	224.72
4-	5	6	30	- 6.6	43.56	217.80
8-	8	10	80	- 2.6	6.76	54.08
12-	15	14	210	1.4	1.96	29.40
16-20	10	18	180	5.4	29.16	291.60
Sets	40		504			817.6

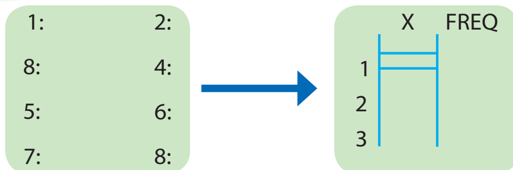
The mean $\bar{x} = \frac{504}{40} = 12.6$

The standard deviation $\sigma = \sqrt{\frac{817.6}{40}} = \sqrt{20.44} \simeq 4.52$ marks

You can use the calculator [Fx-82ES, Fx-83ES, Fx-85ES, Fx-300ES, Fx-350ES] to check the standard deviation.

First: State the calculator on statistical system to enter data

On Mode 2 (Stat) 1 (1-VAR)

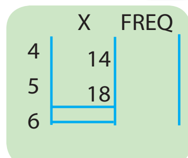


Second: Calculate the standard deviation to the frequency distribution (Example 2)

1. **Enter the centers of sets 2, 6, 10, 14, 18**

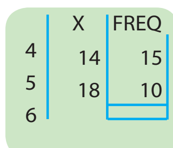
On Mode 2 (Stat) 1 (1-VAR)

2 = 6 = 1 0 = 1 4 = 1 8 =



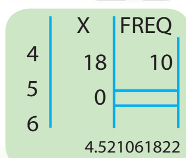
2. **Go to the initial of the second column (FREQ) and enter the coresponding frequency for each set 2, 5, 8, 15, 10**

► ▲ 2 = 5 = 8 = 1 5 = 1 0 =



3. **Recall sum (standard deviation) then $\sigma \simeq$ 4.521**

Shift 1 5 (VAR) 3 (X σn) = ▲



4. **Go back to the original system and switch off the calculator.**

Notice that :

(1) The standard deviation is affected by the deviations of all the values and its value is affected by the outlier.

(2) The standard deviation has the same measuring units of the original data , so it is used to compare the dispersion of sets which have the same measuring units when the mean is equal in the mean . The set which contains more standard deviation is more dispersion.

Drill

The two frequency tables represent the marks of students of two classes A and B in third prepartory in an exam:

Class A

Sets of marks	0-	10-	20-	30-	40-50	Sum
Number of students	2	5	11	15	7	40

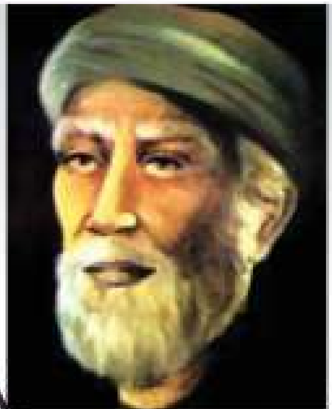
Class B

Sets of marks	0-	10-	20-	30-	40-50	Sum
Number of students	2	3	18	7	10	40

1. *Represent* both distribution using the frequency polygon in one figure.
2. *Find* the mean and standard deviation for both frequency distributions.
3. Which class is more homogeneous in getting marks?

For More Exercises, go to MOE website

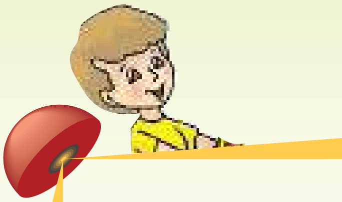
Unit 4 : Trigonometry



Trigonometry is a branch of mathematics that concerned with studying relationships among sides and angles of triangles. Ancient Egyptians were the first to apply the rules of trigonometry in constructing their immortal pyramids and temples as well as applying in astronomy and in calculating geographical distances. Further more Babylonians had also measured the angles in degrees, minutes and seconds. Abou Alryhan Albyrony had settled a table for tangents of angles . Al tousi had deduced that the cosinese of the angles are in proportion with the legs opposite. West civilization learned about what Arab and Muslims wrote through translating the Arab astronomy books by the German Scientist Yohan Muller

Abou Alrayhan Albyrony
Was a great scientist born in
Algorithm in 973 and died in
1048 AD

4 - 1 The main trigonometrical ratios of the acute angle



What you'll learn

- ☆ Ratios of the acute angle in the right angled triangle.

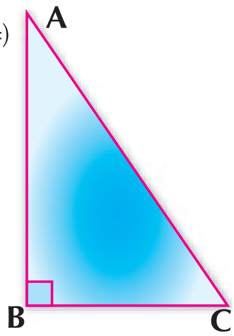
Key Terms

- ☆ Circular measure
- ☆ Sine angle
- ☆ Cosine angle
- ☆ Tangent angle

Think and Discuss

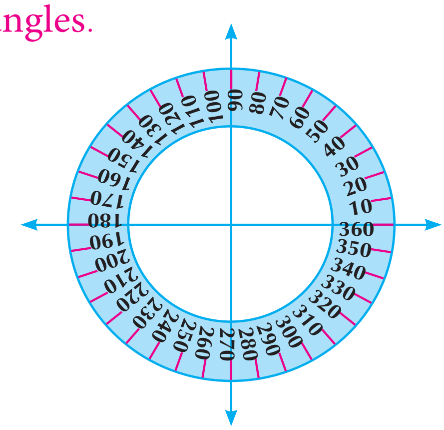
Use the right angled triangle a, b and c shown in the figure opposite ,

Complete using one of these symbols (> or < or =)



1. If m(∠C) > m (∠A) then AB ...... BC
2. $\frac{AB}{AC}$ 1
3. $\frac{AC}{BC}$ 1
4. $\frac{AB}{AC} \div \frac{BC}{AC}$ ...... $\frac{AB}{BC}$
5. $\frac{AB}{AC} + \frac{BC}{AC}$ 1
6. $\frac{(AB)^2}{(AC)^2} + \frac{(BC)^2}{(AC)^2}$ 1

Circular measure of the angles.



We studied that the product of the accummlative angles around a point equals 360°, if you divide the angles into four equal quadrants then a quadrant includes 90° (right angle); and a degree is the circular measuring unit. Similarly, parts of a degree are as follows:

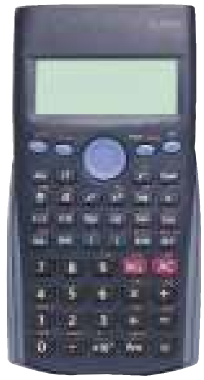
degree = 60 minutes , minute = 60 seconds

35 degrees , 24 minutes ,42 seconds written

as the follows : **35°, 24` ,42``** you can convert minutes and seconds into parts of the degree in one of the following two ways:

First: Convert **24`** to minutes $24` = \frac{24}{60} = 0{,}4$, and convert **42``** first into minutes then into degrees : $42`` = \frac{42}{60} = 0{,}7`$

$0{,}7` = \frac{0{,}7}{60} = 0{,}0116667$



then the sum is $35^\circ\ 24`\ 42`` = 35 + 0{,}4 + 0{,}0116667 = 35{,}4116667^\circ$

Second: Use the calculator as follows :

The sum is : 35,4116667°**equals** 35 [°′″] 24 [°′″] 42 [°′″]

Similarly, convert the fractions of degree into minutes and seconds.

For example: 54. 36° You can convert into degrees , minutes and seconds by using the following keys:

The sum is : $54^\circ\ 21`\ 36``$ [°′″] [SHIFT] [=] 54,36

Drill

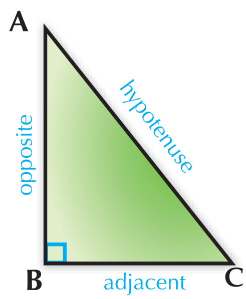
1 **Write each of the following angles in degrees:**

A 76° 16`  B 45° 3` 56``  C 85° 38` 8`` D 65° 26` 43``

2 **Write each of the following angles in degrees, minutes and seconds.**

A 34,6° B 78,08° C 56,18° D 83,246°

The main trigonometrical ratios of the acute angles:



The figure opposite:

The triangle ABC represents the right angled triangle at B where A and C are two complementary acute angles, the side opposite angle C is called leg opposite, the side adjacent to angle C is called adjacent and the side opposite to the right angle is called hypotenuse.

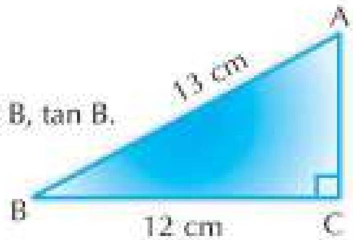
We will know the trigonometrical ratios of the acute angles as the following :

1. **Sine angle**: is denoted by the symbol [sin] .
2. **Cosine angle**: is denoted by the symbol [cos] .
3. **Tangential angle**: is denoted by the symbol [tan] .

sin C	=	$\frac{\text{opposite}}{\text{hypotenuse}}$	=	$\frac{AB}{AC}$
cos C	=	$\frac{\text{adjacent}}{\text{hypotenuse}}$	=	$\frac{BC}{AC}$
Tan C	=	$\frac{\text{opposite}}{\text{adjacent}}$	=	$\frac{AB}{BC}$

Example

1 ABC is a right angled triangle at C, A B = 13 cm, B C = 12 cm



A *Find* the length $\overline{AC}$

B *Find each of the following:* sin A, cos A, tan A, sin B, cos B, tan B.

C *Prove that :* sin A cos B + cos A sin B = 1

D *Find :* $1 + \tan^2 A$

Solution

A $\because$ ABC is a right angled triangle at C $\quad\therefore (AC)^2 = (AB)^2 - (BC)^2$

$\therefore (AC)^2 = (13)^2 - (12)^2 = (13 + 12)(13 - 12) = 25$

$\therefore AC = 5$ cm

B $\sin A = \frac{12}{13}$, $\cos A = \frac{5}{13}$, $\tan A = \frac{12}{5}$, $\sin B = \frac{5}{13}$, $\cos B = \frac{12}{13}$, $\tan B = \frac{5}{12}$

C **The right side** = sin A cos B + cos A sin B

$$\frac{12}{13} \times \frac{12}{13} + \frac{5}{13} \times \frac{5}{13} = \frac{144}{169} + \frac{25}{169} = \frac{144 + 25}{169} = 1$$

D $1 + \tan^2 A = 1 + (\frac{12}{5})^2 = 1 + \frac{144}{25} = \frac{169}{25}$

Drill

ABC is a triangle in which AB = AC =10 cm , BC = 12 cm, drawn $\overrightarrow{AD} \perp \overline{BC}$, $\overrightarrow{AD} \cap \overline{BC} = \{D\}$

First: find the value of sin (CAD) , cos (CAD) , tan (CAD)

Second: Prove that : **A** $\sin^2 C + \sin^2 C = 1$ **B** sin B + cos C > 1

For More Exercises, go to MOE website

4 - 2 The main trigonometrical ratios of some angles

What you'll learn

- ☆ Finding the trigonometric ratios of angles
- ☆ (30° , 45° , 60°)

Key Terms

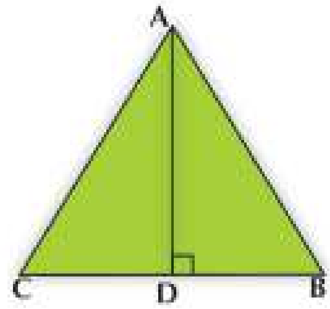
- ☆ Trigonometric ratios
- ☆ Special angles

Think and Discuss

1 In the figure opposite :

A B C is an a equilateral triangle of side length 2L, is and $\overline{AD} \perp \overline{BC}$

Complete:



1. m (∠B) = ...°
2. m (∠B A D) = ...°
3. B D = ... and A D = ... (by L)
4. B D : A B : A D = ... : ...... : ......

From the previous, we notice that :

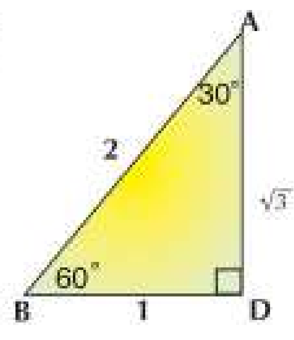
the triangle A B C is 30° , 60° and the ratio between the lengths of the triangle sides are BD : AB : AD = 1 : 2 : $\sqrt{3}$ So you can find the basic trigonometric ratios of the angles 30° , 60° as **follows:**

$\sin 30° = \frac{BD}{AB} = \frac{1}{2}$ and $\cos 30° = \frac{AD}{AB} = \frac{\sqrt{3}}{2}$

$\tan 30° = \frac{BD}{AD} = \frac{1}{\sqrt{3}}$

$\sin 60° = \frac{AD}{AB} = \frac{\sqrt{3}}{2}$

$\cos 60° = \frac{BD}{AB} = \frac{1}{2}$ and $\tan 60° = \frac{AD}{BD} = \sqrt{3}$



Drill

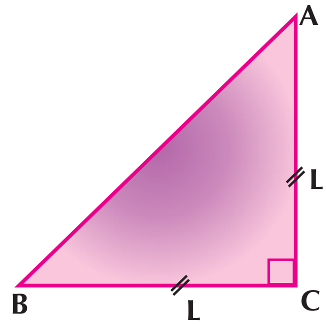
Complete: $\sin 30° = \cos$ ......°, $\tan 30° = \frac{1}{...}$, $\cos 30° = \sin$ ....°

Think and Discuss

1 **In the figure opposite:**

ABC is an isosceles triangle and a right angled triangle at C. The length of each leg is L.

Complete:



1 $m(\angle A) = ...^\circ$, $m(\angle B) = ...^\circ$

2 $\because (AB)^2 = (AC)^2 + ...$ $\therefore (AB)^2 = L^2 + ...$

3 $AC : BC : AB = ... : .... : ....$

$\therefore (AB)^2 = 2L^2$ $\therefore AB = \sqrt{2}\ L$

From the previous, we notice that :

ABC is a triangle in which $m(\angle A) = m(\angle B) = 45^\circ$ and the ratio between the lengths of its sides are $AC : BC : AB = 1 : 1 : \sqrt{2}$ So you can find the trigonometrical ratios of the angle 45° **as follows:**

$\sin 45^\circ = \frac{AC}{AB} = \frac{1}{\sqrt{2}}$ and $\cos 45 = \frac{BC}{AB} = \frac{1}{\sqrt{2}}$, $\tan 45^\circ = \frac{AC}{BC} = 1$

You can put the previous trigonometrical ratios in the following table:

m angle / ratio	30°	60°	45°
Sin	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$
Cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
Tan	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	1

Remarks:

1 **From the previous, we find that :** **(sine)** any angle equals **(cosine)** the supplementary angle of this angle and vice versa .

for example: $\sin 30^\circ = \cos 60^\circ$, $\cos 30^\circ = \sin 60^\circ$ and $\sin 45^\circ = \cos 45^\circ$.

2 For any angle A : $\tan A = \frac{\sin A}{\cos A}$.

Example

1 **Find the value of the following :**

A $\cos 60^\circ \sin 30^\circ - \sin 60^\circ \tan 60^\circ + \cos^2 30^\circ$

B $\dfrac{\cos^2 60^\circ + \cos^2 30^\circ + \tan^2 45^\circ}{\sin 60^\circ \tan 60^\circ - \sin 30^\circ}$

Solution

A The expression $= \cos 60^\circ \sin 30^\circ - \sin 60^\circ \tan 60^\circ + \cos^2 30^\circ$

$$= \frac{1}{2} \times \frac{1}{2} - \frac{\sqrt{3}}{2} \times \sqrt{3} + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} - \frac{3}{2} + \frac{3}{4} = -\frac{1}{2}$$

B The expression $= \dfrac{\cos^2 60^\circ + \cos^2 30^\circ + \tan^2 45^\circ}{\sin 60^\circ \tan 60^\circ - \sin 30^\circ} = \dfrac{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + (1)^2}{\frac{\sqrt{3}}{2} \times \sqrt{3} - \left(\frac{1}{2}\right)} = \dfrac{\frac{1}{4} + \frac{3}{4} + 1}{\frac{3}{2} - \frac{1}{2}} = \dfrac{1+1}{1} = 2$

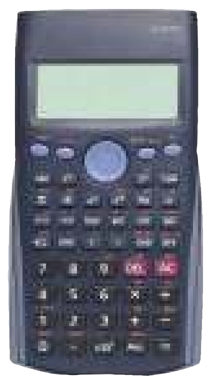
Drill

Prove that:

A $\sin^2 30^\circ = 5 \cos^2 60^\circ - \tan^2 45^\circ$

B $\tan^2 60^\circ - \tan^2 30^\circ = (1 + \tan 60^\circ \tan 30^\circ) \div \cos^2 30^\circ$

Example



2 **Find the following trigonometrical ratios :**

$\sin 43^\circ$, $\cos 53^\circ 28'$, $\tan 64^\circ 37' 49''$

Roundig the sum to the nearest four decimal numbers .

Solution

Start → [sin] 43 = $\qquad \sin 43^\circ \simeq 0,6820$

Start → [cos] 53 [°'"] 28 [°'"] = $\qquad \cos 53^\circ 28' \simeq 0,5953$

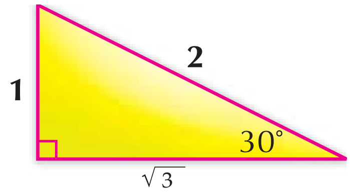
Start → [tan] 64 [°'"] 37 [°'"] 49 [°'"] = $\qquad \text{Tan } 64^\circ 37' 49'' \simeq 2,1089$

Finding the angle given its trigonometrical ratio :

You learned that if you have a given angle, you can find its trigonometrical ratios.

For example: If the measure of an angle is 30° then sin 30° $= \frac{1}{2}$ and similarly, if the angle measure is 33°, then sin 33° = 0,544639035

sin 33° = 0,544639035



Now, we want to identify the angle given its trigonometerical ratio.

for example: If cos C = 0,544639035 find the value of C .

Use the calculator as follows :

Start → SHIFT sin 0,544639035 = °''' 33°

Example

3 **Find m (∠E) in each of the following :**

sin E = 0,6 , cos E = 0,6217 , tan E= 1,0823

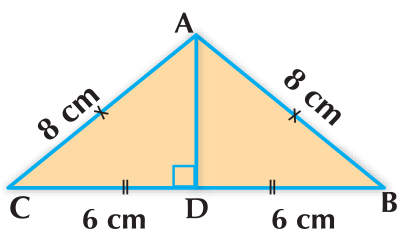
Solution

∵ sin E = 0,6 ∴ m (∠E) = 36° 52` 12`` °''' = 0,6 sin SHIFT

∵ cos E = 0,6217 ∴ m (∠E) = 51° 33` 35`` °''' = 0,6217 cos SHIFT

∵ tan E = 1,0823 ∴ m (∠E) = 47° 15` 48`` °''' = 1,0823 tan SHIFT

4 **Connecting with Geometry:** ABC is an isosceles triangle in which AB = AC = 8 cm and BC = 12 cm .



Find : First: m (∠B)

Second: The area of the surface of the triangle to the nearest two decimal numbers.

Solution

Draw $\overrightarrow{AD} \perp \overline{BC}$

∵ The triangle ABC is an isosceles triangle.

∴ D the midpoint of $\overline{BC}$ and BD = CD = 6 cm

∵ $\cos B = \frac{6}{8} = \frac{3}{4} = 0.75$

Using the calculator :

SHIFT cos 0.75 = °'"

∴ m (∠B) = 41° 24' 35" *(Q.E.D 1)*

To find the surface area of the triangle : find AD **(From Phythegoran's theorem)**

∵ $(AD)^2 = (AB)^2 - (BD)^2$

∴ $(AD)^2 = 64 - 36 = 28$ ∴ $AD = 2\sqrt{7}$

∵ The area of the triangle ABC $= \frac{1}{2} \times BC \times AD = \frac{1}{2} \times 12 \times 2\sqrt{7}$

$= 12\sqrt{7}\ cm^2 \simeq 31.75 cm^2$ *(Q.E.D. 2)*

Another solution for the second part:

∵ $\sin B = \frac{AD}{AB}$ ∴ $\sin B = \frac{AD}{8}$

∴ AD = 8 sin (41° 24' 35") (1)

The area of the triangle ABC $= \frac{1}{2} \times$ BC ×AD **substitute from (1) in this relation**

∴ The area of the triangle ABC $= \frac{1}{2} \times 12 \times 8 \sin (41° 24' 35'') \simeq 31.75 cm^2$.

Use the calculator as follows :

start → 1 ÷ 2 × 12 × 8 × sin 41 °'" 24 °'" 35 °'" =

Drill

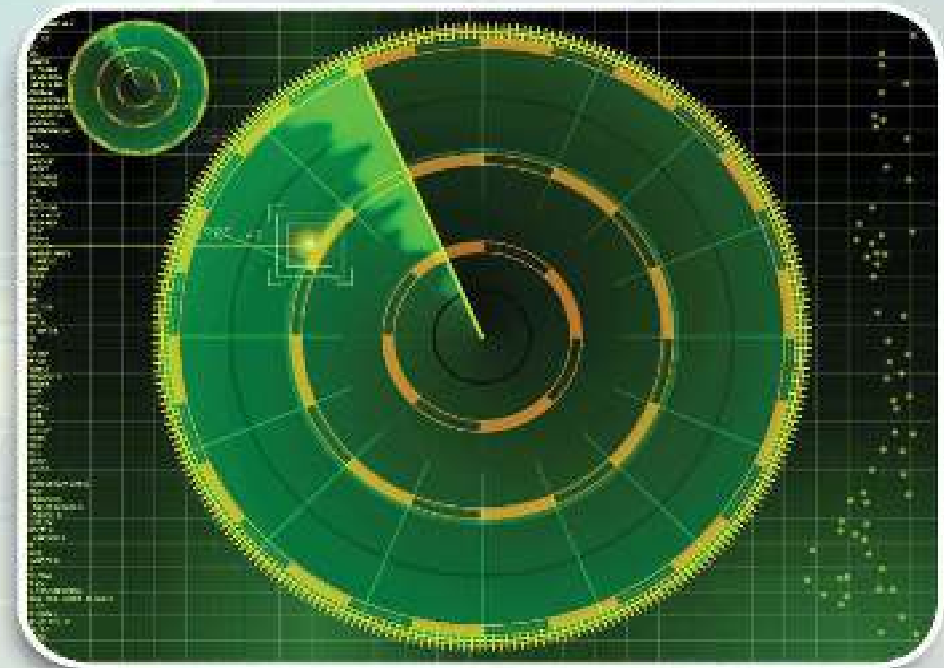
Complete the following :

1. If $\sin X = \frac{1}{2}$ where X is an acute angle then m (∠X) = ...
2. If $\sin \frac{X}{2} = \frac{1}{2}$ where X is an acute angle then m (∠X) = ...
3. sin 60° + cos 30° - Tan 60° = ...
4. If tan (X + 10) = $\sqrt{3}$ where X . IS an acute angle then m (∠X) = ...
5. If tan 2 X = $\sqrt{3}$ where X is an acute angle then m (∠X) = ...

For More Exercises, go to MOE website

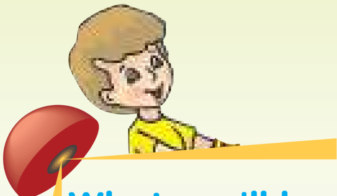
Coordinate geometry

Unit 5: Coordinate geometry



The Radar is used for identifying the range, height, direction and velocity of moving objects like airplanes and ships.

The radar tower receives the reflected waves. The radar screens can determine the coordinates of the target's location (airplane-ship- ....).



5 - 1 Distance between two points

What you'll learn

☆ Finding the distance between two points by using the distance rule.

Key terms

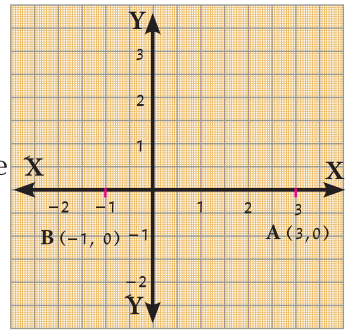
☆ Coordinate plane
☆ Ordered pair
☆ Distance between two points.

Think and Discuss

You represented the ordered pair on the coordinate plane.
Now can you find the distance between the pairs of the following points?

1. A (3, 0) , B (-1, 0)
2. C (0, -3), D (0, -1)
3. M (3, 2), N (7, 5)

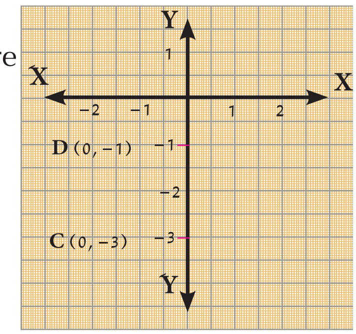
From the previous, we notice that :



1. The two points A (3, 0), B (-1, 0) are both located on x - axis, so :

   $A B = |-1 - 3| = |-4|$

   So $A B = 4$ unit length .

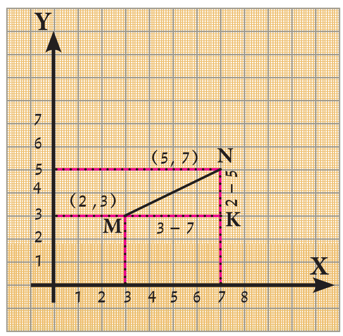


2. The two points C (0, -3), D (0, -1) are both located in the y - axis, so;

   $C D = |-3 - (-1)|$

   $= |-3 + 1| = |-2|$

   $C D = 2$ unit length .



3. The two point M (3, 2), N (7, 5) can be represented graphically as in the following figure opposite. To find The length of $\overline{MN}$ we find;

   $M K = |7 - 3| = 4$ unit length,

   $N K = |5 - 2| = 3$ unit length .

   $\triangle$ M K N is right angle at K

   $\therefore (M N)^2 = (M K)^2 + (K N)^2$

   (Pythagoren theory)

   $(M N)^2 = (3)^2 + (4)^2$ $(L M)^2 = 9 + 16$

   $(M N)^2 = 25$ $\therefore (M N) = 5$ unit length

In general :

If M (x_1, y_1), N (x_2, y_2) are two points on the coordinate plane

then: K M = |O B - O A|

$= |x_2 - x_1|$

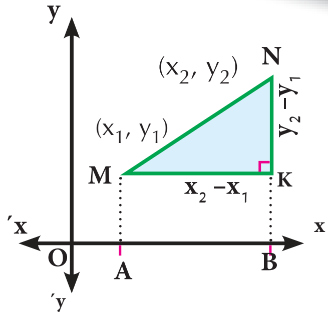
K N = |N B - K B| = $|y_2 - y_1|$

∵ △ N K M is a right angle in K **(pythagoean theory)**

∴ $(MN)^2 = (KM)^2 + (KN)^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

∴ $MN = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

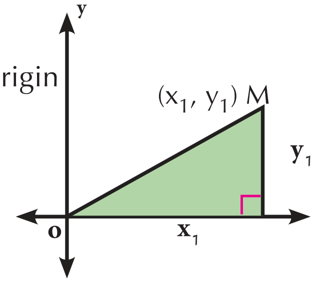


The distance between two points (x_1,y_1), (x_2,y_2) = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The distance between two points = $\sqrt{\text{square difference in the x - axis + square differance in y - axis}}$

Remark:

In the figure opposite the distance of a point M (x_1, y_1) fom the origin point O (0, 0) , O M = $\sqrt{x_1^2 + y_1^2}$



Drill

If A , B , C and D are four given points in the perpendicular coordinate plane, mention the conditions which make those points vertices for each of the following geometrical shapes:

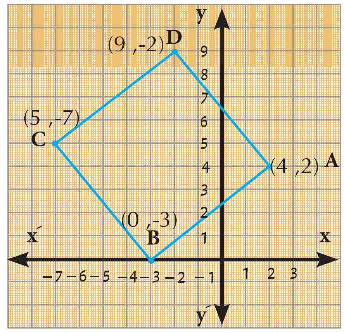
1. Paralleogram
2. Rectangle
3. rhombus
4. Square

Example

1. ABCD is a quadrilateral where, A (2, 4), B (-3, 0), C(-7, 5) and D(-2, 9). Prove that ABCD is a square.

Solution

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[-3-2]^2 + [0-4]^2} = \sqrt{(-5)^2 + (-4)^2} = \sqrt{41}$



$B C = \sqrt{[-7-(-3)]^2 + [5-0]^2} = \sqrt{(-4)^2 + (5)^2} = \sqrt{41}$

$C D = \sqrt{[-2-(-7)]^2 + [9-5]^2} = \sqrt{(5)^2 + (4)^2} = \sqrt{41}$

$D A = \sqrt{[2-(-2)]^2 + [4-9]^2} = \sqrt{(4)^2 + (-5)^2} = \sqrt{41}$

$\because$ A B = BC = DC =D A= $\sqrt{41}$

$\therefore$ Figure A B C D whether a square ar arhombus

to prove that the figure A B C D is a square, find the lengths of the two diagonal $\overline{AC}$, $\overline{BD}$

$AC = \sqrt{[-7-2]^2 + [5-4]^2} = \sqrt{(-9)^2 + 1} = \sqrt{82}$

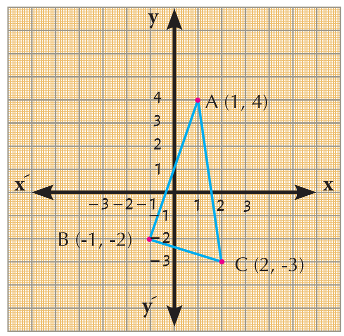
$BD = \sqrt{[-2-(-3)]^2 + [9-0]^2} = \sqrt{(-1)^2 + (9)^2} = \sqrt{82}$

$\because$ AC = B D = $\sqrt{82}$ and the sides of the figure A B C D is equal in length

$\therefore$ Figure A B C D is a square.

2 Prove that the triangle of the vertices A (1, 4), B (- 1, -2), C (2, -3) is a right angle. Find its surface area.

Solution



$(A B)^2 = (-1 - 1)^2 + (-2 - 4)^2 = 4 + 36 = 40$

$(B C)^2 = [2 - (-1)]^2 + [-3 -(-2)]^2 = 9 + 1 = 10$

$(A C)^2 = (2 - 1)^2 + (-3 - 4)^2 = 1 + 49 = 50$

$(A B)^2 + (B C)^2 = 40 + 10 = 50$, $(A C)^2 = 50$

$\because (A C)^2 = (A B)^2 + (B C)^2$

$\therefore$ M ($\angle$ B) = 90° **(The converse to the pytheogeran theary)**

$\therefore$ M ($\triangle$ A B C) = $\frac{1}{2}$ A B × B C = $\frac{1}{2} \times \sqrt{40} \times \sqrt{10} = \frac{1}{2} \times 2\sqrt{10} \times \sqrt{10}$ = 10 square units

3 Prove that the points A (3, -1), B (- 4, 6) and C (2, -2), are located in circle whose center is the point M (-1, 2), , then find the circumference of the circle.

Solution

$A M = \sqrt{(-1-3)^2 + [2-(-1)]^2} = \sqrt{(-4)^2 + (3)^2} = \sqrt{25} = 5$

$B M = \sqrt{[-1-(-4)]^2 + [2-6]^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$

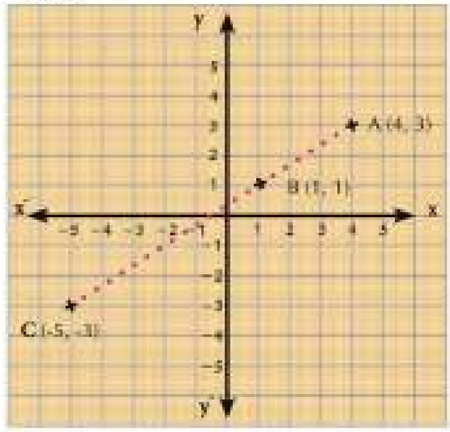
$C M = \sqrt{(-1-2)^2 + [2-(-2)]^2} = \sqrt{(-3)^2 + (4)^2} = \sqrt{25} = 5$

$\because$ A M = B M = C M = 5 $\therefore$ A , B and c are located in a circle whose center is M.

Drill

Prove that the points: A (4, 3), B(1, 1) and C (-5, -3) are collinear.

Complete :



$AB = \sqrt{(1-4)^2 + (1-3)^2}$ = ...

$BC = \sqrt{(-5-1)^2 + (-3-1)^2}$ = ...

$AC = \sqrt{(-5-4)^2 + (-3-3)^2}$ = ...

$\therefore$ A B + B C = ... + ... = ...

$\because$ A B + ... = A C

$\therefore$ The points A , B and C are collinear.

For More Exercises, go to MOE website

5 - 2

The Two Coordinates of the midpoint segment

Think and Discuss

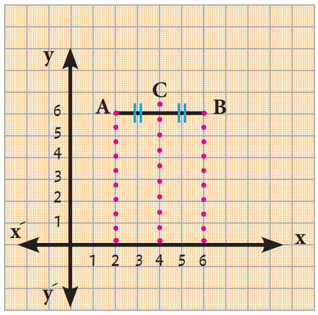
On a perpendicular coordinate plane, find the two coordinates of the midpoint on C straight segment $\overline{AB}$:

First : A (2, 6) and B (6, 6)

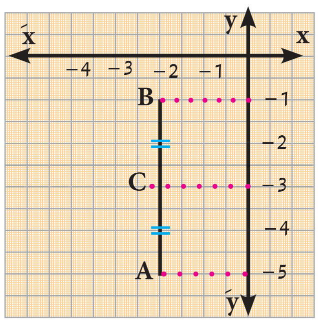
Second : A (-2 , - 5) and B (- 2 , - 1),

Third : A (1, 2) and B (5, 6)

First: The line segement, which its end are the two points (2, 6), B (6, 6), is parallel to the x-axes and the two coordinate of the point of its midpoint C (4, 6)



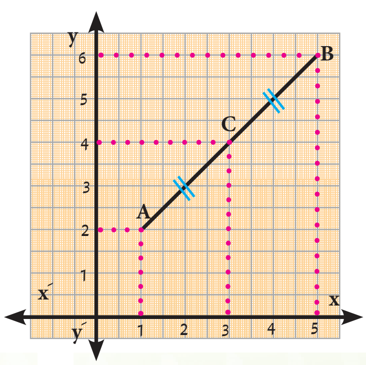
Second : The straight segment with the two ends A (-2, -5), B (-2, -1) is parallel to the y-coordinate. The two coordinates of its midpoints C are (-2, -3) .



Third : In the figure opposite :

Consider that the C is the midpoint of the straight segment with the two ends A(1, 2), B (5, 6) from the drawing, we find that the two coordinates of C are (3, 4).

i. e C $\left(\frac{1+5}{2}, \frac{2+6}{2}\right)$ i. e. C (3, 4)



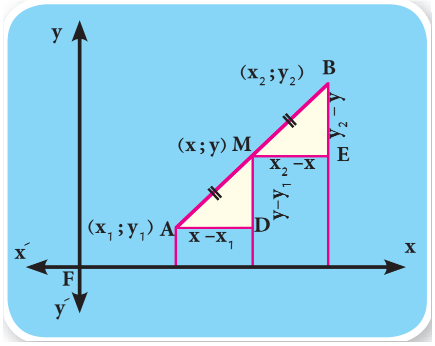
What you'll learn

- ☆ Finding the two coordinates of the midpoint of a straight segment .

Key terms

- ☆ The two ends of the line segment
- ☆ The two coordinates of the midpoint of a straight segment .

In general. you can duduce the law of the coordinate of the midpoint of a straight segment as follows.



If A (x_1, y_1), B (x_2, y_2), M (x, y) where M in the midpoint of $\overline{AB}$ that: △ B E M , △ M D A are cogruent

we find: that A D = M E

$\therefore x - x_1 = x_2 - x$

$\therefore 2x = x_1 + x_2$ $\qquad \therefore x = \frac{x_1 + x_2}{2}$

Similarly: M D = B E $\quad \therefore y - y_1 = y_2 - y$

$\therefore 2y = y_1 + y_2$ $\qquad \therefore y = \frac{y_1 + y_2}{2}$

$M \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Example : If C is the midpoint of $\overline{AB}$ and A (3, - 7) , B (- 5, - 3)

Then the coordinates of midpoint of $\overline{AB}$ are $\left(\frac{3 - 5}{2}, \frac{-7 - 3}{2}\right)$ i.e. (- 1, - 5)

Drill

Calculate the coordinates of point C the midpoint of $\overline{AB}$ in the following cases :

1. A(2, 4), B (6, 0)
2. A (7, -5), B (-3, 5)
3. A (-3, 6), B (3, -6)
4. A (7, -6), B (-1, 0)

Examples

1. If C (6, -4) is the midpoint of $\overline{AB}$ where: A (5, -3) then find the coordinates of a point B .

Solution

Consider that B (x_2, y_2), A (5, -3), and the midpoint of $\overline{AB}$ is the point C (6 - 4)

$\because x = \frac{x_1 + x_2}{2}, \; y = \frac{y_1 + y_2}{2}$

$\therefore 6 = \frac{5 + x_2}{2}$ $\qquad \therefore 5 + x_2 = 12$ $\qquad \therefore x_2 = 12 - 5 = 7$

$-4 = \frac{-3 + y_2}{2}$ $\qquad \therefore -3 + y_2 = -8$

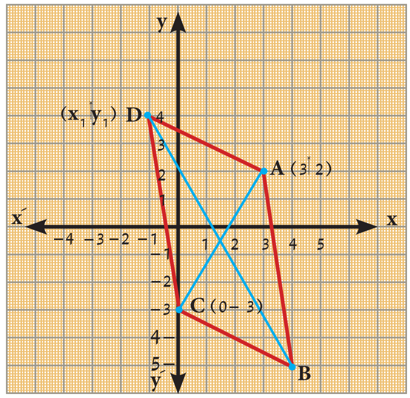
$y_2 = -8 + 3$ $\qquad y_2 = -5$ $\qquad \therefore$ B (7, -5)

2 A B C D is a parallorogram, A (3, 2), B (4, -5), C (0- 3) - Find the two coordinates of the point at which the two diagonals intersect. Then find the coordinates of point D.

Solution

The figure A B C D is a parallelogram , M is the intersection point of its diagonal.

consider D (x_1 , y_1)



∵ M is the mid of $\overline{AC}$ ∴ $M(\frac{3+0}{2}, \frac{2-3}{2})$

∴ $M(\frac{3}{2}, -\frac{1}{2})$

∴ $M(\frac{4+x_1}{2}, \frac{-5+y_1}{2})$

∴ $\frac{3}{2} = \frac{4+x_1}{2}$ ∴ $3 = 4 + x_1$

∴ $x_1 = -1$

, $-\frac{1}{2} = \frac{-5+y_1}{2}$ ∴ $-1 = -5 + y_1$

∴ $y_1 = 4$

∴ The coordinates of the point D are (-1, 4)

For More Exercises, go to MOE website

5 - 3 The slope of the straight line

What you'll learn

- The relation between the slope of two parallel straight lines.
- The relation between the slope of two perpendicular, straight lines.

Key terms

- A Positive measure of the angle
- A negative measure of the angle
- The slope of the straight line
- Two parallel straight lines
- Two perpendicular straight lines.

You know that the slope of the straight line passing through two points (x_1 , y_1), (x_2 , y_2) equals $\frac{y_2 - y_1}{x_2 - x_1}$

Think and Discuss

Find the slope of the straight line passing through each pair of the following ordered pairs :

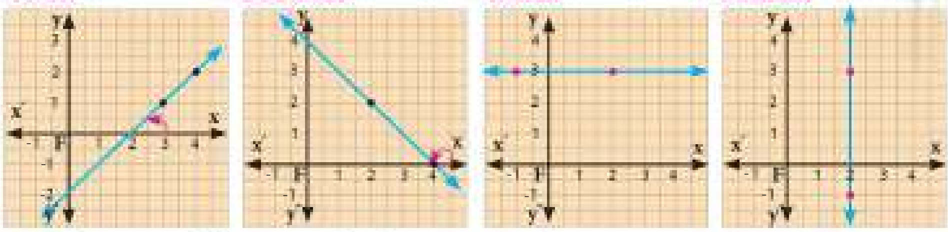
First: (3, 1), (4, 2) **Second:** (4, 0), (2, 2)

Third: (-1, 3), (2, 3) **Fourth:** (2, -1), (2, 3)

What do you notice?

From the previous, you can draw the straight lines passing through the previous pairs of points in the perpendicular coordinate plane as in the following figure:

First: **Second:** **Third:** **Fourth:**



The positive and the negative measure of the angle :

An angle is positive when it is formed by a counter anticlockwise rotation and it is negative when it is formed by a clockwise rotation.

From the previous figures, we deduce that:

The figure number	The slope $\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$	The type of the positive angle that the straight line makes in the positive direction to the x-coordinates	The slope of the straight line
1	$m = \frac{2-1}{4-3} = 1$	acute	Larger than zero
2	$m = \frac{2-0}{2-4} = -1$	obtuse	Smaller than zero
3	$m = \frac{3-3}{2+1} = 0$	zero	equal to zero
4	$m = \frac{3+1}{2-2}$ (unidentified)	right	unidentified

We can deduce the slope of the straight line as follows:

Slope of the straight line is the tangent of the positive angle which the straight line makes with the positive direction to x axis.

i.e slope of a straight line = tan E, where E is the positive angle that the straight line makes with the positive direction of the x axis.

Examples

1. Find the slope of the straight line which makes an angle of a measure 56° 12` 48`` in the positive direction to the x-axes.
2. Find the measure of the positive angle that the straight line makes to the x - axis if m =1.4865 (where m is the slope) .

Solution

1. ∵ m = tan E ∴ m = Tan 56° 12` 48`` = 1.494534405

   Start → [tan] 56 [°‚‚‚] 12 [°‚‚‚] 48 [°‚‚‚] [=]

2. ∵ m = Tan E ∴ Tan E = 1.4865 ∴ m (∠ E) = 56° 4` 13``

   Start → [SHIFT] [tan] 1,4865 [=] [°‚‚‚]

Drill

1. Find the slope of the straight line that makes a positive angle in the positive direction of to the x - axis, its measure:

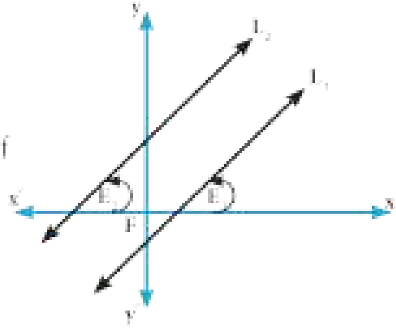
   A 30° B 45° C 60°

2. Using the calculator, find the measure of the positive angle made by the straight line of slope (m) in the positive direction of x-axis in the following cases :

   A m = 0.3673 B m = 1.0246 C m = 3.1648

The relation between the slope of the two parallel straight lines.

Think and Discuss



The figure opposite: Represents two parallel straight lines L_1, L_2 with two slpoes m_1, m_2, making two positive angles of measures E_1, E_2 in the positive direction of the x-axes.

Complete the following :

1. $m(\angle E_1) = m(\angle E_2)$ because ....
2. Tan E_1 .... Tan E_2
3. m_1 .... m_2

from the previous, we deduce that :

IF $L_1 // L_2$ **then** $m_1 = m_2$

i.e.: If two lines are parallel, then their slopes are equal and vice versa .

Thus If $m_1 = m_2$ then $L_1 // L_2$

i.e.: If two lines have equal slopes, then the two lines are parallel.

Examples

1. Prove that the straight line passing through two points (-3, -2), (4, 5) is parallel to the straight line that makes with the positive direction to the x-axes an angle of 45 ° measure

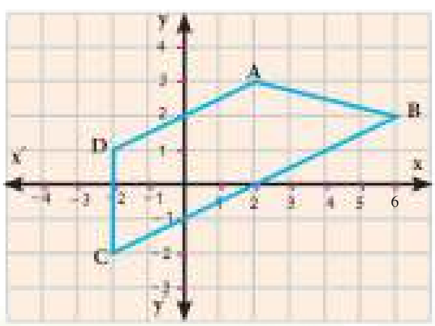
Solution

The slope of the first straight line $(m_1) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-2)}{4-(-3)} = \frac{7}{7} = 1$

The slope of the second straight line (m_2) = Tan 45° = 1 $\qquad \because m_1 = m_2$

$\therefore$ The two straight lines are parallel.

2. Represent graphically the points A (2, 3), B (6, 2) C (-2, -2) and D (- 2, 1), in the coordinate plane then **prove that** the figure A B C D is trapezoid .



Solution

From the drawing, we find that : $\overline{AD} // \overline{BC}$

To prove that anaclytically, we find the slope of each of both: $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$.

The slope of $\overline{AD}$ **(Let it be m1)**

$\because m = \frac{y_2 - y_1}{x_2 - x_1}$ $\therefore m_1 = \frac{3-1}{2+2} = \frac{2}{4} = \frac{1}{2}$

and the slope of $\overline{BC}$ **(Let it be M2)**

$m_2 = \frac{2+2}{6+2} = \frac{4}{8} = \frac{1}{2}$ $\because m_1 = m_2$ $\therefore \overline{AD} \parallel \overline{BC}$

∴ The figure A B C D is a trapezoid unles the points A, B, C, D are collinear ..... (1)

∵ The slope of $\overline{AB} = \frac{3-2}{2-6} = \frac{1}{-4}$, the slope of $\overline{CD} = \frac{2+1}{-2+2}$ .... (unknown)

∴ The two straight lines are not parallel............ (2)

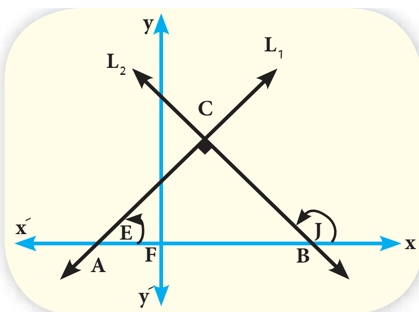
From (1), (2)   ∴ The figure A B C D is a trapezoid .

Drill

1. Prove that the straight line passsing through the two points (2, 3), (0, 0) is parallel to the straight line passing through the two points (- 1, 4), (1, 7).
2. Prove that the straight line passing through the two points (2, -1), (6, 3) is parallel to the straight line that makes an angle its of 45° measure with the positive direction to the x-axis.
3. If the straight line $\overleftrightarrow{AB}$ // the y-axis where A (x, 7), B (3, 5) , then find the value of x.
4. If the straight line $\overleftrightarrow{CD}$ // the x-axis where C (4, 2), D (- 5,y) then find the value of y.

The relation between the slope of the two perpendicular straight lines.

Think and Discuss



The figure opposite : represents the two straight lines L_1, L_2 which their two slopes are m_1, m_2 where $L_1 \perp L_2$.

Find the relation between ∠ E , ∠ J)

Then complete the following table :

Values of E	20°	40°	.....	.....	....
Values of J	......	.....	140°	150°	.....
Tan E_1 × Tan J_2	.....	.....	.....	.....	.....

From the previous table, we deduce that :

Tan E_1 × Tan J_2 = -1   **i. e :** $m_1 \times m_2 = -1$

If L_1, L_2 are two straight lines of slopes m_1, m_2, where $m_1, m_2 \in R^*$
If $L_1 \perp L_2$ then $m_1 \times m_2 = -1$
i. e: The product of multiplying the slopes of the two perpendicular straight, lines = -1 and vice versa, if $m_1 \times m_2 = -1$, then $L_1 \perp L_2$
i. e: If the product of multiplying the slopes of two straights lines = -1, then the two straight lines are perpendiculars.

Examples

1 Prove that the straight line passing through the two points $(4, 3\sqrt{3})$, $(5, 2\sqrt{3})$ is perpendicular on the straight line that makes with the positive direction to the x-axes to an angle of 30° measure.

Solution

Consider that the slope of first straight line is m_1 and the slope of the second straight line is m_2.

$\because m = \frac{y_2 - y_1}{x_2 - x_1}$ $\qquad \therefore m_1 = \frac{3\sqrt{3} - 2\sqrt{3}}{4 - 5} = -\sqrt{3}$

$\because m = \tan E$ $\qquad \therefore m_2 = \text{Tan } 30° = \frac{1}{\sqrt{3}}$

$\therefore m_1 \times m_2 = -\sqrt{3} \times \frac{1}{\sqrt{3}} = -1$ $\qquad \therefore$ The two straight line are perpendicular .

2 If a triangle with vertices y(4, 2),x (3, 5), Z (-5, A) is right angle at y then find the value of A.

Solution

Find the slope of $\overleftrightarrow{xy}$ thus $m_1 = \frac{5 - 2}{3 - 4} = \frac{3}{-1} = -3$, find the slope of thus $m_2 = \frac{A - 2}{-5 - 4} = \frac{A - 2}{-9}$

$\because \triangle x\,y\,z$ is a right angle at y $\qquad \therefore m_1 \times m_2 = -1$

$\therefore -3 \times \frac{A - 2}{-9} = -1$ $\qquad \therefore \frac{(A - 2)}{3} = -1$

$\therefore A - 2 = -3$ $\qquad \therefore A = 2 - 3$ $\qquad \therefore A = -1$

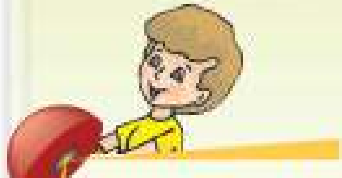
Drill

Find the slope of the perpendicular straight line on the straight line through the two points (3, -2), (5, 1) .

For More Exercises, go to MOE website

The Equation of the straight line given its slope and its y - intercept

5 - 4



What you'll learn

- ☆ Finding the equation of the straight line with given the slope and the intersected part from the y - axis.
- ☆ Finding the equation of the straight line given its slope and its Y-intercept.

Key terms

- ☆ Equation of straight line.
- ☆ Slope of a straight line.
- ☆ y - intercept .

Think and Discuss

You learned the linear relation between two variables x , y, it is :

A x + B y+ C = 0 where A, B (each of both) $\neq 0$

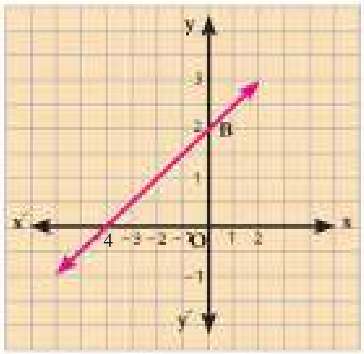
Is represented graphically by a straight line .

Example

Represent the relation :

x -2y+ 4 = 0 graphically .

From the graphical figure, calculate:



A The slope of the straight line .

B The lenght of the vertical part included between the origin point and the intesection point of the straight line with y - axis.

Solution to make the drawing easier, select the intersection point of the 2 axes: as follows :

y= 0	$\therefore$ x + 4 = 0	
$\therefore$ x = -4	(-4 , 0)	satisfies the relation.
x = 0	$\therefore$ -2y+ 4 = 0	
$\therefore$ 2y= 4	(0, 2)	satisfies the relation

From the drawing we find that the slope of the straight line

(m) > 0 ..... (why?) thus, m = $\frac{\dots\dots}{\dots\dots} = \frac{\dots\dots}{\dots\dots}$

The distance between the 2 points o and B are called the y - intercept .

intercept and is equal to 2 unit lenght and is denoted by the symbol (b).

The previous equation is written as: y = m x + b

thus, 2y = x + 4 and by dividing both sides by 2

$\therefore y = \frac{1}{2} x + 2$

We notice in this form that:

The slope the straight line (m) which is the coefficient of x equals $\frac{1}{2}$, and the lenght of y- intercept b = 2 and these are the same results we got the previous drawing .

Defintion

The equation of the straight line with respect to its slope (m) and the y - intercept (b).

Is $y = m x + b$ where $m \in R$

Notice that : The equation of the straight line is written: $ax + by + c = \text{zero}, b \neq 0$

In the fromula: $y = m x + b$ as the following :

$ax + by + c = \text{zero}$ thus $by = ax - c$

$\therefore y = -\frac{a}{b} x - \frac{c}{b}$ and it is in the formula: $y = m x + c$

Where $m = \frac{-a}{b} = \frac{\text{- Coefficient x}}{\text{Coefficient y}}$ Where c is the lenght of the y - intercept .

Examples

1 Find the slope of the straight line $3x + 4y - 5 = \text{zero}$ in two different methods then find the lenght of the y intercept .

Solution

$\because$ The equation of the straight line in the formula of $ax + by + c = 0, b \neq 0$

$\therefore$ The slope of the straight line $= \frac{-a}{b}$

$\therefore$ The slope of the straight line $= \frac{-3}{4}$

or : it is written in the formula of $y = mx + c$

$\therefore 4y = -3x + 5$

$y = \frac{-3}{4} x + \frac{5}{4}$

$\therefore$ The slope of the straight line $= \frac{-3}{4}$

$\therefore$ The length of y - intercept $= \frac{5}{4}$

2 Find the equation of the straight line passing through the point (1, 2) and perpendicular on the straight line passing through the two points A (2, -3), B (5, -4) .

Solution

$\because$ The slope of the straight line passing through the two points a, b $= \frac{-4 -(-3)}{5 - 2} = \frac{-4 + 3}{5 - 2} = \frac{-1}{3}$

thus, the slope of the straight line is perpendicular on = 3

$\therefore$ The equation of the straight line is written in the formula: $y = 3x + c$

$\because$ The straight line passes through the point (1, 2) so, it satisfies the equation .

$\therefore 2 = 3 \times 2 + c$

$\therefore c = 2 - 6 = -4$

$\therefore$ The equation of the straight is written in this formula : $y = 3x - 4$

3 If A (- 3, 4), B (5, - 1), C (3, 5) find the equation of the straight line passing through the vertex A and bisecting $\overline{BC}$.

Solution

The midpoint of $\overline{BC} = (\frac{3+5}{2}, \frac{5-1}{2}) = (\frac{8}{2}, \frac{4}{2}) = (4, 2)$

$\therefore$ The slope of the required straight line $= \frac{2-4}{4+3} = \frac{-2}{7}$

$\because y = mx + c$ $\therefore y = \frac{-2}{7} x + c$

$\because$ The point of A (-3, 4) passes through the straight line, so it satisfies the equation.

$\therefore 4 = \frac{-2}{7} \times -3 + c$ $\therefore 4 = \frac{6}{7} + c$ $\therefore c = \frac{22}{7}$

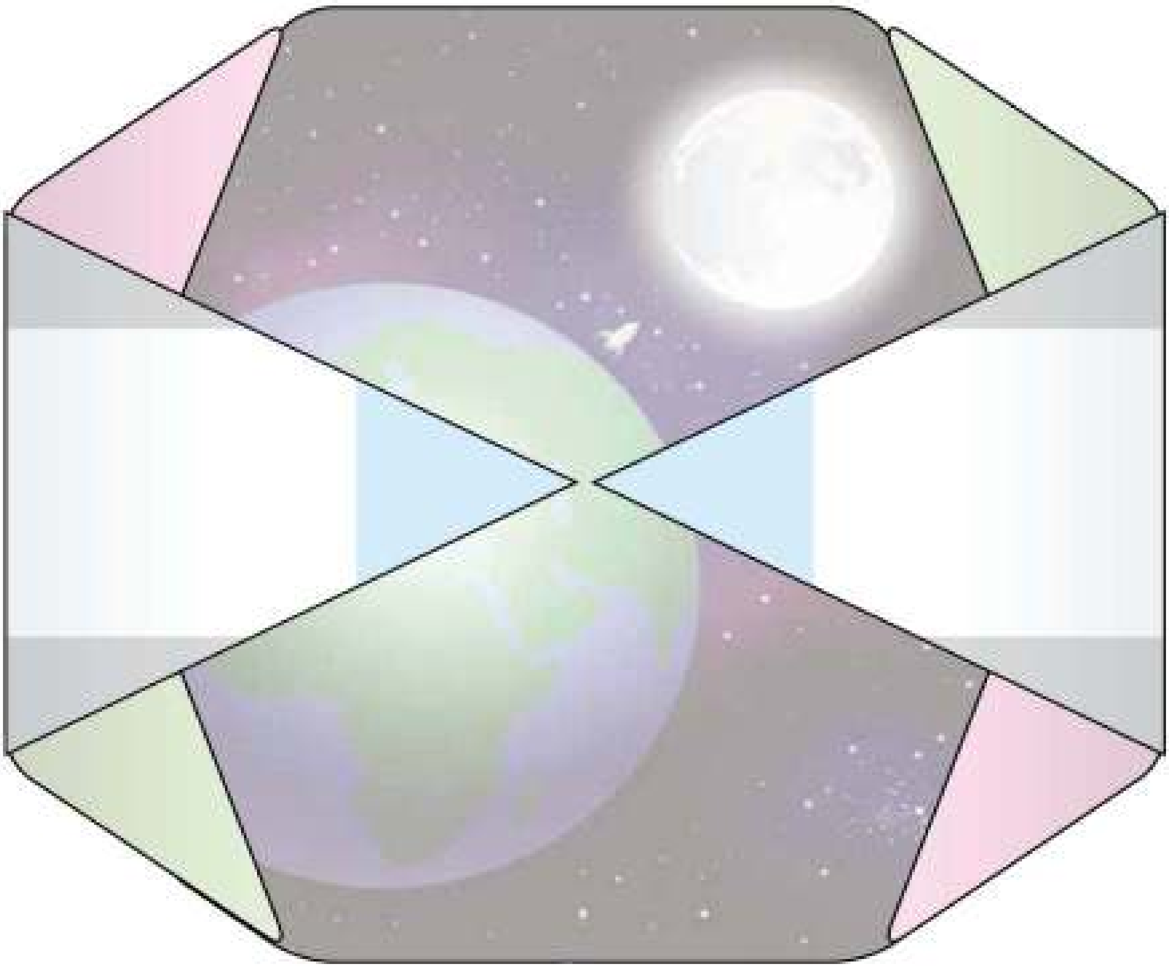
$\therefore$ The equation of the straight line is written as in the formula:$y = \frac{-2}{7}x + \frac{22}{7}$ and by the multiyipling two sides in 7

$\therefore 7y = -2x + 22$ i . e the equation is : $2x + 7y - 22 = 0$

For More Exercises, go to MOE website

3

Second Term



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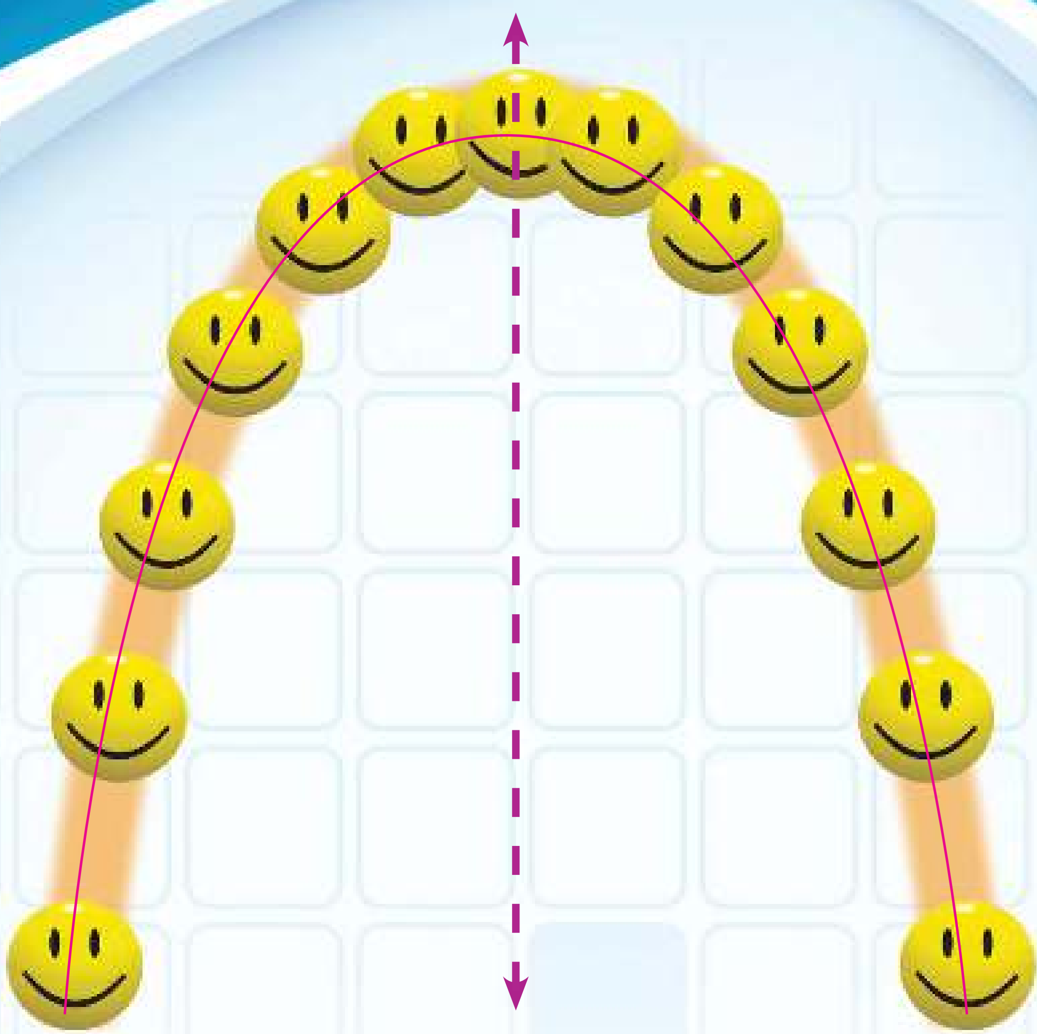
Unit (1)

Equations



One of the players threw the ball so, it took the direction shown in the figure.

This figure represents one of the functions which you will study and is called “a quadratic function”.

1 - 1

Solving two equations of first degree in two variables graphically and algebraically

What you'll learn

- ☆ Solving two equations of first degree in two variables.

Key terms

- ☆ Equation of first degree.
- ☆ Graphical solution.
- ☆ Substitution set.
- ☆ Algebraic solution.
- ☆ Solution set.

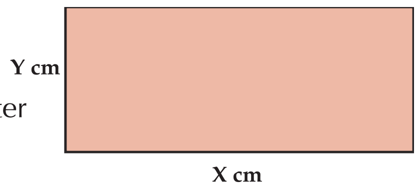
Think and Discuss

A rectangle of a perimeter 30cm. What are the possible values of its length and width. If the length of the rectangle = x cm and the width of the rectangle = y cm

then :

the length + width = $\frac{1}{2}$ the perimeter

$\therefore\ x + y = 15$



- This equation is called the equation of first degree in two variables.
- Solving this equation means finding an ordered pair of the real number is satisfying equation.
- Can (-5 .20) be a solution of the previous equation. Explain your answer. Dear student: Solve this problem after the following.
- You can solve this equation by putting it in one of the two forms:

(1) $y = 15 - x$ or (2) $x = 15 - y$

By giving one of the two variables any value, you can calculate the value of the other variable.

If $x \in R$ then the substitution set is $R \times R$ thus there are infinite number of solutions of the equation of the first degree, in which each of them is in an ordered pair. (x , y) where its first projection x and its second projection y.

when $x = 8 \therefore\ y = 15 - 8 = 7\ \therefore\ (8 , 7)$ is a solution of the equation
when $x = 9.5\ \therefore\ y = 15 - 9{,}5 = 5{,}5\ \therefore\ (9{,}5 , 5{,}5)$ is a solution of the equation
when $x = 4\sqrt{7}\ \therefore\ y = 15 - 4\sqrt{7}$
$\therefore\ (4\sqrt{7} , 15 - 4\sqrt{7})$ is a solution of the equation

First: Solving equations of the first degree in two variables graphically :

Examples

(1) Find the solution set of the equation $2x - y = 1$

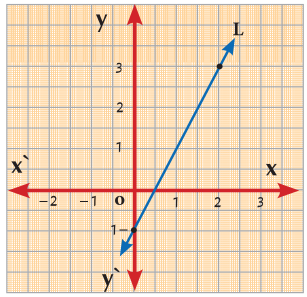
Solution

Write the equation in the form y = 2x - 1

By putting x = 0 ∴ y = -1 ∴ (0, -1) is a solution of the equation

By putting x = 2 ∴ y = 3 ∴ (2, 3) is a solution of the equation

And by drawing the straight line L passing throught the two represented points of the two ordered pairs (0 , -1) ,(2 , 3).



We find that every point ∈ L is a solution to the equation.

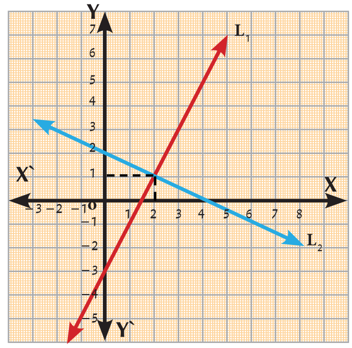
i.e for the equation 2x - y = 1 their is an infinite number of solutions

Tell another four solutions for this equation?

2 Find the solution set of the following two equations graphically:

L_1 : y = 2x - 3 , L_2 : x + 2y = 4

Solution



In the equation y = 2x - 3

By putting X= 0 ∴ y = -3 ∴ (0, -3) is a solution of this equation

By putting X = 4 ∴ y = 5 ∴ (4, 5) is a solution of this equation

Thus: L_1 in the opposite figure represents the solution set of this of equation **(1)**

By putting the equation x + 2y = 4 int the form x = 4 – 2y

By putting y = 0 ∴ x = 4 ∴ (4 , 0) is a solution of this equation

By putting y = 1 ∴ x = 2 ∴ (2 , 1) is a solution of this equation

This : L_2 in the opposite figure represents the solution set of the equation **(2)**

In the figure $L_1 \cap L_2$ is the point (2 , 1)

∴ The solution set of the two equations is {(2 , 1)}

Drill

Find the solution set for each pair in the following equations graphically :

1 2x + y = 0 x + 2y = 3

2 y = 3x - 1 x - y + 1 = 0

Example 3

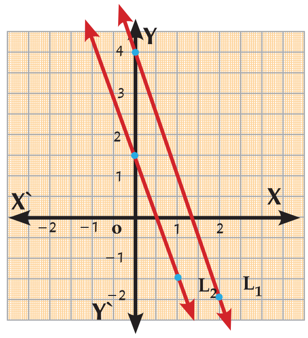
Find graphically the solution set for each pair of the following equations:

First: $3x + y = 4$ (1) , $2y + 6x = 3$ (2)

Second: $3x + 2y = 6$ (1) , $y = 3 - \frac{3}{2}x$ (2)

Solution

First:



Put the equation (1) in the form $y = 4 - 3x$

By Putting $x = 0 \therefore y = 4$ thus, (0, 4) is a solution of the equation

By Putting $x = 2 \therefore y = -2$ thus, (2, -2) is a solution of the equation

L_1 represents a solution set of the equation (1)

By putting the equation (2) in the form $y = \frac{3 - 6x}{2}$

By Putting $x = 0$ $\therefore y = \frac{3}{2}$ thus, $(0, \frac{3}{2})$ is a solution of the equation

By Putting $x = 1$ $\therefore y = \frac{-3}{2}$ thus, $(1, \frac{-3}{2})$ is a solution of the equation

and L_2 is a solution of the equation (2)

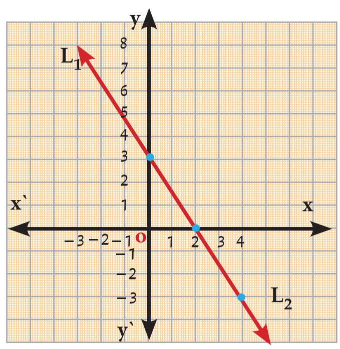
$\because L_1 \cap L_2 = \phi$ $\therefore$ No solution for the two equations together.

i.e there is no solution of the two equations **(1)** , **(2)** when $L_1 // L_2$

From the Analytical Geometry :

The slope of $L_1 = \frac{-3}{1} = -3$ The slope of $L_2 = \frac{-6}{2} = -3$ $\therefore L_1 // L_2$

Second:



By Putting the equation **(2)** in the form of $2y = 6 - 3X$

i.e. $3X + 2Y = 6$ is the same as equation (1) the graph shown illustrates the graphical representation of the two equations by two coincident straight lines.

We say that: The two equations **(1)** and **(2)** have an infinite number of solutions. The solution set is $\{(x, y): y = 3 - \frac{3}{2}x\}$

Drill

Graphically find the solution set for each pair in the following equations:

1. $3x + y = 5$, $y + 3x = 8$
2. $2x + y = 4$, $8 - 2y = 4x$

Second: Solving two equations of first degree in two variables algabrically.

Solving two simultaneous equations of first degree in two variables is being done by removing one of the two variables where we get an equation of first degree in one variable. Solving this equation gives the value of this variable and by substituting in one of the given equations we get the value of the other which we removed.

Example 4

Find the solution set of the two equations

$2x - y = 3$ (1) , $x + 2y = 4$ (2)

Solution **(Substitution method)**

From the equation **(1)** , $y = 2x - 3$

by substitution in the equation **(2)** $\therefore\ x + 2(2x - 3) = 4$

thus : $x + 4x - 6 = 4$ $\therefore\ 5x = 10$ $\therefore\ x = 2$

Substituting in equation **(1)** $\therefore\ y = 2 \times 2 - 3$ $\therefore\ y = 1$

$\therefore$ The common solution set of the two equations = {(2, 1)}

Another solution **(Omitting method)**

Omitting one of the two variables in the two equations (by adding or subtracting) to get a third equation in one variable, and by solving the resulted equation we find the value of this variable.

$2x - y = 3$ (1) , $x + 2y = 4$ (2)

By multiplying the two sides of the equation (1) × 2 $\therefore\ 4x - 2y = 6$ (3)

Adding (2) and (3) $\therefore\ 5x = 10$ $\therefore\ x = 2$

Substituting in (1) $\therefore\ 2 \times 2 - y = 3$ $\therefore\ y = 1$

$\therefore$ The common solution set of the two equations is = {(2, 1)}.

Drill

1. Find algebraically, the solution set of each pair of the following equations:
   - A) $3x + 4y = 24$, $x - 2y + 2 = 0$
   - B) $3x + 2y = 4$, $x - 3y = 5$
2. What is the number of solutions of each pair in the following equations:
   - A) $7x + 4y = 6$, $5x - 2y = 14$
   - B) $3x + 4y = -4$, $5x - 2y = 15$
   - C) $9x + 6y = 24$, $3x + 2y = 8$

Example 5

Find the values of a , b knowing that (3 , -1) is the solution of the two equations.

$a x + b y - 5 = 0$, $3 a x + b y = 17$

Solution

∵ (3, -1) is the solution of the two equations

∴ (3, -1) is the solution of the equations $a x + b y - 5 = 0$

∴ $3 a - b - 5 = 0$ **i.e. :** $3a - b = 5$ (1)

, (3, -1) is the solution of the equations $3 a x + b y = 17$

∴ $9 a - b = 17$ (2)

Substracting both sides of equation (1) from both sides of equation (2) we get :

$6 a = 12$ ∴ $a = 2$

Substituting in equation (1)

$3 \times 2 - b = 5$ ∴ $b = 1$

Example 6

A two-digit number of sum of its digits is 11. If the two digits are reversed, then the resulted number is 27 more than the original number. What is the original number ?

Solution

Consider that the units digit is x and the tens digit is y .

∴ $x + y = 11$ .... (1)

	units digit	tens digit	the value of the number
The original number	x	y	$x + 10 y$
The sum after reversing digits	y	x	$y + 10 x$

The resulted number after reversed its two digits - the original number = 27

∴ $(y + 10 x) - (x + 10 y) = 27$ ∴ $y + 10 x - x - 10 y = 27$

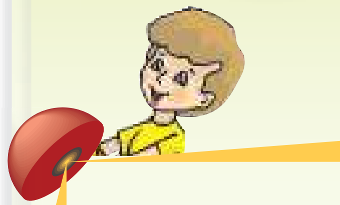
∴ $9 x - 9 y = 27$ **by dividing by 9** ∴ $x - y = 3$ .... (2)

By adding both equations (1) and (2)

∴ $2x = 14$ ∴ $x = 7$ substituting in the equation .... (1)

∴ $7 + y = 11$ ∴ $y = 4$ ∴ the number is 47

Solving an equation of second degree in one unknown graphically and Algebraically



What you'll learn

- (2) Solving an equation of second degree in one unknown graphically and Algebraically.

Key terms

- Graphical solution
- Algebraic solution
- Solution set

Think and Discuss

We have reperesented graphically the quadratic function f where :

$f(x) = a x^2 + b x + c$, a, b , c $\in$ R, $a \neq 0$

The corresponding equation is $f(x) = 0 \Rightarrow ax^2 + b x + c = 0$

You have previously solved this equation by factorizing.

To solve the equation : $x^2 - 4x + 3 = 0$

We factorize the left side of the equation to be :

(x - ....) (x .... 1) = 0

$\therefore$ x - .... = 0 or (x - 1) = 0

$\therefore$ x = ... or x = ...

$\therefore$ The solution set is { .... , ... }

First: the graphical solution:

To solve $a x^2 + b x + c = 0$ graphically we follow the steps:

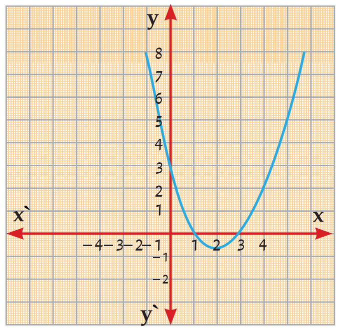
- We draw the function curve of $f(x) = a x^2 + b x + c$ where $a \neq 0$
- Identify the set of x coordinates of the points of intersection of the function curve with the x-axis, thus we get the solution of the equation.

Example 1

Draw the graphical representation of the function f where $f(x) = x^2 - 4 x + 3$ in the interval [- 1, 5]

From the drawing, find the solution set of the equation $x^2 - 4 x + 3 = 0$

Solution



Identify some ordered pairs (x, y) which belong to the function f, whose first projection $x \in [-1, 5]$

f (-1) = 8, f (0) = 3, f (1) = 0,

f (2) = -1, f (3) = 0, f (4) = 3, f (5) = 8

Put the ordered pairs in a table as follows:

x	5	4	3	2	1	0	-1
y = f(x)	8	3	0	-1	0	3	8

Plot on the coordinate plane the points which represent these ordered pairs, then draw a curve passing through these points.

From the drawing we find that the function curve f intersects the x-axis in two points (3, 0), (1, 0) the two numbers 1, 3 are called the two roots of the equation $x^2 - 4x + 3 = 0$.

Thus, the solution set of the equation is {1 , 3}

Drill

1. Draw the graphical form of the function f where $f(x) = x^2 + 2x + 1$ in the interval [-4, 2] and from the drawing find the solution set of the equation: $x^2 + 2x + 1 = 0$

2. Draw the graphical form of the function f where $f(x) = -x^2 + 6x - 11$ in the interval [0, 6] and from the drawing find the solution set of the equation: $x^2 - 6x + 11 = 0$

Second : The algebraic solution by using the general rule:

Think and Discuss

Solving the equation : $x^2 - 6x + 7 = 0$ using the idea of completing the square.

Complete : $\because$ $x^2 - 6x + 9 + 7 - 9 = 0$

$\therefore$ (x - .... $)^2 - 2 = 0$ (x - .... $)^2 = 2$

x - .... $= \sqrt{2}$ or x .... $= -\sqrt{2}$

x = ... $+ \sqrt{2}$ or x = ... $- \sqrt{2}$

$\therefore$ x = ... $\pm\sqrt{2}$

You can solve an equation of second degree : $a x^2 + b x + c = 0$ where $a, b, c \in R$, $a \neq 0$ using the rule

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{where } a \neq 0,\ a, b, c \in R$$

Examples

2 Find the solution set of the equation $3x^2 = 5x - 1$ rounding the results to two decimal places.

Solution

$\because 3x^2 = 5x - 1 \qquad \therefore 3x^2 - 5x + 1 = 0$

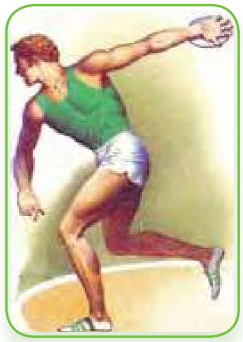
$\therefore a = 3, b = -5, c = 1$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25 - 4 \times 3 \times 1}}{2 \times 3} = \frac{5 \pm \sqrt{13}}{6} = \frac{5 \pm 3.61}{6}$$

As for $x = \frac{5 + 3.61}{6} = 1.44$ or $x = \frac{5 - 3.61}{6} = 0.23$

$\therefore$ The solution set is : {1.44, 0.23}

3 In a disk throwing race the path way of the disk to one of the players follows the relation : $y = -0.043x^2 + 4.9x + 13$ where x represents the horizontal distance in meters, y represents the disk height from the floor surface. Find the horizontal distance at which the disk falls to the nearset hundred.



Solution

$\because a = -0.043, b = 4.9, c = 13$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(4.9) \pm \sqrt{(4.9)^2 - 4 \times (-0.043) \times 13}}{2 \times (-0.043)}$$

$$= \frac{(-4.9) \pm \sqrt{26.246}}{-0.086} = \frac{-4.9 \pm 5.123}{-0.086}$$

$\therefore x = \frac{-4.9 + 5.123}{-0.086} = -2.59$ (refused) why ?

or $x = \frac{-4.9 - 5.123}{-0.086} = 116.5465116$ meters

$\therefore$ The horizontal distance where the disk lands is 116.55 meters

For More Exercises, go to MOE website

Solving two equations in two variables, one of them is of the first degree and the other is of the second degree

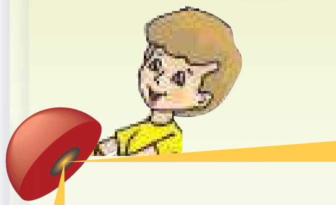
1 - 3

Introductions:

You know that the equation $2x - y = 3$ is an equation of the first degree in two variables while the equations: $x^2 + y = 5$ and $xy = 2$ are equations of the second degree in two veriables. why?

We will solve the two equations in two varibles one of them is of the first degree and the other of the second degree, by the substitution method as shown in the following examples.

Mental Math: If $x + y = 10$ and $x^2 - y^2 = 40$ then find $x - y$.



What you'll learn

- ☆ Solving two equations in two variables one of them is of the first degree and the other of the second degree.

Key terms

- ☆ Equation of the first degree
- ☆ Equation of the second degree
- ☆ Solution set

Examples

1 Find algebraically the solution set of the two equations:

$y + 2x + 1 = 0$, $4x^2 + y^2 - 3xy = 1$

Solution

From the first equation: $y = -(2x + 1)$

Substituting in second equation.

$\therefore 4x^2 + [-(2x + 1)]^2 - 3x[-(2x + 1)] = 1$

$\therefore 4x^2 + 4x^2 + 4x + 1 + 6x^2 + 3x - 1 = 0$

$\therefore 14x^2 + 7x = 0$ $\therefore 7x(2x + 1) = 0$

$\therefore x = 0$ or $2x + 1 = 0$ i.e $x = \frac{-1}{2}$

Substituting for the values of x in first equation :

When $x = 0$ $\therefore y = -(0 + 1) = -1$,

When $x = \frac{-1}{2}$ $\therefore y = -(2 \times \frac{-1}{2} + 1) = 0$

$\therefore$ The solution set is : $\{(0, -1), (\frac{-1}{2}, 0)\}$

2 A rectangle of a perimeter 14 cm and area 12 cm^2. Find its two dimensions.

Solution

Supose the two dimensions of the rectangle are x and y.

∵ The rectangle perimeter = **2 (Length + Width)**

∴ $14 = 2(x + y)$ ..... **divide both sides by 2**

∴ $x + y = 7$ i.e. $y = 7 - x$ .... (1)

∵ The rectangle area = **length** × **width** ∴ $xy = 12$ .... (2)

Substituting from equation (1) in equation (2)

∴ $x(7 - x) = 12$ $7x - x^2 = 12$

∴ $x^2 - 7x + 12 = 0$ $(x - 3)(x - 4) = 0$

∴ $x = 3$ or $x = 4$ substitute in equation (1)

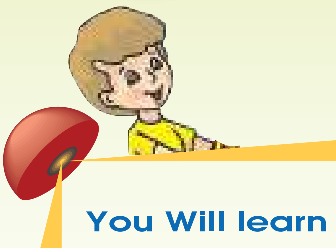
when : $x = 3$ ∴ $y = 7 - 3 = 4$,

when : $x = 4$ ∴ $y = 7 - 4 = 3$ the lenght and width of the rectangle are 3 cm and 4 cm.

For More Exercises, go to MOE website

2 - 1

Set of zeroes of a polynomial function



You Will learn

- ☆ Find zeroes of the polynomial function.

Key terms

- ☆ Polynomial function.
- ☆ Set of zeroes of the polynomial function.

Think and Discuss

if f : R ⟶ R where $f(x) = x^3 - 3x^2 + 2x$ is a polynomial function of third degree in X. calculate : f (0) , f (1), and f (2) **what do you notice?**

We notice that : f (0) = 0 , f (1) = 0 , f (2) = 0

So 0 , 1 and 2 are called the set of zeroes of the function.

Generally if f : R ⟶ R is a polynomial in x, then the set of values of x which makes f (x) = 0 is called the set of zeroes of the function f and its denoted by the sumbol Z (f).

i.e : Z (f) is the solution set of the equation f (x) = 0

In general, to get the zeros of the function f, put f(x) = 0 and solve the resulted equation to find the set of values of x.

Example

Find Z(f) for each of the following polynomial :

1. $f_1(x) = 2x - 4$
2. $f_2(x) = x^2 - 9$
3. $f_3(x) = 5$
4. $f_4(x) = 0$
5. $f_5(x) = x^2 + 4$
6. $f_6(x) = x^6 - 32x$
7. $f_7(x) = x^2 + x + 1$

Solution

1. $f_1(x) = 2x - 4$ put $f_1(x) = 0$ ∴ $2x - 4 = 0$

   i.e $2x = 4$ ∴ $x = 2$ ∴ $z(f_1) = \{2\}$.

2 $f_2(x) = x^2 - 9$

put $f_2(x) = 0$ $\quad \therefore\ x^2 - 9 = 0$

i.e $x^2 = 9$ $\quad \therefore\ x = \pm 3$ $\quad \therefore\ z(f_2) = \{-3, 3\}$.

3 $f_3(x) = 5$

$\therefore$ there is no real number that makes $f_3(x) = 0$ $\quad \therefore\ z(f_3)$ is ϕ

4 $f_4(x) = 0$

$\therefore$ all the real numbers R are zeroes to this function $\quad \therefore\ z(f_4)$ is R

5 put $x^2 + 4 = 0$

$\therefore\ x^2 = -4$ $\quad \therefore\ x = \pm\sqrt{-4} \notin R$ $\quad \therefore\ z(f_5)$ is ϕ

6 put $x^6 - 32x = 0$

$\therefore\ x(x^5 - 32) = 0$ $\quad \therefore\ x = 0$ $\quad , x^5 = 32$

when $x^5 = 2^5$ $\quad \therefore\ x = 2$ $\quad \therefore\ z(f_6) = \{0, 2\}$

7 put $x^2 + x + 1 = 0$

the expression $x^2 + x + 1$ could not be factorized so we use the rule to solve the quadratic

equation $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $a = 1, b = 1, c = 1$

$\therefore\ x = \dfrac{-1 \pm \sqrt{-3}}{2} \notin R$

$\therefore$ there is no solutions then $z(f_7) = \phi$

Drill

1 **Find the set of zeroes of the following functions :**

a $f(x) = x^3 - 4x^2$ **b** $f(x) = x^2 - 2x + 1$ **c** $f(x) = x^2 - 2x - 1$

d $f(x) = x^4 - x^2$ **e** $f(x) = x^2 - x + 1$ **f** $f(x) = x^2 - 2$

For More Exercises, go to MOE website

Algebraic fractional function

2 - 2

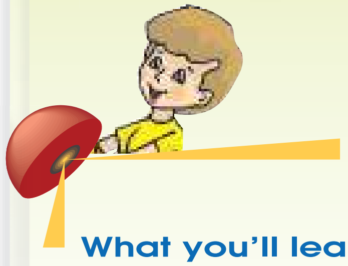
Think and Discuss

you have previously learned the rational number which is in the form $\frac{a}{b}$ where $a , b \in Z,\ b \neq 0$

if $p: R \longrightarrow R$, $p(x) = x + 3$,

$f: R \longrightarrow R$, $f(x) = x^2 - 4$.

1. *Find* the domain of f and p .
2. If $n(x) = \frac{p(x)}{f(x)}$ **can** you find the domain of n when you know the domain of each of p and f ?



What you'll learn

- Algebraic fractional function.

Key terms

- Polynomial function.
- The domain of algebraic fraction.
- The common domain for two algebraic fractions.

From the previous, we deduce the following:

n is called an algebraic fractional function or an algebraic fraction where $n(x) = \frac{x + 3}{x^2 - 4}$

The domain in this case is R except for the values of x which makes the fraction unknown (set of zeroes of the denominator).

i.e : **the domain of n(x) is R - {-2 , 2}**

If p and f are two polynomail functions and z (f) is the set of zeroes of f, then the function n where

$$n : R - z(f) \longrightarrow R\ ,\ n(x) = \frac{p(x)}{f(x)}$$

is called real algebraic fractional function or briefly called an algebraic fraction.

Note that : **the domain of algebraic fractional function = R - the set of zeroes of the denominator.**

Drill

1 **Identify the domain of each of the following algebraic fractional function then** ***find*** **n(0), n (2), n (-2):**

a) $n(x) = \frac{x+3}{4}$ b) $n(x) = \frac{x-2}{2x}$ c) $n(x) = \frac{1}{x+2}$

d) $n(x) = \frac{x^2+9}{x^2-16}$ e) $n(x) = \frac{x^2+1}{x^2-x}$ f) $n(x) = \frac{x^2-1}{x^2+1}$

2 **If the domain of the function n :** $n(x) = \frac{x-1}{x^2 - ax + 9}$ is R - {3} ***then find*** the value of a.

The common domain of two or more algebraic fraction:

The set of real numbers where the fractions are identified together completely (at the same time).

Example

If n_1 , n_2 are two algebraic fractions where :

$n_1(x) = \frac{1}{x-1}$, $n_2(x) = \frac{3}{x^2-4}$ ***then calculate*** the common domain of n_1 , n_2

Solution

Let m_1 the domain of n_1 , m_2 the domain of n_2.

$\therefore$ m_1 = R - {1} , m_2 = R - {-2 , 2} then the common domain of the two fractions n_1 , n_2 = $m_1 \cap m_2$

where : $m_1 \cap m_2$ = {(R - {1}} ∩ {R - {-2 , 2}} = R - {-2 , 1 , 2}

Remark : For any value of the variable x which belongs to the common domain then , each of $n_1(x)$ and $n_2(x)$ are defined (existed).

Generally :

If n_1 and n_2 are two algebraic fractions, and if the domain of $n_1 = R - X_1$ (where X_1 , the set of zeroes of the denominator of n_1) of the domain $n_2 = R - X_2$ (where X_2 , the set of zeroes of the denominator of n_2)

then the common domain of the two fractions n_1 and $n_2 = R - (X_1 \cup X_2)$

= R - the set of zeroes of the two denominators of the two fractions.

$\therefore$ the common domain of a number of algebraic fractions

= R - the set of zeroes of the denoinators of these fractions

Drill

Find the common domain for each of the following :

1. $n_1(x) = \frac{1}{x}$, $n_2(x) = \frac{2}{x + 1}$
2. $n_1(x) = \frac{3}{x^2 - x}$, $n_2(x) = \frac{2x - 3}{x^2 - 1}$
3. $n_1(x) = \frac{3}{x - 2}$, $n_2(x) \frac{5}{x + 2}$, $n_3(x) = \frac{x}{x^3 - 4x}$
4. $n_1(x) = \frac{x^2 - 4}{x^2 - 5x + 6}$, $n_2(x) = \frac{3x}{x^2 - x}$, $n_3(x) = \frac{x^2 - 3x - 4}{x^2 + x - 2}$

For More Exercises, go to MOE website

2 - 3 Equality of two algebraic fractions

What you'll learn

☆ The concept the equality of two algebraic fractions.

☆ How to determine when two algebraic fractions are equal.

Key terms

☆ Reducing an algebraic fraction.

☆ Equality of two algebraic fractions.

Reducing the algebraic fraction

Think and Discuss

If n is an algebraic fraction where : $n(x) = \dfrac{x^2 + x}{x^2 - 1}$

Complete :

1. The domain of n = ...
2. The common factor between the numerator and denominator after factorizing both of them perfect facterization is .... ≠ zero where x doesn't take the value of ....
3. The algebraic fraction in the simplest form after removing the common factor = ...
4. Does the domain of the algebraic fraction change after putting it in the simplest form ?

From the previous, we deduce that :

Putting the algebraic fraction in the simplest form is called reducing the algebraic fraction.
Follow the following steps to reduce an algebraic fraction :

1. *Factorize both the numerator and denominator perfectly.*
2. *Identify the domain of the algebraic fraction before removing the common factors in the numerator and denominator.*
3. *Remove the common factor in both the numerator and denominator to get the simplest form.*

Definition: It is said that the algebraic fraction is in its simplest form if there are no common fractons between its numerator and denominator.

Example 1

If $n(x) = \dfrac{x^3 + x^2 - 6x}{x^4 - 13x^2 + 36}$ then reduce n(x) in the simplest form showing the domain of n.

Solution

$\because$ $n(x) = \dfrac{x^3 + x^2 - 6x}{x^4 - 13x^2 + 36} = \dfrac{x(x^2 + x - 6)}{(x^2 - 4)(x^2 - 9)} = \dfrac{x(x + 3)(x - 2)}{(x + 2)(x - 2)(x + 3)(x - 3)}$

$\therefore$ the domain of $n(x) = R - \{-3, -2, 2, 3\}$.

$\therefore$ $n(x) = \dfrac{x}{(x + 2)(x - 3)}$ then cancel $(x + 3)$, $(x - 2)$ from the numerator and denominator.

Equality of two algebraic fraction to be equal

Think and Discuss

Find **$n_1(x)$ and $n_2(x)$ in the simplest form showing the domain of the following :**

1. $n_1(x) = \dfrac{x + 3}{x^2 - 9}$, $n_2(x) = \dfrac{2}{2x - 6}$
2. $n_1(x) = \dfrac{2x}{2x + 4}$, $n_2(x) = \dfrac{x^2 + 2x}{x^2 + 4x + 4}$

Does $n_1 = n_2$ in each case ? Explain your answer.

From the previous we deduce that :

1. $n_1(x) = \dfrac{x + 3}{(x + 3)(x - 3)} = \dfrac{1}{x - 3}$ **and the domain of** $n_1 = R - \{-3, 3\}$

   $n_2(x) = \dfrac{2}{2(x - 3)} = \dfrac{1}{x - 3}$ **and the domain of** $n_2 = R - \{-3\}$

   i.e.: n_1 and n_2 are reduced to the same fraction but the domain of $n_1 \neq$ the domain n_2

2. $n_1(x) = \dfrac{2x}{2(x + 2)} = \dfrac{x}{x + 2}$ **and the domain of** $n_1 = R - \{-2\}$

   $n_2(x) = \dfrac{x(x + 2)}{(x + 2)^2} = \dfrac{x}{x + 2}$ **and the domain of** $n_2 = R - \{-2\}$

   i.e.: n_1 and n_2 are reduced to the same form, **and the domain of** $n_1 =$ **and the domain of** n_2

From the previous, we deduce that :

> *It is said that the two algebraic fractions n_1 and n_2 are equal (i.e: $n_1 = n_2$) if the two following conditions are satisfied.*
> *the domain of n_1 = the domain of n_2 , $n_1(x) = n_2(x)$ for each $x \in$ the common domain.*

Examples

2 **If** $n_1(x) = \frac{x^2}{x^3 - x^2}$, $n_2(x) = \frac{x^3 + x^2 + x}{x^4 - x}$ **prove that :** $n_1 = n_2$

Solution

$\because$ $n_1(x) = \frac{x^2}{x^3 - x^2} = \frac{x^2}{x^2(x-1)}$ $\therefore$ $n_1(x) = \frac{1}{x-1}$

the domain of n_1 = R - {0 , 1} (1)

$\because$ $n_2(x) = \frac{x^3 + x^2 + x}{x^4 - x} = \frac{x(x^2 + x + 1)}{x(x^3 - 1)} = \frac{x(x^2 + x + 1)}{x(x-1)(x^2 + x + 1)}$

$\therefore$ $n_2(x) = \frac{1}{x-1}$

the domain of n_2 = R - {0 , 1} (2)

from (1) **and** (2)

$\because$ the domain of n_1 = the domain of n_2 , $n_1(x) = n_2(x)$ for each $x \in R - \{0 , 1\}$

$\therefore$ $n_1 = n_2$

3 **If** $n_1(x) = \frac{x^2 - 4}{x^2 + x - 6}$, $n_2(x) = \frac{x^3 - x^2 - 6x}{x^3 - 9x}$

prove that $n_1(x) = n_2(x)$ for the values of x which belong to the common domain and find the domain.

Solution

$\because$ $n_1(x) = \frac{x^2 - 4}{x^2 + x - 6} = \frac{(x+2)(x-2)}{(x+3)(x-2)} = \frac{x+2}{x+3}$

the domain of n_1 = R - {-3 , 2} (1)

$\because\ n_2(x) = \frac{x^3 - x^2 - 6x}{x^3 - 9x} = \frac{x(x-3)(x+2)}{x(x+3)(x-3)} = \frac{x+2}{x+3}$

and the domain of n_2 = R - {-3 , 0, 3} (2)

from (1) **and** (2)

we notice that : $n_1(x)$, $n_2(x)$ are reduced to the same fraction $\frac{x+2}{x+3}$.

but the domain of $n_1 \neq$ domain of n_2 so $n_1 \neq n_2$.

we can say that : $n_1(x) = n_2(x)$ take the same values if x belongs to the common domain for the two functions n_1 , n_2 R - {-3 , 0 , 2 , 3} .

Drill

Complete the following :

1. The simplest from of the function $f(x) = \frac{4x^2 - 2x}{2x}$, $x \neq 0$ is ....
2. The common domain of the function n_1 , n_2 where $n_1(x) = \frac{x-2}{x^2-4}$,

   $n_2(x) = \frac{1}{x+1}$ is ....
3. if $n_1(x) = \frac{1+a}{x-2}$, $n_2(x) = \frac{4}{x-2}$ and $n_1(x) = n_2(x)$ then a = ...
4. If the simplest form of the algebraical fraction $n(x) = \frac{x^2 - 4x + 4}{x^2 - a}$ is $n(x) = \frac{x-2}{x+2}$ then a = ...
5. If $n_1(x) = \frac{-7}{x+2}$, $n_2(x) = \frac{x}{x-k}$ and the common domain of two function n_1 , n_2

   is R - {-2 , 7} then k = ...

For More Exercises, go to MOE website

2 - 4

Operations on Algebraic fractions

First : Adding and subtracting the algebraic fractions

Think and Discuss

1. **If $\frac{a}{b}$, $\frac{c}{b}$ are two rational numbers then find each of the following :** $\frac{a}{b} + \frac{c}{b}$, $\frac{a}{b} - \frac{c}{b}$

2. **If $\frac{a}{b}$, $\frac{c}{d}$ two rational numbers then find each of the following :** $\frac{a}{b} + \frac{c}{d}$, $\frac{a}{b} - \frac{c}{d}$

From the previous, we can do the operation of adding or subtracting of two algebraic fractions :

If $x \in$ *the common domain of the two algebraic fractions* n_1, n_2 *where :*

1. $n_1(x) = \frac{f_1(x)}{f_2(x)}$, $n_2(x) = \frac{f_3(x)}{f_2(x)}$

(two algebraic fractions having a common denominator)

then : $n_1(x) + n_2(x) = \frac{f_1(x)}{f_2(x)} + \frac{f_3(x)}{f_2(x)} = \frac{f_1(x) + f_3(x)}{f_2(x)}$,

$n_1(x) - n_2(x) = \frac{f_1(x)}{f_2(x)} - \frac{f_3(x)}{f_2(x)} = \frac{f_1(x)}{f_2(x)} + \frac{-f_3(x)}{f_2(x)}$

2. $n_1(x) = \frac{f_1(x)}{f_2(x)}$, $n_2(x) = \frac{f_3(x)}{f_4(x)}$

(two algebraica fractions having two different denominators)

then : $n_1(x) + n_2(x) = \frac{f_1(x)}{f_2(x)} + \frac{f_3(x)}{f_4(x)}$

$= \frac{f_1(x) \times f_4(x) + f_3(x) \times f_2(x)}{f_2(x) \times f_4(x)}$,

$n_1(x) - n_2(x) = \frac{f_1(x)}{f_2(x)} - \frac{f_3(x)}{f_4(x)} = \frac{f_1(x) \times f_4(x) - f_3(x) \times f_2(x)}{f_2(x) \times f_4(x)}$

What you'll learn

- ☆ Doing the operations of (+ , - , × , ÷) on the algebraic fractions

Key terms

- ☆ Additive inverse of the algebraic fractions.
- ☆ multiplicative inverse on the algebraic fractions.

Examples

1 **If** $n_1(x) = \frac{x}{x^2 + 2x}$, $n_2(x) = \frac{x + 2}{x^2 - 4}$

Find **$n(x) = n_1(x) + n_2(x)$ show the domain of n.**

Solution

$\because$ $n(x) = n_1(x) + n_2(x)$

$\therefore$ $n(x) = \frac{x}{x^2 + 2x} + \frac{x + 2}{x^2 - 4} = \frac{x}{x(x + 2)} + \frac{x + 2}{(x - 2)(x + 2)}$

domain n = R - {-2 , 0 , 2}

$\therefore$ $n(x) = \frac{1}{x + 2} + \frac{1}{x - 2} = \frac{x - 2 + x + 2}{(x + 2)(x - 2)} = \frac{2x}{(x + 2)(x - 2)}$

2 *Find :* **n(x) in the simplest form showing the domain of n where :**

$n(x) = \frac{3x - 4}{x^2 - 5x + 6} + \frac{2x + 6}{x^2 + x - 6}$

Solution

$\because$ $n(x) = \frac{3x - 4}{(x - 2)(x - 3)} + \frac{2(x + 3)}{(x - 2)(x + 3)}$

domain n = R - {-3 , 2 , 3}

$\therefore$ $n(x) = \frac{3x - 4}{(x - 2)(x - 3)} + \frac{2}{x - 2}$

$\because$ L.C.M. of denominators = (x - 3)(x - 2) by multiplying the two terms of the second fraction in (x - 3)

$\therefore$ $n(x) = \frac{3x - 4}{(x - 2)(x - 3)} + \frac{2(x - 3)}{(x - 2)(x - 3)} = \frac{3x - 4 + 2x - 6}{(x - 2)(x - 3)}$

$= \frac{5x - 10}{(x - 2)(x - 3)} = \frac{5(x - 2)}{(x - 2)(x - 3)} = \frac{5}{x - 3}$

3 *Find* **n(x) in the simplest form showing the domain of n where :**

$n(x) = \frac{12}{12x^2 - 3} + \frac{2}{2x - 4x^2}$, then find n(0) , n(-1) if possible.

Solution

$\because$ $n(x) = \frac{12}{12x^2 - 3} + \frac{2}{-4x^2 + 2x}$

$= \frac{12}{12x^2 - 3} + \frac{2}{-(4x^2 - 2x)}$ *(descending order) according to the powers of x*

$= \frac{12}{3(4x^2 - 1)} - \frac{2}{2x(2x - 1)} = \frac{4}{(2x + 1)(2x - 1)} - \frac{1}{x(2x - 1)}$ *(Factorize)*

domain n = R - $\{\frac{-1}{2}, 0, \frac{1}{2}\}$

L.C.M of denominators = x (2x + 1) (2x - 1)

$\therefore$ $n(x) = \frac{4x}{x(2x + 1)(2x - 1)} - \frac{2x + 1}{x(2x + 1)(2x - 1)}$

$\therefore$ $n(x) = \frac{4x - (2x + 1)}{x(2x + 1)(2x - 1)} = \frac{4x - 2x - 1}{x(2x + 1)(2x - 1)}$

$= \frac{2x - 1}{x(2x + 1)(2x - 1)} = \frac{1}{x(2x + 1)}$

n(0) does not exist because zero $\notin$ the function domain of n,

$n(-1) = \frac{1}{-1 \times (-2 + 1)} = \frac{1}{-1 \times -1} = 1$

Drill

Find **n(x) in the simplest form showing its domain where :**

1. $n(x) = \frac{x - 2}{x} + \frac{3 + x}{2x}$
2. $n(x) = \frac{2x}{x + 2} + \frac{4}{x + 2}$
3. $n(x) = \frac{2}{x + 3} + \frac{x + 3}{x^2 + 3x}$
4. $n(x) = \frac{x}{x - 4} - \frac{x + 4}{x^2 - 16}$
5. $n(x) = \frac{3}{x - 1} - \frac{2}{x - 1}$
6. $n(x) = \frac{5}{x - 3} + \frac{4}{3 - x}$
7. $n(x) = \frac{x^2}{x - 1} + \frac{x}{1 - x}$
8. $n(x) = \frac{x}{x - 2} - \frac{x}{x + 2}$
9. $n(x) = \frac{x + 3}{2x} - \frac{x}{2x - 1}$
10. $n(x) = \frac{3}{x + 1} + \frac{2x + 1}{1 - x^2}$

Second: Multiplying and dividing the algebraic fractions

Think and Discuss

For each algebric fraction n (x) ≠ 0, there is a multiplicative inverse. It is the reciprocal of the fraction and denoted by n^{-1} (X).

If $n(x) = \frac{x+2}{x+5}$ **, then** $n^{-1}(x) = \frac{x+5}{x+2}$ **where the domain of n = R - {-5}, the domain of** n^{-1} **= R -{-2, -5} and then** $n(x) \times n^{-1}(x) = 1$

From the previous, we can do a multiplication or division of two algebraic fractions as follows :

If n_1 *,* n_2 *are two algebraic fractions where:*

$n_1(x) = \frac{f_1(x)}{f_2(x)}$, $n_2(x) = \frac{f_3(x)}{f_4(x)}$ *then :*

1 $n_1(x) \times n_2(x) = \frac{f_1(x)}{f_2(x)} \times \frac{f_3(x)}{f_4(x)} = \frac{f_1(x) \times f_3(x)}{f_2(x) \times f_4(x)}$

where $x \in$ *the common domain of the two algebraic fractions* n_1 *,* n_2

i.e. $R - (Z(f_2) \cup Z(f_4))$

2 $n_1(x) \div n_2(x) = \frac{f_1(x)}{f_2(x)} \div \frac{f_3(x)}{f_4(x)} = \frac{f_1(x)}{f_2(x)} \times \frac{f_4(x)}{f_3(x)}$

then, the domain of $n_1 \div n_2$ *is the common domain of* n_1 *,* n_2 *,* n_2^{-1}

i.e. $R - (Z(f_2) \cup Z(f_3) \cup Z(f_4))$

Examples

4 **If** $f(x) = \frac{x+1}{x^2 - x - 2} \times \frac{x^2 + 3x - 10}{3x^2 + 16x + 5}$

then find **f(x) in the simplest form and identify its domain, then find f(0) , f(-1) if possible.**

Solution

$f(x) = \frac{x+1}{(x-2)(x+1)} \times \frac{(x+5)(x-2)}{(3x+1)(x+5)}$

$= \frac{(x+1)(x+5)(x-2)}{(x-2)(x+1)(3x+1)(x+5)} = \frac{1}{3x+1}$ *(The simplest form)*

the domain f = R - $\{-5 , -1 , -\frac{1}{3} , 2\}$, f(0) = 1,

f(-1) it is not exist because -1 ∉ the domain of f .

5 **If** $f(x) = \dfrac{x^2 - 9}{2x^2 + 3x} \div \dfrac{3x^2 + 6x - 45}{4x^2 - 9}$

then find **n(x) in the simplest form showing the domain of n .**

Solution

$\because \; n(x) = \dfrac{x^2 - 9}{2x^2 + 3x} \div \dfrac{3(x^2 + 2x - 15)}{4x^2 - 9}$

$\therefore \; n(x) = \dfrac{(x + 3)(x - 3)}{x(2x + 3)} \div \dfrac{3(x + 5)(x - 3)}{(2x + 3)(2x - 3)}$

domain of n $= R - \{0 , -\frac{3}{2}, \frac{3}{2}, -5 , 3\}$

$\therefore \; n(x) = \dfrac{(x + 3)(x - 3)}{x(2x + 3)} \times \dfrac{(2x + 3)(2x - 3)}{3(x + 5)(x - 3)}$

$= \dfrac{(x + 3)(x - 3)(2x + 3)(2x - 3)}{3x(2x + 3)(x + 5)(x - 3)} = \dfrac{(x + 3)(2x - 3)}{3x(x + 5)}$

Drill

Third: Find n(x) in the simplest form identifying a domain in each of the following:

1 $n(x) = \dfrac{x^2 + x + 1}{x} \times \dfrac{x^2 - x}{x^3 - 1}$

2 $n(x) = \dfrac{x^3 - 1}{x^2 - x} \times \dfrac{x + 3}{x^2 + x + 1}$

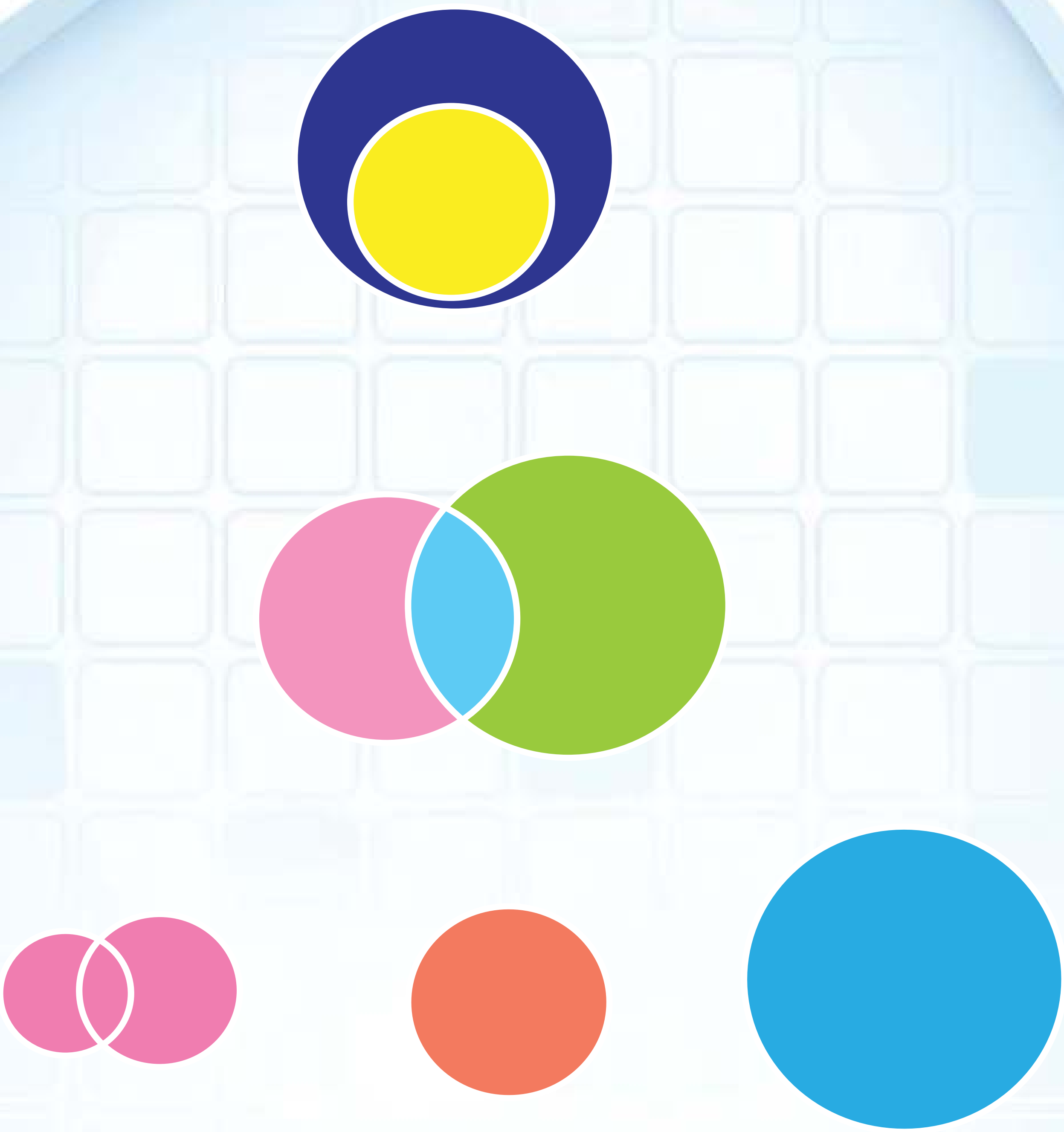
3 $n(x) = \dfrac{3x - 15}{x + 3} \div \dfrac{5x - 25}{4x + 12}$

4 $n(x) = \dfrac{x^2 + 2x - 3}{x + 3} \div \dfrac{x^2 - 1}{x + 1}$

For More Exercises, go to MOE website

Probability

Unit 3 : Probability



3 - 1

Operation on events

What you'll learn

- ☆ Do operations on events (intersection, union).

Key terms

- ☆ Union
- ☆ Intersection
- ☆ Two mutually exclusive events.
- ☆ Venn diagram

Think and Discuss

A regular dice is rolled once randomly and the upper face is observed as :

1. Sample space (S) = { ...., ..., ..., ..., ..., ...}.
2. The event of having 7 is .... and the event is called .... and the probability of appearnce = ...
3. The event of getting a number less than 9 is .... and the event is called .... and the probability of appearance = ...
4. The event of getting a prime even number is .... and it is a subset of .... and the probability of occurrance = $\frac{\dots\dots}{\dots\dots}$

If A is an event of S i e $A \subset S$ then $P(A) = \frac{n\ (A)}{n\ (S)}$

where n (A): number of elements of the event A, n (S) is the number of elements of sample space S, and P (A) is the probability of occurring event (A).

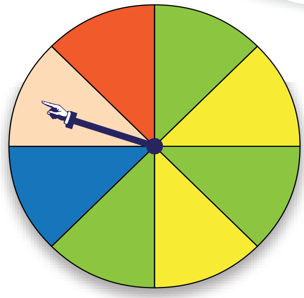
we notice that : probability can be written as a fraction or percentage as follows :

impossible event	less likely	Equally likely as unlikely	More likely	Certain event
0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
0%	25%	50%	75%	100%

Drill

1. A box contains 3 white balls and 4 red balls. If a ball is randomly drawn, then calculate the probability that the ball drawn is...... :
   - A. white.
   - B. white or red.
   - C. blue.

2 **The opposite figure is a spinner divided into eight equal colored sectors *Find*** the probability that the indicater stops on :



A the green color.

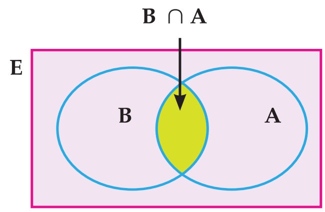
B the yellow color.

C the blue color.

Operations on events :

Events are subset of the sample space (S), so oparetions on events are similar to the operations on sets such as union and intersection. When the sample sapce (S) is considered the universal set, we can represent events and operations on the sample space by using Venn diagrams:

First: intersection

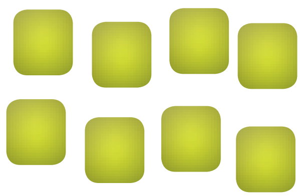


If A and B are two events from a sample space (S), then the intersection of the two events A and B which are denoted by the symbol $A \cap B$ means the events A and B occur together.

Note that : It is said that an event occured if the outcome of the experiment is an element of the elements of the set expressing this event.

Example

A set of identical cards numbered from 1 to 8 with no repetition mixed up and well, if a card is drawn randomly.



1 write down the sample space.

2 write down the following events.

A Event A : The drawn card has an even number.

B Event B : The drawn card has a prime number.

C Event C : The drawn card has a number divisible by 4.

3 Use Venn diagram to calculate the probability of :

A occurring A and B together.

B occurring A and C together.

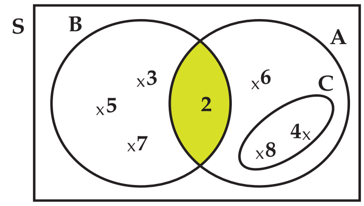
C occurring B and C together.

Solution

1 $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $n(S) = 8$

2 A $A = \{2, 4, 6, 8\}$ B $B = \{2, 3, 5, 7\}$ C $C = \{4, 8\}$

3 Use the venn diagram opposite and find :



A The probability of the occurrence of events A and B together means $A \cap B$ where :

B $A \cap B = \{2\}$ it is a one element set ∴ $n(A \cap B) = 1$

∴ the probability of the occurrence of events A and B together = $P(A \cap B)$

$$= \frac{n(A \cap B)}{n(S)} = \frac{1}{8}$$

C The probability of the occurrence of the events A and C together means $A \cap C$ where :

$A \cap C = \{4, 8\}$ ∴ $n(A \cap C) = 2$

∴ The probability of the occurrence of the events A and C together = $P(A \cap C)$

$$= \frac{n(A \cap C)}{n(S)} = \frac{2}{8} = \frac{1}{4}$$

the probability of the occurrence of the events B and C together means $B \cap C$ where :

$B \cap C = \phi$ (because B and C are two separate or distant sets), $n(B \cap C)$ = zero

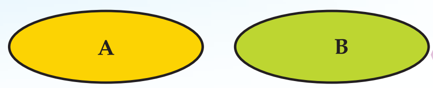
∴ The probability of the occurrence of two events B and C together = $P(B \cap C)$

$$= \frac{n(B \cap C)}{n(S)} = \frac{0}{8} = \text{zero}$$

Notice that : the two events B and C cannot occur at the same time so we say A and B are mutually exclusive events.

Mutually exclusive events.

It is said that A and B are mutually exclusive events if $A \cap B = \phi$



and it is said that a set of events are mutually exclusive if every pair is mutually exclusive.

Drill

A regular dice is rolled once :

1 Write down the sample space.

2 Write the following events:

A A = the event of getting an even number. **B** B = the event of getting an odd number.

C C = the event of getting an a prime even number.

3 Find the following probabilities of :

A The occurrence of two events A and B together.

C The occurrence of two events A and C together.

Second : Union

If A and B are two events from the sample sapce (S) then the union of the two events which is denoted by the symbol $A \cup B$ means the occurrance of the two events A or B or both i.e occurance of at least one event.

Example

1 9 identical cards numbered from 1 to 9 a card was drawn randomally.

First Write down the sample space.

Second Write down the following events :

A Getting a card with an even number.

B Getting a card with a number divisible by 3.

C Getting a card with a prime number greater than by 5.

Third use the venn diagram to calculate the probability of :

a Occurrence of A or B

b Occurrence of A or C

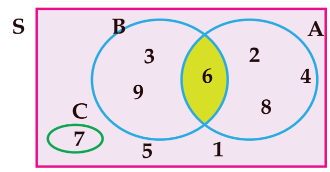
c Find $P(A) + P(B) - P(A \cap B)$, $P(A \cup B)$ what do you notice ?

Solution

First $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $n(S) = 9$

Second $A = \{2, 4, 6, 8\}$, $n(A) = 4$, $B = \{3, 6, 9\}$, $n(B) = 3$, $C = \{7\}$, $n(C) = 1$

Third **In the venn opposite diagram :**



A Occurrence of A or B means $A \cup B$

where : $A \cup B = \{2, 3, 4, 6, 8, 9\}$, $n(A \cup B) = 6$

$\therefore$ probability of the occurrence of A or B = $P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{6}{9} = \frac{2}{3}$

B Occurrence of A or C means $A \cup C$ they are two distant sets.

then $A \cup C = \{2, 4, 6, 7, 8\}$, $n(A \cup C) = 5$

$\therefore$ probability of the occurrence of A or C = $P(A \cup C) = \frac{n(A \cup C)}{n(S)} = \frac{5}{9}$

C $P(A) = \frac{n(A)}{n(S)} = \frac{4}{9}$, $P(B) = \frac{n(B)}{n(S)} = \frac{3}{9}$

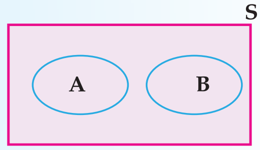
$A \cap B = \{6\}$ $\therefore P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{9}$

$P(A) + P(B) - P(A \cap B) = \frac{4}{9} + \frac{3}{9} - \frac{1}{9} = \frac{2}{3}$ **(1)**

, $P(A \cup B) = \frac{2}{3}$ **(2)**

from **(1), and (2)** we get $P(A) + P(B) - P(A \cap B) = P(A \cup B)$

Remark: From the opposite figure, A and B are mutually exclusive events from the sample space S, then :



1. $A \cap B = \phi$
2. $P(A \cap B) = \dfrac{\text{number of elements of } \phi}{\text{number of elements of S}} = \dfrac{\text{Zero}}{\text{number of elements of S}} = \text{Zero}$

Notic that A and B are mutually exclusive events.

Then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ but $(A \cap B) =$ zero

$\therefore P(A \cup B) = \frac{4}{9} + \frac{1}{9} -$ zero

$= \frac{5}{9}$ As previously found

i.e if A and B are two mutually exclusive events then $P(A \cup B) = P(A) + P(B)$

Drill

1. If A and B are two events in the sample space of a random experiment complete :

   A) $P(A) = 0.2$, $P(B) = 0.6$, $P(A \cap B) = 0.3$, $P(A \cup B) = ...$

   B) $P(A) = 0.55$, $P(B) = \frac{3}{10}$, $P(A \cap B) = ...$, $P(A \cup B) = \frac{13}{20}$

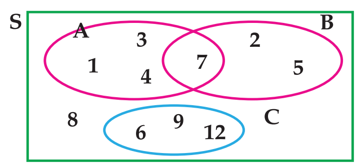
   C) $P(A) = ...$, $P(B) = \frac{1}{4}$, $P(A \cap B) =$ zero, $P(A \cup B) = 0.9$

2. use the venn opposite diagram to find:

   A) $P(A \cap B)$, $P(A \cup B)$

   B) $P(A \cap C)$, $P(A \cup C)$

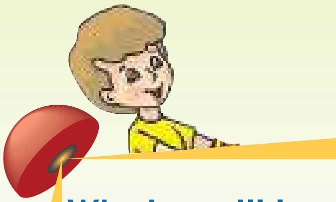
   C) $P(B \cap C)$, $P(B \cup C)$



For More Exercises, go to MOE website

3 - 2

Complementary event and the difference between two events



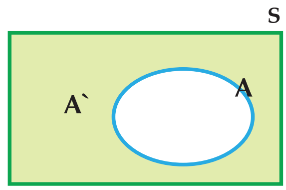
What you'll learn

- ☆ The concept of the complementary even
- ☆ The concept of the difference between two events.

Key terms

- ☆ complementary event
- ☆ difference between two events.

Think and Discuss



In the venn diagram opposite :

If S is the universal set, $A \subset S$

then the complementery set of A is A`

Complete :

1. A ∪ A` = ......... , A ∩ A` = ...
2. If S = {1, 2, 3, 4, 5, 6, 7} A = {2, 4, 6} then: A` = {.............}.

From the previous, we notice that : If S is the sample space of a random experiment and one ball is drawn from a box having identical balls numbered from 1 to 7 and observing the number on it.

A is the event of getting even number : A = {2, 4, 6}

A` is the event of getting an odd number : A` = {1, 3, 5, 7} and it is a complementry event to A .

The complementry event :

i.e : If $A \subset S$ then A` is the complementry event to event A

where A ∪ A` = S , A ∩ A` = ϕ

i.e the event and the complementary event are two mutually exclusive events.

Drill

If S the sample space of a random experiment, $A \subset S$, A`is the complementry event to the event A and S = { 1 , 2 , 3 , 4 , 5 , 6 }.

Complete the following table and record your observation.

event A	event A`	P (A)	P (A`)	P (A) + P (A`)
{2, 4, 6}				
	{3, 6}			
{5}				
{1, 2, 3, 4, 5, 6}				

From the previous table, notice that : P (A) + P (A`) = 1 **then :** P (A`) = 1 - P (A) , P (A) = 1 - P (A`)

Note : P (A) + P (A`) = P (S) = 1

Example

1 A classroom contains 40 students. 18 of them read Al-Akhbar newspaper, 15 read Al Ahram news paper and 8 read both newspapers. If a student is selected randomly calculate the probability that the student :

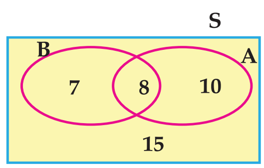
A reads Al-Akhbar newspaper

B doesn't read Al-Akhbar newspaper

C reads Al-Ahram newspaper

D reads both newspaper.

Solution



Let the event A be reading Al Akhbar newspaper and the event B reading Al Ahram newspaper. then A ∩ B is the event of reading both newspapers.

then n (S) = 40 , n (A) = 18 , n (B) = 15 , n(A ∩ B) = 8

A event A : Read Al Akhbar newspaper then $P(A) = \frac{n(A)}{n(S)} = \frac{18}{40} = \frac{9}{20}$

B Does not read Al Akhbar is the complementary event of the event A and it is A`.

$\therefore P(A`) = \frac{\text{number of elements of s set A`}}{n(S)} = \frac{15+7}{40} = \frac{22}{40} = \frac{11}{20}$

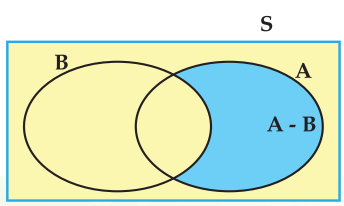
Another solution : $P(A`) = 1 - P(A) = 1 - \frac{9}{20} = \frac{11}{20}$

C event B : read Al Ahram newspaper then : $P(B) = \frac{n(B)}{n(S)} = \frac{15}{40} = \frac{3}{8}$

D event A ∩ B means reading both newspaper

$\therefore P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{8}{40} = \frac{1}{5}$

Think : Does the event reading Al Akhbar newspaper mean to read Al Akhbar newspaper only? Explain your answer.



Notice that : The event of reading Al Akhbar newspaper is represented by venn opposite diagram by set A while the event of reading Al Akhbar only but not other newspaper is represented by the set A - B **and read as A difference B**

The difference between two events

If A , B are events of s, then A-B is the event of the occurrence of A and the non-occurrence of B, i.e., the occurrence of the event A only. Note that : $(A - B) \quad (A \cap B) = A$

Drill **In the previous example Find :**

1. the probability that the student reads Al - Akhbar newspaper only.
2. the probability that the student reads Al - Ahram newspaper only.
3. the probability that the student reads Al - Akhbar only or Al - Ahram only.

For More Exercises, go to MOE website

Unit (4): The Circle

Drivers are to be familiar with traffic signs well and to distinguish between them.

Search in the different knowledge resources (traffic department - library - internet) for traffic signs.

4 - 1 Basic Definitions and Concepts

What you'll learn

- ☆ The basic concepts related to the circle.
- ☆ The concept of axis of symmetry in the circle.

Key terms

- ☆ Circle
- ☆ Surface of a circle
- ☆ Radius
- ☆ chord
- ☆ Diameter
- ☆ Axis of symmetry in a circle

Think and Discuss

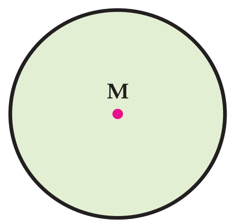
Yousef used the program, **Google Earth** , on his computer to study the geography of Egypt.
Yousef noticed some green, circular areas next to the desert areas so, he asked his father about them.

The father Said: You learn that a drop of water means the source of life. Therefore, we should minimize the consumption of water in order to irrigate the land by the central irrigation method (sprinkle irrigation) in which, the wheels of the irrigation machine circle around a fixex point which draws those circles.

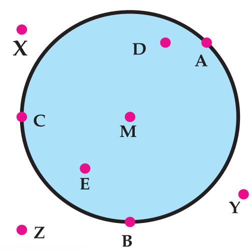
1. How can you draw the circle of a football field?
2. What is your role in minimizing the consumption of water?

The circle: is the set of points of a plane which are at constant distance from a fixed point in the same plane. The fixed point is called the centre of the circle and the constant distance is called the radius length.

The circle is usually denoted by its center. So we say, circle M to mean the circle which its center is point M, as in the figure opposite.



When drawing circle M in a plane, it divides the points of the plane in to three sets of points as in the figure, and they are :



1. The set of points inside the circle like points: M, D, E, ...
2. The set of points on the circle like points: A, B, C, ...
3. The set of points outside the circle like points: X, Y, Z, ...

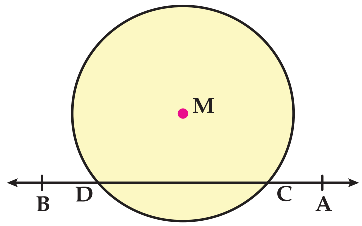
Surface of the circle :

set of points of the circle ∪ the set of points inside the circle

Drill

Inthe figure opposite, complete :

1. $\overleftrightarrow{AB}$ ∩ circle M = ...
2. $\overleftrightarrow{AB}$ ∩ surface of circle M = ...
3. M ∉ circle M, M ∈ ....

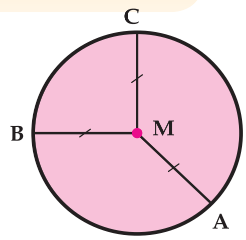


Radius of a circle : is a line segment with one endpoint at the center and the other endpoint on the circle.

In the figure opposite $\overline{MA}$, $\overline{MB}$, $\overline{MC}$ are radii for circle M where :

MA = MB = MC = radius length of the circle (r)

Two circles are congruent if their radii are equal in length.

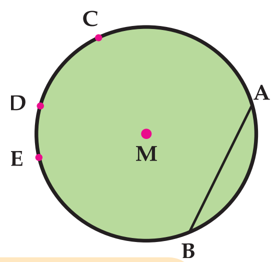


The chord : is a line segment whose end points are any two points on the circle.

Drill

In the figure opposite :

Draw all the chords of the circle which pass through the pairs of points A, B, C, D, E.



Diameter : is the chord passing through the center of the circle.

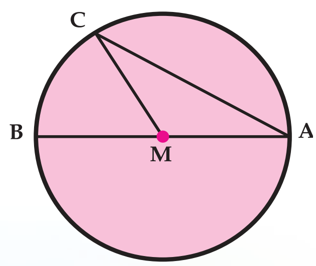
Drill

1. Which chord in the following figure is a diameter in circle M ?
2. What are the number of diameters in any circle ?
3. To prove that the diameter of a circle is its largest chord in length, complete :

   In the triangle A M C : AM + MC > ....

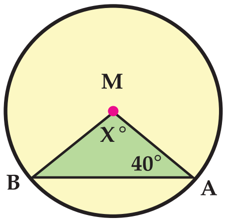
   In circle M : CM = BM (radii)

   Thus : AM + ... > .... ∴ AB > ....

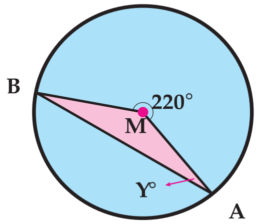


4. If the radius length of a circle = r then the diameter length = ... and the perimeter of the circle = ... , area of the circle = ...

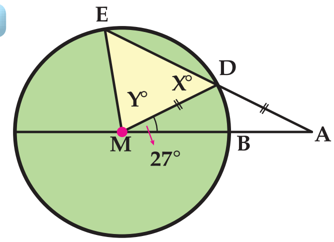
5. **In each of the following figures find the value of the used symbol in measuring :**

a


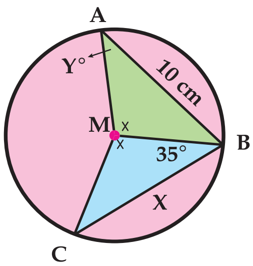
X = ...

b


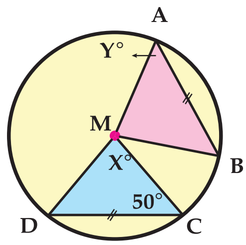
Y = ...

c


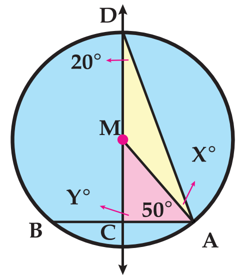
X = ...,
Y = ...

d


X = ...,
Y = ...

e


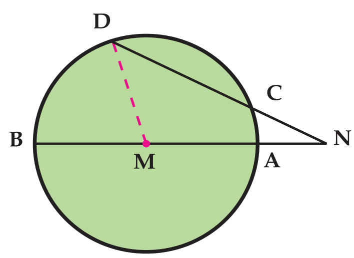
X = ...,
Y = ...

f


X = ...,
Y = ...

Example 1

In the figure opposite : $\overline{AB}$ is a diameter in circle M. $\overrightarrow{BA} \cap \overrightarrow{DC} = \{N\}$. ***Prove that :*** NB > ND.



Solution

Draw a radius $\overline{MD}$ in Δ N M D :

M N + M D > N D

∵ M B = M D (radii)

∴ M N + M B > N D

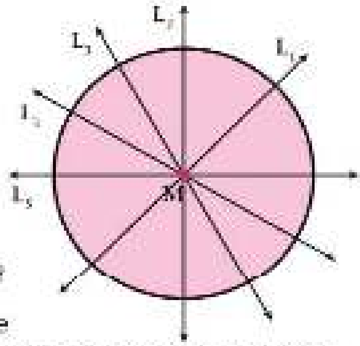
∴ N B > N D **(Q.E.D.)**

Drill

In the previous example, prove that : N C > N A.

Symmetry in the circle

Activity 1



1. Draw circle M on a transparent paper using compasses.
2. Draw the straight line L_1 passing through the center of the circle and dividing it in to two arcs.
3. Fold the paper around the straight line L_1, what do you notice?
4. Draw another straight line L_2 passing through the center of the circle and, then fold the paper around it - repeat this step a number of times by drawing the straight lines L_3, L_4, ... , ... what do you notice in each case ?

From the previous activity we deduce that :

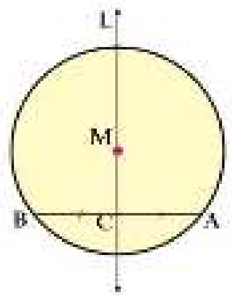
> ***Any straight line passing through the center of a circle is an axis of symmetry of it.***

Think : What is the number of axes of symmetry in the circle?

Activity 2

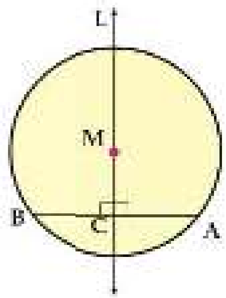
Study each of the following figures (as given in the drawing). What do you deduce?

1.



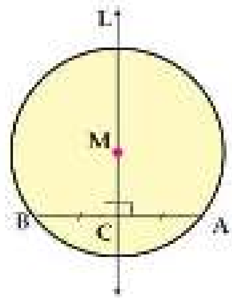
Deduction : ....

2.



Deduction : ....

3.



Deduction : ....

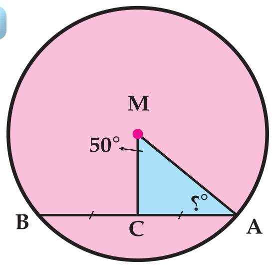
Important Corollaries

From 1	the straight line passing through the center of the circle and the midpoint of any chord of it is perpendicular to this chord.
From 2	the straight line passing through the center of a circle and perpendicular to any chord of it bisects this chord.
From 3	the perpendicular bisector of any chord of a circle passes through the center of the circle.

Drill

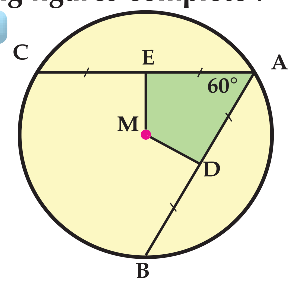
1 **M circle in each of the following figures complete :**

A



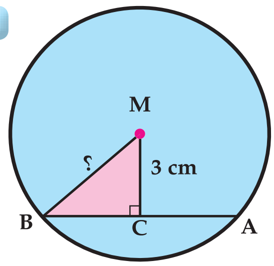
m (∠ M A C) = ...

B



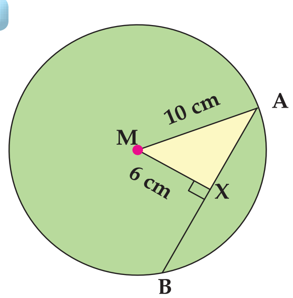
m (∠ D M E) =................

C



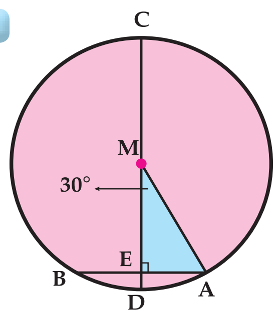
If A B = 8 cm,
then M B = ...

D



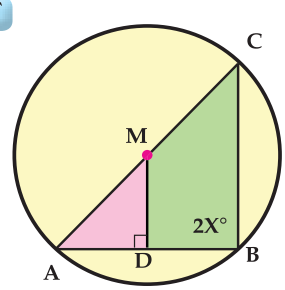
A B = ...

E



If A B = 10 cm,
then C D = ...

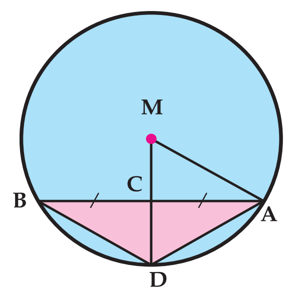
F



X = ...

2 **In the figure opposite :** M circle with radius length 13 cm, $\overline{AB}$ is a chord of length 24 cm, C is the midpoint of $\overline{AB}$, $\overrightarrow{MC}$ ∩ circle M = {D}.

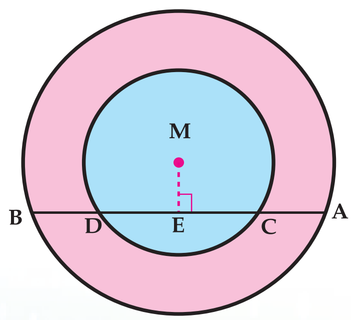
Find the area of the triangle A D B.



Example 2

In the figure opposite: two concentric circles M, $\overline{AB}$ is a chord in the larger circle intersecting the smaller circle at C and D :

Prove that : AC = BD.



Solution

Given : $\overline{AB}$ ∩ the smaller circle = {C, D}

R. T. P : A C = B D

Construction : Draw $\overline{ME} \perp \overline{AB}$ to intersect it at E .

Proof : In the larger circle $\overleftrightarrow{ME} \perp \overline{AB}$ $\therefore$ E A = E B **(1) (corollary)**

In the smaller circle $\overleftrightarrow{ME} \perp \overline{CD}$ $\therefore$ E C = E D **(2) (corollary)**

By subtracting (2) from (1), we get:

E A - E C = E B - E D $\therefore$ A C = B D **(Q . E . D.)**

Drill

In the figures opposite :
What are the lines segments that are equal in length? Explain your answer.

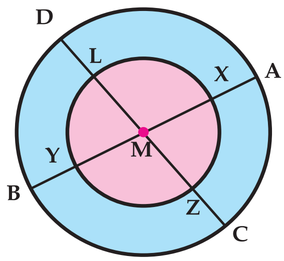


fig (2)

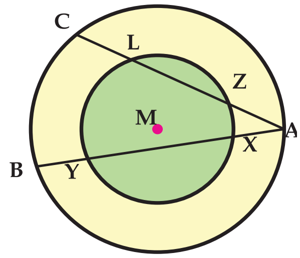
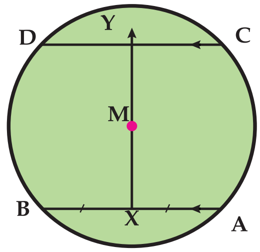


fig (1)

Example 3

In the figure opposite : M circle, $\overline{AB} \parallel \overline{CD}$, X is the midpoint of $\overline{AB}$, $\overrightarrow{XM}$ is drawn to intersect $\overline{CD}$ at Y. ***Prove that*** Y is the midpoint of $\overline{CD}$



Solution

Given : $\overline{AB} \parallel \overline{CD}$, A X = B X

R . T . P : CY = DY

Proof : $\because$ X is the midpoint of $\overline{AB}$ $\therefore$ $\overleftrightarrow{MX} \perp \overline{AB}$

$\because$ $\overline{AB} \parallel \overline{CD}$, $\overleftrightarrow{XY}$ intersects them $\therefore$ m ($\angle$ D Y X) = m ($\angle$ A XY) = 90° alternating angles

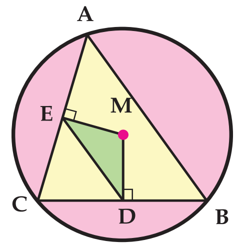
$\therefore$ $\overleftrightarrow{MY} \perp \overline{CD}$ $\therefore$ Y is the midpoint of $\overline{CD}$ **(Q.E.D)**

Drill

$\overline{AB}$ and $\overline{CD}$ are two parallel chords in circle M. AB = 12 cm , CD = 16 cm. Find the distance between those two chords if the radius length of circle M equals 10 cm. Are there any other answers? Explain your answer.

Think If $\overline{AB}$ and $\overline{CD}$ are two parallel chords in a circle where AB > CD, which chord is closer to the center of the circle ? Explain your answer.

Example 4



In the figure opposite : ABC triangle is an inscribed triangle inside a cricle with center M, $\overline{MD} \perp \overline{BC}$, $\overline{ME} \perp \overline{AC}$.

Prove that : First : $\overline{ED} \parallel \overline{AB}$

Second : Perimeter $\Delta CDE = \frac{1}{2}$ Perimeter ΔABC

Solution

Given : $\overline{MD} \perp \overline{BC}$ and $\overline{ME} \perp \overline{AC}$

R . T . P : First: $\overline{ED} \parallel \overline{AB}$

Second : Perimeter $\Delta CDE = \frac{1}{2}$ Perimeter ΔABC

Proof :

First: $\because \overline{MD} \perp \overline{BC}$ $\therefore$ D is the midpoint of $\overline{BC}$ **(1)**

$\because \overline{ME} \perp \overline{AC}$ $\therefore$ E is the midpoint of $\overline{AC}$ **(2)**

in ΔABC, D is the midpoint of $\overline{BC}$ and E is the midpoint of $\overline{AC}$

$\therefore \overline{DE} \parallel \overline{AB}$ **(Q.E.D 1)**

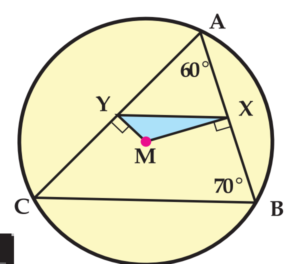
$DE = \frac{1}{2} AB$ **(3)**

Second : **From (1), (2), (3) :**

$\therefore$ Perimeter $\Delta CDE = CD + CE + ED = \frac{1}{2} CB + \frac{1}{2} AC + \frac{1}{2} AB$

$= \frac{1}{2}(CB + AC + AB) = \frac{1}{2}$ Perimeter ΔABC

Drill



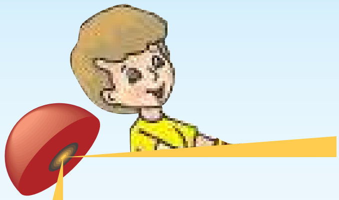
In the figure opposite : In circle M, $\overline{MX} \perp \overline{AB}$, $\overline{MY} \perp \overline{AC}$,

$m(\angle A) = 60°$, $m(\angle B) = 70°$.

Find : the measures of the angles of the triangle M X Y.

For More Exercises, go to MOE website

4 - 2 Positions of a point, a straight Line and a circle with respect to a circle.



What you'll learn

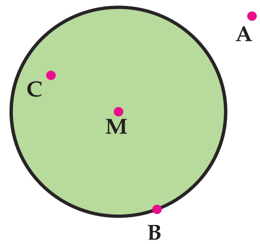
- ☆ Identifying the position of a point with respect to a circle.
- ☆ Position of a straight line with respect to a circle.
- ☆ Relation of the tangent with the radius of a circle.
- ☆ Position of a circle with respect to another circle.
- ☆ Relation of the line of centers with the common chord and the common tangent.

Key terms

- ☆ Point is outside a circle.
- ☆ Point is on the circle
- ☆ Point is inside a circle.
- ☆ Two distant circles.
- ☆ Two intersecting circle.
- ☆ Two circles touching
- ☆ Common tangent
- ☆ Line of centers
- ☆ Common chord

First: Position of a point with respect to a circle.

Think and Discuss

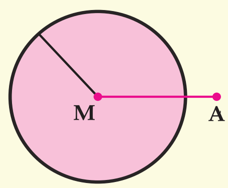


In the figure opposite, circle M divides the points of the plane in to three sets of points.

1. How can you determine the position of the points: A, B, and C with respect to circle M ?
2. What is the relation between (MA, r), (MB , r) and (MC , r) ?

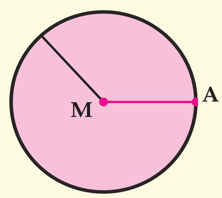
If M circle with radius length r and A was a point on the circle plane, then:

1 A is outside the circle



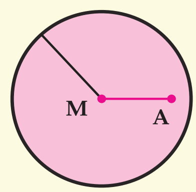
So : $MA > r$
and vise versa

2 A is on the circle



So : $MA = r$
and vise versa

3 A is inside the circle



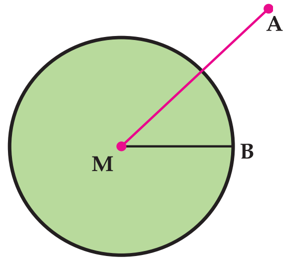
So : $MA < r$
and vise versa

Drill

If M circle with radius length = 4 cm and A is a point in its plane, Complete :

1. ***IF:*** $MA = 4$ cm, then A is .... circle M, because ....
   ....
2. ***IF:*** $MA = 2\sqrt{3}$ cm, then A is ....circle M, because ....
   ....
3. ***IF:*** $MA = 3\sqrt{2}$ cm, then A is ....circle M, because ....
   ....
4. ***IF:*** MA = zero, then A is..............circle M and represented by ....

Example 1



If circle M with radius length 5 cm, A is a point in its plane and MA = 2x - 3 cm. ***Find*** the values of X, if A is located outside the circle.

Solution

$\because$ Point A is located outside the circle M $\therefore$ MA > 5 So : 2X - 3 > 5 i.e. 2X > 8 $\therefore$ X > 4

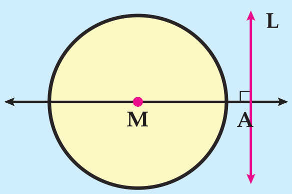
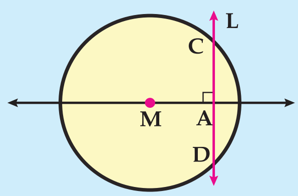
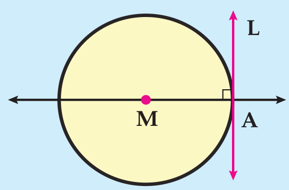
Drill

From the previous example, find the value of X in the following cases :

1. MA = 2x + 1, point A on the circle.
2. MA = 8x – 27, point A inside the circle.

Second: Position of a straight line with respect to a circle :

If M circle with radius length of r, L is a straight line on its plane, $\overleftrightarrow{MA} \perp L$ where $\overleftrightarrow{MA} \cap L = \{A\}$, Then:

1	2	3
the straight line L is located outside the circle M L ∩ circle M = ϕ	the straight line L is a secant to the circle M L ∩ circle M = {C, D}	the straight line L is tangent to circle M L ∩ the circle = {A}
		
So : MA > r and vise verse	So : MA < r and vise verse	So : MA = r and vise verse

Think : In each of the following cases, Find L ∩ surface of circle M.

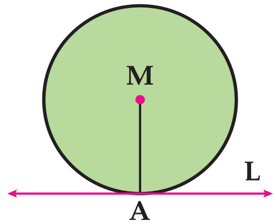
Drill

If M circle with radius length 7 cm and $\overline{MA} \perp L$ where $A \in L$. Complete the following:

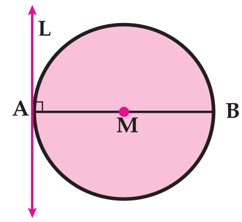
1. ***If*** MA = $4\sqrt{3}$ cm Then the straight line L ....
2. ***If*** MA = $3\sqrt{7}$ cm Then the straight line L ....
3. ***If*** 2 MA - 5 = 9 Then the straight line L ....
4. ***If*** the straight line L intersects circle M and MA = 3X -5 Then X ∈ ....
5. ***If*** the straight line L tangent to circle M and MA = $X^2 - 2$ Then X ∈ ....

Important facts

1 A tangent to a circle is perpendicular to the radius at its point of tangency.



2 If a straight line is perpendicular to a diameter of a circle at one of its endpoints, then it is a tangent to the circle.

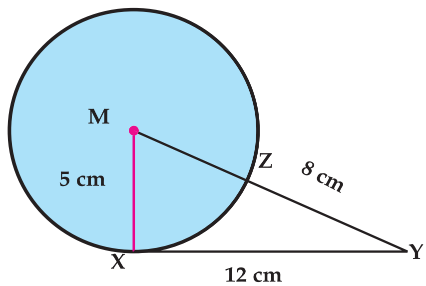


Think

1. **How many tangents can be drawn to circle M ?**
   First : from a point on the circle.
   Second: from a point outside the circle.
2. What is the relation between the two drawn tangents to the circle from the two end points of any diameter in it ?

Example 2

In the figure opposite: M circle with radius length of 5 cm, XY = 12 cm, $\overline{MY} \cap$ circle M = {Z} and ZY = 8 cm.



Prove that : $\overleftrightarrow{XY}$ is a tangent to circle M at X.

Solution

$\because$ $\overline{MY} \cap$ circle M = {Z} $\quad\therefore$ MY = M Z + ZY

$\because$ M Z = M X = 5 cm (radii) $\quad\therefore$ MY = 5 + 8 = 13 cm

$\because$ $(MY)^2 = (13)^2 = 169$, $(MX)^2 = (5)^2 = 25$, $(XY)^2 = (12)^2 = 144$

$\therefore$ $(MX)^2 + (XY)^2 = 25 + 144 = 169 = (MY)^2$

$\therefore$ m ($\angle$ M XY) = 90° **(The converse of the pythagorean theorem)**

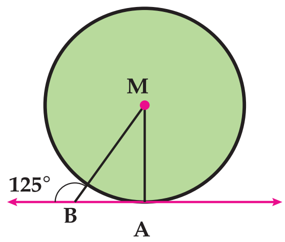
$\because$ $\overleftrightarrow{XY} \perp \overline{MX}$

$\therefore$ $\overleftrightarrow{XY}$ is a tangent to the circle at X. **(Q.E.D)**

Drill

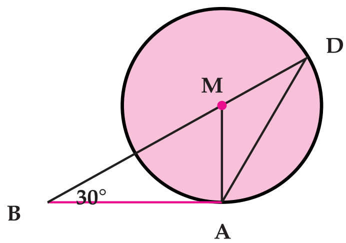
1 **M circle is in each of the following figures and $\overleftrightarrow{AB}$ is a tangent : Complete :**

A



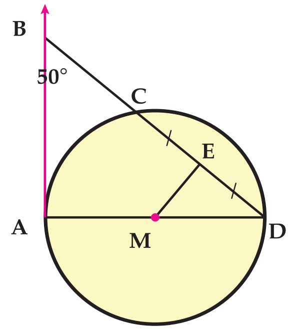
m (∠ A M B) = ...

B



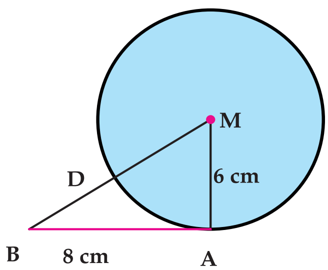
m (∠ A D B) = ...

C



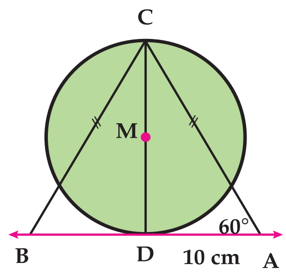
m (∠ A M E) = ...

D



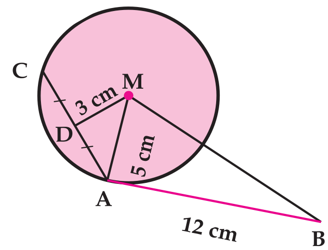
D B = ... cm

E



Perimeter Δ A B C = ... cm

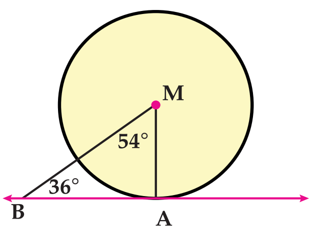
F



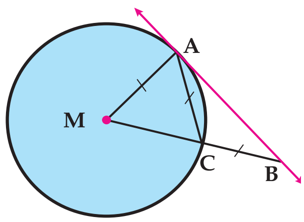
Perimeter of the figure A B M D = ... cm

2 **In each of the following figures, explain why $\overleftrightarrow{AB}$ is a tangent to circle M :**

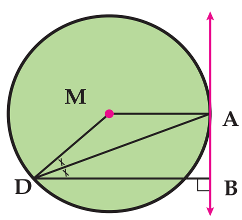
A



B

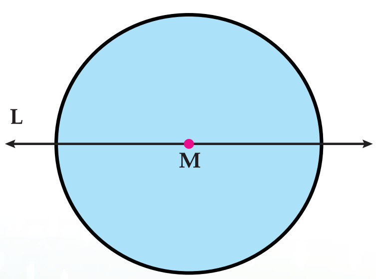


C



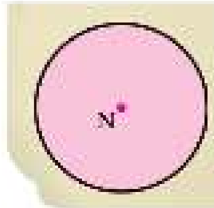
Third : Position of a circle with respect to another circle.

Activity



1. Draw a circle with center M and with an appropriate radius length = r_1 cm.
2. Draw one of the axes of symmetry of circle M. Let it be the straight line L as in the figure opposite.

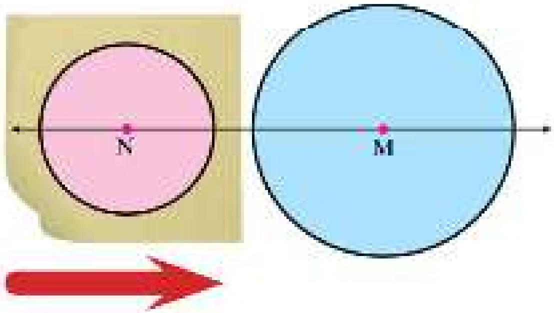
3. On a transparent paper draw a circle with center N and with an appropriate radius length $= r_2$ cm where $r_2 < r_1$.



4. Put the transparent paper where point N belongs to the straight line L.

Notice that : the straight line $L = \overleftrightarrow{MN}$ is called $\overleftrightarrow{MN}$ the line of centers of the two circles M and N and it is an axis of symmetry for both of them.

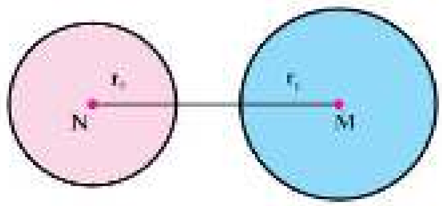
5. Move the transparent paper towords circle M where N remains $\in$ L to see different positions of the two circles. Measure the two circles in relation to each other. Mesure length of $\overline{MN}$ in each case.
What is the relation between the length of $\overline{MN}$ **(the distance between the centers of the two circles M and N)**, $r_1 + r_2$ or $r_1 - r_2$ in each position.



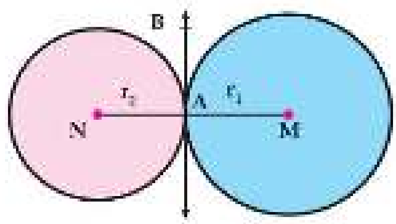
Drill

If M and N are two circles on the plane, their two radii are r_1 and r_2 respectively where $r_1 > r_2$. Complete:

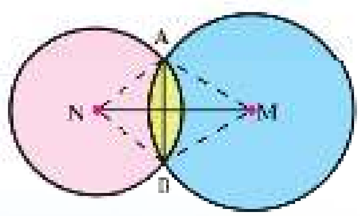
1. If $MN > r_1 + r_2$, then $M \cap N =$ ...., surface of circle $M \cap$ surface of circle N = ... and the two circles are distant.



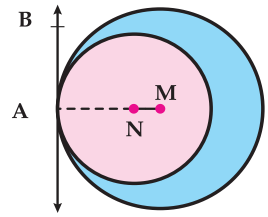
2. If $MN = r_1 + r_2$, then $M \cap N =$ .... surface of circle $M \cap$ surface of circle N = ... and the two circles are touching externally.



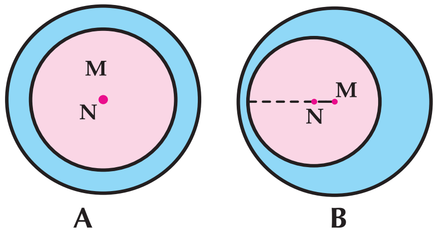
3. If $r_1 - r_2 < MN < r_1 + r_2$, then $M \cap N =$ ...., surface of circle $M \cap$ surface of circle N = the surface of the yellow area and the two circles are intersecting.



4 If: $MN = r_1 - r_2$, then $M \cap N$ = ..., surface of circle M ∩ surface of circle N = ... and the two circles are touching internally.



5 If: $MN < r_1 - r_2$ and then $M \cap N$ =.........., surface of circle M ∩ surface of circle N = ... and the two circles are intersecting as in figure .... when MN = zero, the two circles are concentric. as in figure ....



Corollaries

1 The line of centers of two touching circles passes through a point of tangency and is perpendicular to the common tangent.

2 The line of centers of two intersecting circles is perpendicular to the common chord and bisects it.

Example 3

Two circles M and N with radii length of 9 cm and 4 cm respectively. Show the position of each of them with respect to the other in the following cases:

A MN = 13 cm
B MN = 5 cm
C MN = 3 cm
D MN = zero
E MN = 10 cm
F MN = 15 cm

Solution

∵ $r_1 = 9$ cm, $r_2 = 4$ cm ∴ $r_1 + r_2 = 13$ cm and $r_1 - r_2 = 5$ cm

A MN = 13 cm ∴ $MN = r_1 + r_2$ ∴ the two circles are touching externally.

B MN = 5 cm ∴ $MN = r_1 - r_2$ ∴ the two circles are touching internally.

C MN = 3 cm ∴ $MN < r_1 - r_2$, $MN \neq 0$ ∴ circle N is inside circle M.

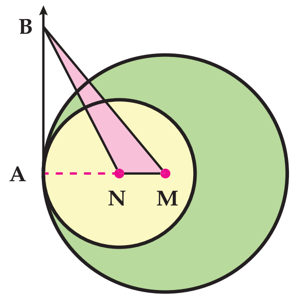
D MN = zero ∴ the two circles are concentric.

E MN = 10 cm ∴ $r_1 - r_2 < MN < r_1 + r_2$ ∴ the two circles are intersecting.

F MN = 15 cm ∴ $MN > r_1 + r_2$ ∴ the two circles are distant.

Example 4

M and N are two circles with radii length of 10 cm and 6 cm respectively and are both touching internally at A, $\overleftrightarrow{AB}$ is a common tangent for both at A. If the area of the triangle B M N = 24 cm^2 ***Find*** the length of $\overline{AB}$.



Solution

$\because$ The two circles are touching internally at A $\quad \therefore A \in \overleftrightarrow{MN}$, $\overleftrightarrow{MN} \perp \overleftrightarrow{AB}$

then the length of $\overline{AB}$ is the height of the triangle B M N whose base is $\overline{MN}$

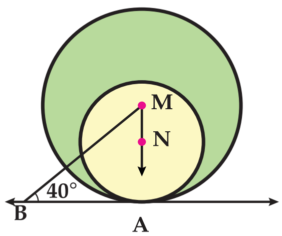
where : M N = 10 - 6 = 4 cm (why ?)

Area Δ B M N = $\frac{1}{2} \times$ M N $\times$ A B $\quad \therefore 24 = \frac{1}{2} \times 4 \times$ A B $\quad \therefore$ A B = 12 cm

Drill

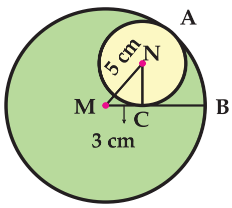
In each of the following figures the circles are touching two - by - two. Use the information of each figure and complete :

1



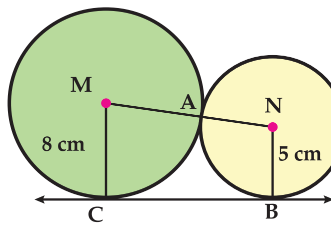
m ($\angle$ B M N) = ...°

2



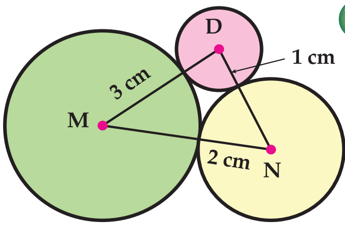
B C = ... cm

3



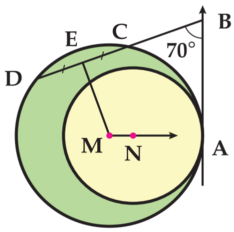
B C = ... cm

4



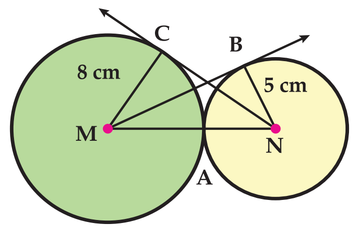
m ($\angle$M D N) = ... °

5



m ($\angle$ E M N) = ...°

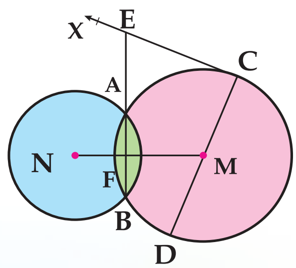
6



M B = ... cm,
N C = ... cm

Example 5

M and N are two intersecting circles at A and B, $\overline{CD}$ is a diameter in circle M and $\overrightarrow{CX}$ is a tangent to the circle M at C where $\overrightarrow{CX} \cap \overrightarrow{BA}$ = {E}, $\overleftrightarrow{MN} \cap \overline{AB}$ = {F}. ***Prove that:*** m ($\angle$ DMN) = m ($\angle$ CEB).



Solution

Given : circle M ∩ circle N = {A, B}, $\overline{CD}$ is a diameter in circle M and $\overrightarrow{CX}$ is a tangent to circle M.

R. T. P: Prove that m (∠ D M N) = m (∠ C E B).

Proof : ∵ the line of centers is perpendicular to the common chord.

∴ $\overleftrightarrow{MN} \perp \overline{AB}$ i.e m(∠ A F M) = 90°

∵ $\overline{CD}$ is a diameter in circle M and $\overrightarrow{CX}$ is a tangent at C

∴ $\overrightarrow{CX} \perp \overline{CD}$ i.e m (∠ E C D) = 90°

∴ m (∠ C E F) + m(∠ C M F) = 360° - (90° + 90°) = 180° **(why ?)**

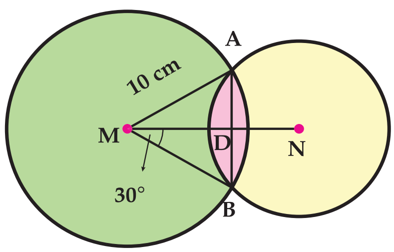
∵ m (∠ D M F) + m (∠ C M F) = 180°

∴ m (∠ D M N) = m (∠ C E F) **(Q.E.D)**

Drill

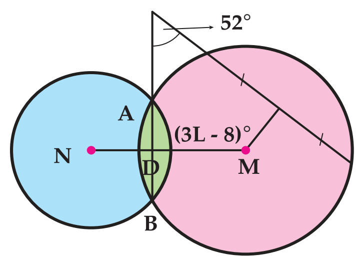
1 **In each of the following figures M and N are two intersecting circles at A and B Complete :**

A



A B = ... cm

B



L = ...

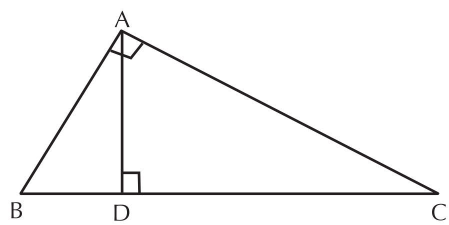
Notice that:

A B C is a right angled triangle at A. If $\overline{AD} \perp \overline{BC}$ then :

$(AB)^2 = BD \times BC$ ***(Euclidean theorem)***

, $(AD)^2 = DB \times DC$ ***(Corollary)***

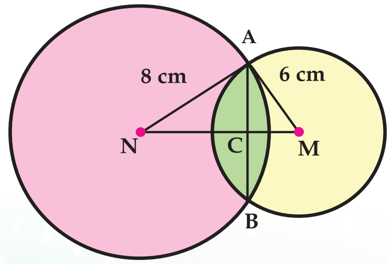
, A D × B C = A B × A C ***Why ?***



2 **In the figure opposite :** M and N are two intersecting circles at A, B

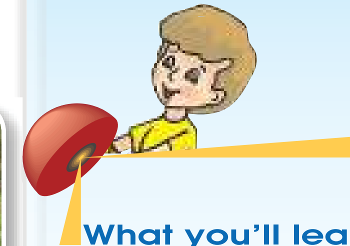
$\overline{MN} \cap \overline{AB}$ = {C}, A M = 6 cm, A N = 8 cm and $\overline{MA} \perp \overline{AN}$.

Find the length of $\overline{AB}$



For More Exercises, go to MOE website

Identifying the circle



4 - 3

What you'll learn

☆ How to draw a circle passing through a given point.
☆ How to draw a circle passing through two given points.
☆ How to draw a circle passing through three given points.

Key terms

☆ Circumcircle

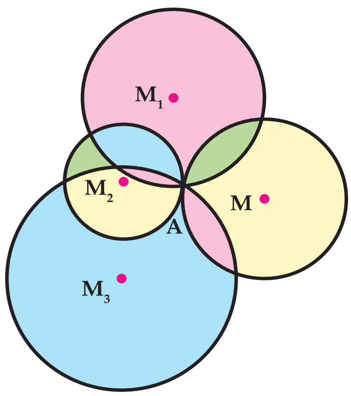
Think and Discuss

- Why is a compass used in drawing a circle?
- What is the axis of the straight segment ?
- Is the center of the circle located on the axis of any chord in it?
- How can you draw (identify) a circle on a plane?

A circle can be drawn (identified) with given terms:

1. Center of the circle.
2. Radius length of the circle.

First: Drawing a circle passing through a given point :



Given : A is a given point on the plane.

R.T.P: Draw a circle passing through point A.

Construction :

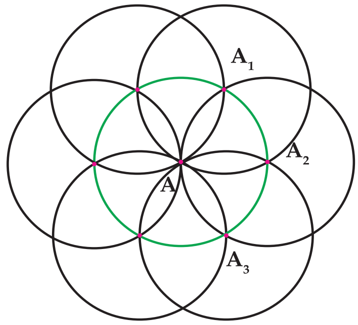
1. Take any choosen point as M on the same plane.
2. State the tip of the compass at M and with an opening equalling MA draw the circle M. The circle lM passes through point A.
3. State the tip of the compass at another point M_1 and with an opening equalling M_1A draw circle M_1. The circle M_1 passes through point A.
4. Repeat the previous work and note :

For each choosen point (center of the circle) it is possible to draw a circle passing through point A.

✍ What are the number of points on the plane? What are the number of circles that can be drawn and pass through point A ?

✍ If the radii of these circles are equal in length, where are their centers located ?

From the previous we deduce that:



1. An infinite number of circles can be drawn passing through a given point as A.
2. If the radii of these circles are equal in length then their centers are located on a congruent circle and its center is point A.

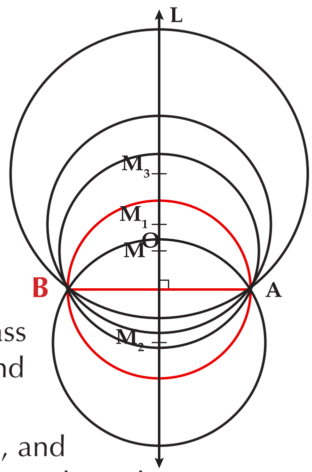
Drill

If L is a straight line on the plane; A is a given point where $A \in L$. Use the geometric tools and draw a circle passing through point A, with radius length 2 cm. How many circles can be drawn? (do not erase the arces).

Second: Drawing a circle passing through two given points:

Given: A and B are two given points in the plane.

R.T.P: Draw circle M passing through the two points A and B i.e $\overline{AB}$ is a chord in circle M.



Construction:

1. Draw the straight segment $\overline{AB}$.
2. Draw the straight line L, the axis of $\overline{AB}$ where $L \cap \overline{AB} = \{F\}$.
   (the center of the circle is on the axis of the chord $\overline{AB}$).
3. Take any chosen point M where $M \in L$, state the tip of the compass at M and with an opening equalling M A, draw the circle M to find that it passes through point B.
4. State the tip of the compass at another point as M_1 where $M_1 \in$ L, and with an opening equalling M_1 , A, draw the circle M_1 where it passes through point B.
5. Repeat the previous work and note :

For each chosen point E on the axis of $\overline{AB}$ (center of the circle), it is possible to draw a circle passing through the two points A and B.

✍ What is the number of points of the straight line L? What is the number of circles that can be drawn andpass through the two points A and B ?

✍ What is the radius length of the smallest circle that can be drawn to pass through the two points A and B ?

✍ Can two circles intersect at more than two points ?

From the previous, we deduce that :

1. An infinite number of circles can be drawn to pass through two given points like A and B.
2. The radius length of the smallest circle can be drawn in order to pass through the two points A and B is equal to $\frac{1}{2}\ \overline{AB}$.
3. Two circles can not be intersected in more than two points.

Drill

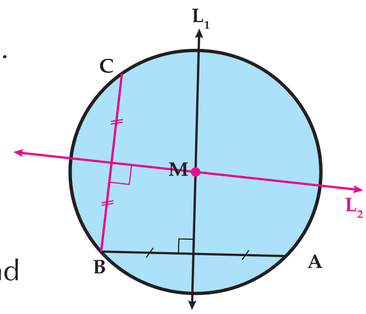
Using your geometric tools and draw $\overline{AB}$ with length 4 cm then draw on one figure :

1. A circle passing through the two points A and B and its diameter length is 5 cm. What are the possible solutions?
2. A circle passing through the two points A and B and its radius length is 2 cm. What are the possible solutions?
3. A circle passing through the two points A and B and its diameter length is 3 cm. What are the possible solutions?

Third : Drawing a circle passing through three given points:

Given: A, B and C are three given points on the plane.

R.T.P: Draw circle M passing through the three points A, B and C.



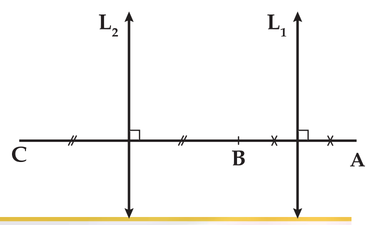
Construction:

1. Draw the straight line L_1 axis of $\overline{AB}$ thus $M \in L_1$.
2. Draw the straight line L_2 axis of $\overline{BC}$ thus $M \in L_2$.
3. If $L_1 \cap L_2 = \{M\}$, state the tip of the compass at point M and with an opening equalling MA. Draw the circle M. You will find it passing through the two points B and C.
4. If $L_1 \cap L_2 = \phi$, can you identify the position of point M ? Explain your answer.

Notice that :

If A, B, and C are collinear then $L_1 \mathbin{//} L_2$ and $L_1 \cap L_2 = \phi$

A circle cannot be drawn passing through the three points A, B, and C.



From the previous, we deduce that :

There is one and only one circle which passes through three noncollinear points.

Drill

Using the geometric tools and draw the triangle A B C in which AB = 4 cm, BC = 5 cm and CA = 6 cm, Draw circle passing through the points A, B and C. What is the kind of triangle ABC with respect to the measures of its angles ? Where is the center of the circle located with respect to the triangle?

Corollaries

Corollary (1) *The circle which passes through the vertices of a triangle is called the circumcircle of this triangle.*

It is said to be that a triangle is inscribed in a circle if its vertices are on the circle.

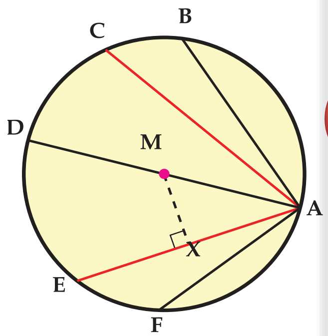
Corollary (2) *The perpendicular bisectors of the sides of a triangle intersect at a point which is the center of the circumcircle of the triangle.*

For More Exercises, go to MOE website

4 - 4

The relation between the chords of a circle and its center

Think and Discuss



In the figure opposite :

A is a point on circle M the chords $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{AE}$, $\overline{AF}$ were inscribed in it .

1. What is the relation between the length of the chord and its distance from the center of the circle ?
2. If the chords are equal in length, what can you conclude ?
3. If the chords are equidistant from the center of the circle, what do we expect ?

Notice that :

The distance of chords $\overline{AE}$, from the center of circle M equal M X where X is the midpoint of the chord $\overline{AE}$, in circle M which its radius length is r.

Thus : $(MX)^2 + (AX)^2 = (AM)^2 = r^2$ **(constant expression)**

i.e :

> *The closer the chord is from the center of the circle, the longer its length is and vise versa.*

What you'll learn

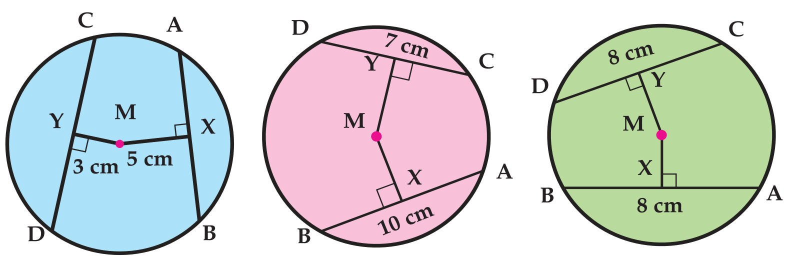
- ☆ Deducing the relation between the chords of a circle and its center.
- ☆ How to solve problems related to the relation between the chords of a circle and its center.

Key terms

- ☆ Equal chords.
- ☆ Congruent circles

Drill

1. **Complete by using the relation (> , < and =) :**



A B .... C D M X .... MY M X .... MY

2 **In the figure opposite M F < M E ,** ***Complete*** **:**

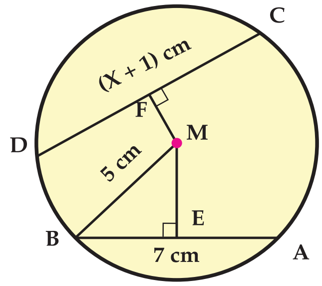
$\because$ M F < M E $\quad\therefore$ C D > ....

$\therefore$ X + 1 > .... $\quad$ X > ....

$\because$ $\overline{CD}$ is a chord in circle M $\quad\therefore$ C D $\leqslant$ ....

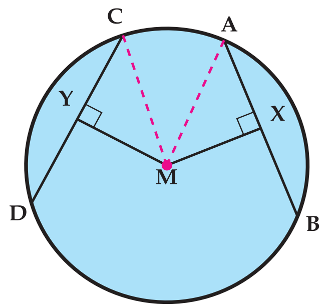
$\therefore$ X $\leqslant$ .... $\quad$ Thus .... < X $\leqslant$ ....

i.e : X $\in$ ....



Theorem

If chords of a circle are equal in length, then thay are equidistant from the center.



Given: A B = C D, $\overline{MX} \perp \overline{AB}$, $\overline{MY} \perp \overline{CD}$

R.T.P: Prove that M X = MY.

Construction: Draw $\overline{MA}$, $\overline{MC}$.

Proof: $\because$ $\overline{MX} \perp \overline{AB}$ $\quad\therefore$ A X = $\frac{1}{2}$ A B.

$\because$ $\overline{MY} \perp \overline{CD}$ $\quad\therefore$ CY = $\frac{1}{2}$ C D.

$\because$ A B = C D $\therefore$ A X = CY.

$\because$ the two triangles A X M and C Y M, both have :

$$\begin{cases} AM = CM \\ m(\angle AXM) = m(\angle CYM) = 90^\circ \\ AX = CY \quad \text{(Proof)} \end{cases}$$

$\therefore$ $\Delta AXM \equiv \Delta CYM$ $\quad$ We get : M X = MY **(Q.E.D.)**

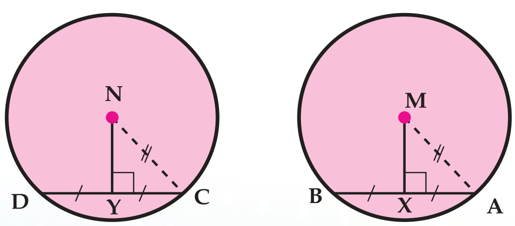
Corollary

In congruent circles, chords which are equal in length, are equidistant from the centers

In the figure opposite :

The two circles M and N are congruent A B = C D,

$\overline{MX} \perp \overline{AB}$, $\overline{NY} \perp \overline{CD}$, then : M X = NY.



Drill

Study the figure then complete :

A ***If :***

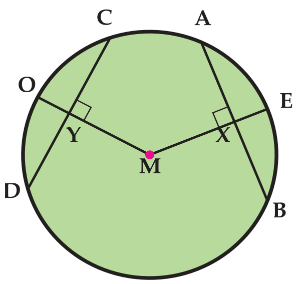
A B = C D,

then :

M X =.........

∵ M E =.........

∴ E X =.........



B ***If :***

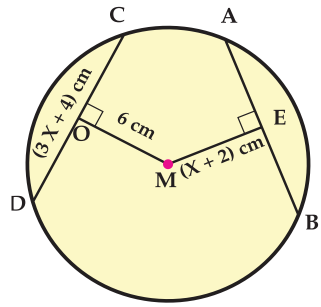
A B = C D,

then :

M E =.........

∴ X =......... cm,

C D =......... cm



C ***If :***

AB = CD,

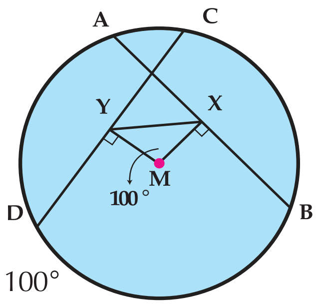
then :

M X =.........

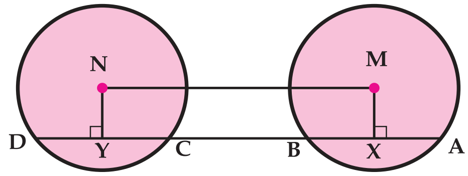
in Δ M XY:

∵ m (∠ X MY) = 100°

∴ m(∠ M XY) =.........°



D

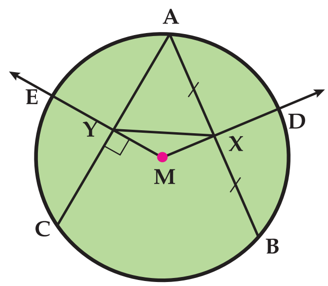


If : M and N are two congruent circles A B = C D

then : M X = ...

and the figure MXYN ....

Example 1



$\overline{AB}$ and $\overline{AC}$ are two equal chords in length in circle M and X is the midpoint of $\overline{AB}$, $\overrightarrow{MX}$ intersects the circle at D, $\overrightarrow{MY} \perp \overline{AC}$ intersects it at Y and intersects the circle at E.

Prove that : First : X D =Y E.

Second: m (∠ Y X B) = m (∠ XY C)

Given : A B = A C, X is the midpoint of $\overline{AB}$, $\overrightarrow{MY} \perp \overline{AC}$

R.T.P : prove that :

First : X D =Y E ***Second:*** m (∠ Y X B) = m (∠ XY C)

Proof : ∵ X is the midpoint of $\overline{AB}$ ∴ $\overrightarrow{MX} \perp \overline{AB}$.

∵ A B = A C, $\overrightarrow{MX} \perp \overline{AB}$, $\overrightarrow{MY} \perp \overline{AC}$ ∴ M X = MY

∵ M D = M E = r

∴ M D - M X = M E - MY ∴ X D =Y E **(Q.E.D 1)**

in Δ M XY ∵ M X = MY ∴ m (∠Y X M) = m (∠ XY M) (1)

∵ $\overrightarrow{MX} \perp \overline{AB}$, $\overrightarrow{MY} \perp \overline{AC}$ ∴ m (∠ M X B) = m (∠ MY C) = 90° (2)

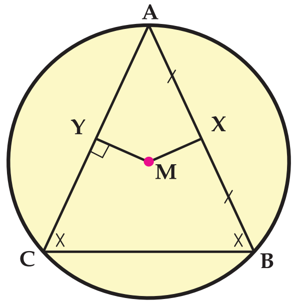
From (1) and (2) we get: m (∠Y X B) = m (∠ XY C) **(Q.E.D 2)**

Drill

In the figure opposite : Triangle A B C is inscribed in circle M, in which :

m (∠ B) = m (∠ C), X is the midpoint of $\overline{AB}$, $\overline{MY} \perp \overline{AC}$.

Prove that : M X = MY



Converse of the theorem

In the same circle (or in congruent circles) chords which are equidistant from the center (s) are equal in length

Drill

Study the figure then complete :

1

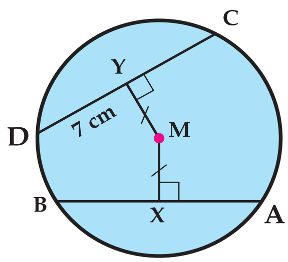
If :

M X = MY ,

Y D = 7 cm,

Then :

A B = ... cm



2

If :

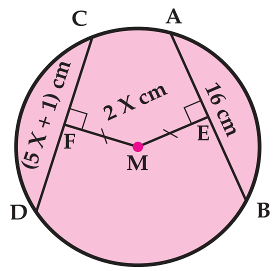
M E = M F,

Then :

C D = ...

∴ X = ...,

E M = ... cm , A M = ... cm



3

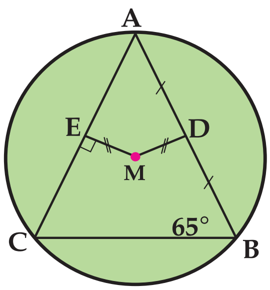
If :

M D = M E et

m (∠ B) = 65°,

Then :

m (∠ A) = ...°



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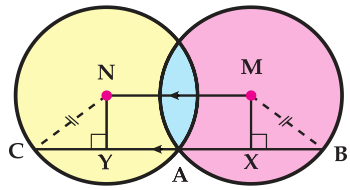


∵ $\overleftrightarrow{MN}$ // $\overleftrightarrow{BC}$ ∴ M X = ...

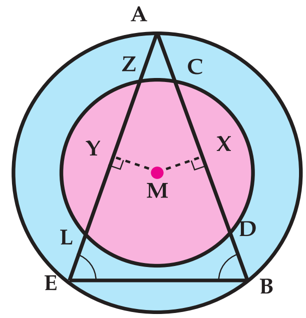
∵ the two circles M, and N .... ,

A ∈ $\overline{BC}$ ∴ A B = ...

Examples

2 Two concentric circles M, $\overline{AB}$ is a chord in the larger circle and intersects the smaller circle at C and D, $\overline{AE}$ is a chord in the larger circle and intersects the smaller circle at Z and L.

If m($\angle$ABE) = m($\angle$AEB), then ***prove that :*** CD = ZL.



Solution

Given: m ($\angle$ A B E) = m ($\angle$ A E B)

R.T.P: prove that C D = Z L

Construction: Draw $\overline{MX} \perp \overline{AB}$ and $\overline{MY} \perp \overline{AE}$

Proof : In Δ ABC : $\because$ m($\angle$ A B E) = m ($\angle$ A E B) $\therefore$ A B = A E.

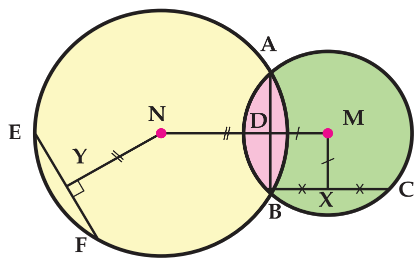
In the larger circle : AB = AE. (proof) $\therefore$ MX = MY (theorem)

$\because$ In the smaller circle MX = MY: (proof)

$\therefore$ CD = ZL ((Converse of the theorem)) **(Q.E.D.)**

3 **In the figure opposite :** M and N are two intersecting circles at A and B.,

$\overleftrightarrow{MN} \cap \overleftrightarrow{AB}$ = {D}, X is the midpoint of $\overline{BC}$, $\overline{NY} \perp \overline{EF}$, M X = M D, NY = N D. ***Prove that :*** B C = E F.



Solution

Given: X is the midpoint of $\overline{BC}$, $\overline{NY} \perp \overline{EF}$, MX = MD , and NY = ND.

R.T.P: BC = EF.

Proof : $\because$ $\overleftrightarrow{MN}$ is the line of centers, $\overline{AB}$ is a common chord for the two circles M and N

In circle M : $\because$ X is the midpoint of $\overline{BC}$

$\therefore$ $\overline{MX} \perp \overline{BC}$, $\overline{MD} \perp \overline{AB}$, MX = MD

$\because$ BC = AB (Converse of the theorem) **(1)**

In circle N : $\because$ $\overline{NY} \perp \overline{EF}$, $\overline{ND} \perp \overline{AB}$ and NY = ND

$\therefore$ EF= AB (Converse of the theorem) **(2)**

From (1) and (2) we get : BC = EF

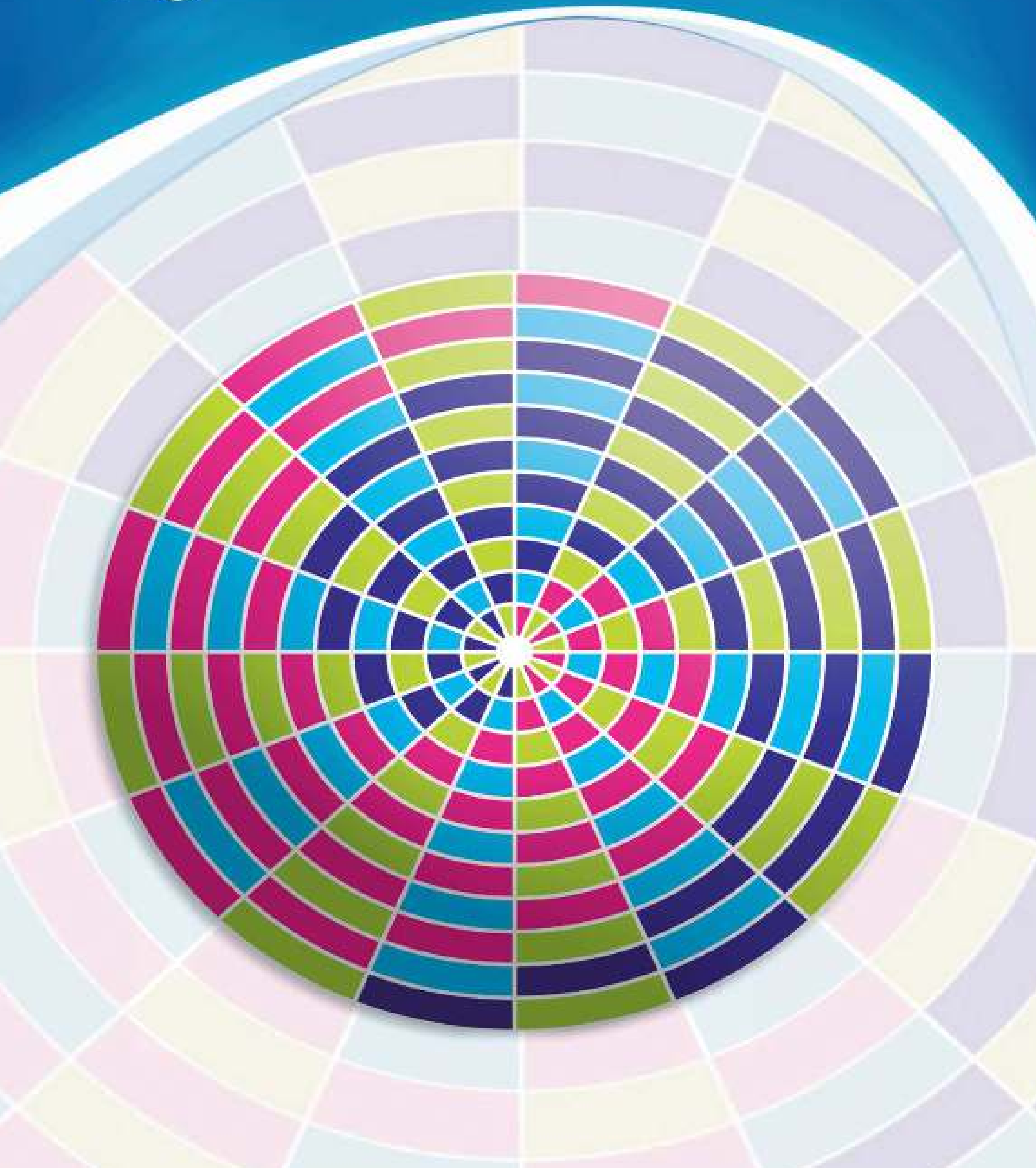
Think

If M and N are two congruent circles and intersecting at A and B. Is AB an axis to $\overleftrightarrow{MN}$? Explain your answer.

For More Exercises, go to MOE website

Geometry

Unit 5: Angles and Arcs in the circle



5 - 1 Central Angles and Measuring Arcs

What you'll learn

- ☆ The concept of arc length.
- ☆ The concept of measuring an arc.
- ☆ How to find the relation between chords of a circle and its arcs.

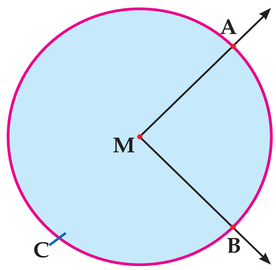
Key terms

- ☆ Central angle
- ☆ Inscribed angle
- ☆ Arc
- ☆ two adjacent arcs
- ☆ Measuring an arc
- ☆ Chord
- ☆ Tangent

Think and Discuss

In the opposite figure :

The two sides of $\angle$ AMB divide the cicle M into two arcs:

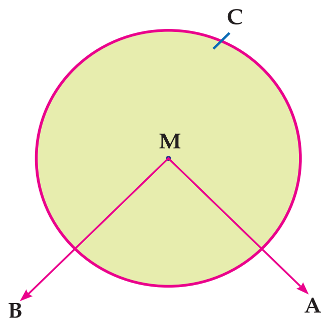


1. The minor arc AB and is denoted by $\overset{\frown}{AB}$.
2. The major arc ACB and is denoted by $\overset{\frown}{ACB}$.

- ◆ What is the position of the points of $\overset{\frown}{AB}$ with respect to $\angle$ AMB ?
- ◆ What is the position of points $\overset{\frown}{ACB}$ with respect to reflected angle of $\angle$AMB ?
- ◆ If $\angle$ AMB is a straight angle, what do you notice ?

Central Angle	It is the angle whose vertex is the center of the circle and the two sides are radii in the circle.

In the opposite figure we notice that :

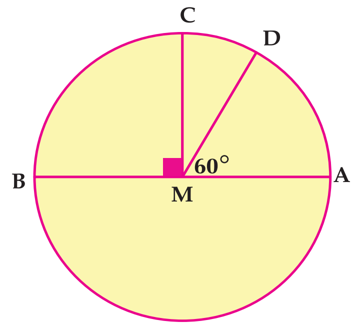


1. $\overset{\frown}{AB}$ is opposite to the central angle $\angle$AMB and $\overset{\frown}{ACB}$ is opposite to the central reflective angle $\angle$AMB.
2. If $\angle$ AMB is a straight angle ($\overline{AB}$ is a diameter in circle M) then $\overset{\frown}{AB}$ is congruent to $\overset{\frown}{ACB}$ and each is called "a semicircle".

Measure of the arc	Is the measure of the central angle opposite to it.

In the opposite figure :

$\overline{AB}$ is a diameter in the circle M, $\overline{MC} \perp \overline{AB}$ $m(\angle AMD) = 60°$



Notice that :

1. $m\,\overset{\frown}{AD} = m(\angle AMD) = 60°$
2. $m\,\overset{\frown}{CB} = m(\angle CMB) = 90°$
3. $m(\overset{\frown}{DC}) = m(\angle DMC) = 30°$ ***(Why?)***
4. $m\,\overset{\frown}{AB} = m(\angle AMB) = 180°$

   i.e. Measure of the semicircle = 180° and measure of a circle = 360°

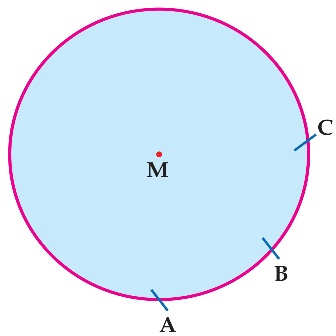
Adjacent arcs	are two arcs in the same circle that have only one point in common.

Represent $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ in the opposite figure:

thus :

$m(\overset{\frown}{AB}) + m(\overset{\frown}{BC}) = m(\overset{\frown}{ABC})$

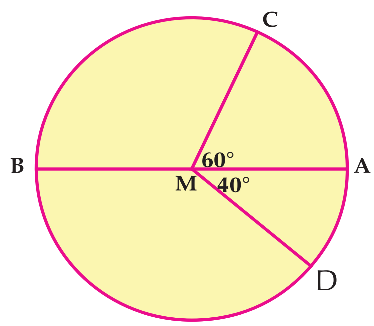
$, m(\overset{\frown}{AB}) = m(\overset{\frown}{ABC}) - m(\overset{\frown}{BC})$



Drill

In the opposite figure :

$\overline{AB}$ is a diameter in the circle M, $m(\angle AMC) = 60°$, $m(\angle AMD) = 40°$.



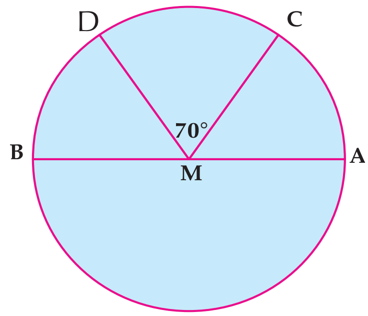
Complete :

1. $m(\overset{\frown}{AD})$ = ...°, $m(\overset{\frown}{AC})$ = ...°
2. $m(\overset{\frown}{CAD}) = m(\overset{\frown}{CA})$ + ...

   = ... + ... = ...°
3. $m(\overset{\frown}{BC}) = m(\overset{\frown}{ACB}) - m(\quad) = 180°$ - .... = ...°

   (Why?)
4. $m(\overset{\frown}{DCB})$ = measure of circle - $m(\overset{\frown}{\quad})$ = ... - .... = ...°

Example 1



$\overline{AB}$ is a diameter in the circle M, m ($\angle$ CMD) = 70° ,
m ($\overset{\frown}{AC}$) : m ($\overset{\frown}{DB}$) = 5 : 6, ***find*** m ($\overset{\frown}{ACD}$) .

Solution

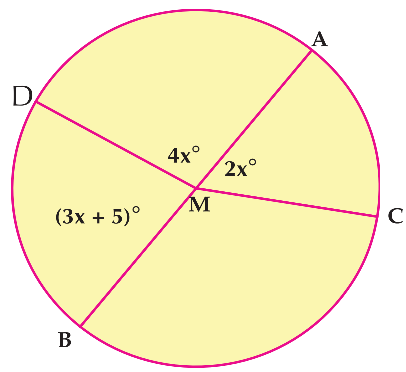
Suppose that m ($\overset{\frown}{AC}$) = 5x $\therefore$ m ($\overset{\frown}{DB}$) = 6x

$\because$ m ($\overset{\frown}{ADB}$) = m ($\overset{\frown}{AC}$) + m ($\overset{\frown}{CD}$) + m ($\overset{\frown}{DB}$) = 180°

$\therefore$ 5x + 70° + 6x = 180° 11x = 110° $\therefore$ x = 10° , m ($\overset{\frown}{AC}$) = 50°

$\therefore$ m ($\overset{\frown}{ACD}$) = m ($\overset{\frown}{AC}$) + m ($\overset{\frown}{CD}$) = 50° + 70° = 120°

Drill



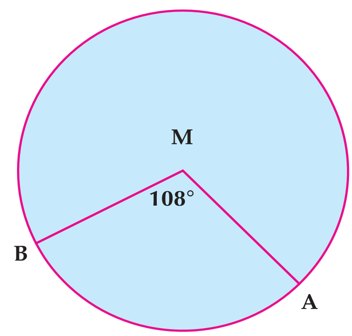
In the opposite figure : $\overline{AB}$ is a diameter of the circle M, study the figure, then complete :

1. x = ...
2. m ($\overset{\frown}{AC}$) = ...°
3. m ($\overset{\frown}{AD}$) = ...
4. m ($\overset{\frown}{BC}$) = ... °
5. m ($\overset{\frown}{CAD}$) = ...
6. m ($\overset{\frown}{CBD}$) = ...
7. m ($\overset{\frown}{ACD}$) = ...
8. m ($\overset{\frown}{ADC}$) = ...

Arc length	**is a part of a circle's circumference proportional with its measure**

Where the arc length = $\dfrac{\textbf{The measure of the arc}}{\textbf{The measure of the circle}}$ × **circumference of the circle**

Example 2



In the opposite figure :

M is a circle with radius length of 5 cm, m ($\overset{\frown}{AB}$) = 108°.

Find : the length of $\overset{\frown}{AB}$ (π = 3.14)

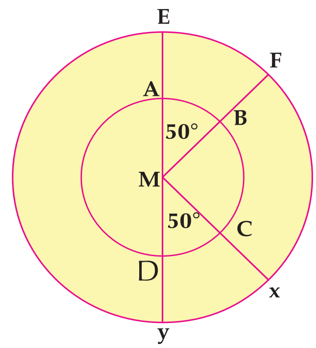
Solution

Arc length = $\dfrac{\text{Measure of the arc}}{\text{Measure of the circle}}$ × circumference of the circle.

$= \dfrac{108}{360} \times 2 \times 3.14 \times 5 = 9.42$cm.

Drill

In the opposite figure : Two concentric circles, the radius length of the minor circle is 7 cm and the radius length of the major circle is 14 cm ($\pi = \frac{22}{7}$)



Complete : **In the minor circle :**

m $(\overset{\frown}{AB})$ = m $(\overset{\frown}{.....})$ = ...°

length of $\overset{\frown}{AB} = \frac{50}{360} \times 2 \times \frac{22}{7} \times$ .... = ... cm

length of $\overset{\frown}{CD}$ = ... × .... = ...cm

∴ $\overset{\frown}{AB}$ (congruent to / not congruent to) $\overset{\frown}{CD}$

In the major circle :

m $(\overset{\frown}{EF})$ = m $(\overset{\frown}{.......})$ = ...° , length of $\overset{\frown}{EF}$ = ... × .... = ... cm

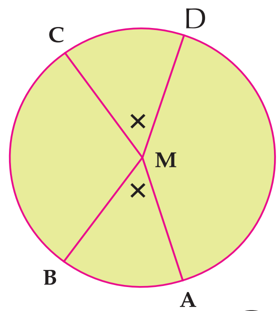
length of $\overset{\frown}{XY}$ = ... × .... = ...cm

∴ $\overset{\frown}{EF}$ (congruent to / not congruent to) $\overset{\frown}{XY}$

- Is $\overset{\frown}{AB}$ congruent to $\overset{\frown}{EF}$? What do you deduce ?

Important corollaries :

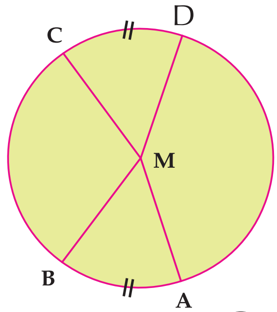
Corollary (1) *In the same circle (or in congruent circles), if the measures of arcs are equal, then the lengths of the arcs are equal, and conversely.*



In circle M

If: m $(\overset{\frown}{AB})$ = m $(\overset{\frown}{CD})$

then: the length of $\overset{\frown}{AB}$ = the length of $\overset{\frown}{CD}$

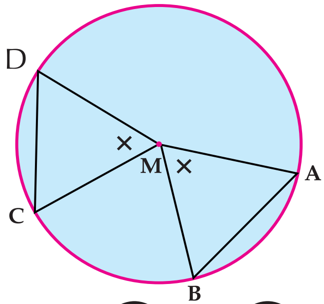


And conversely

If: the length of $\overset{\frown}{AB}$ = the length of $\overset{\frown}{CD}$

then : m $(\overset{\frown}{AB})$ = m $(\overset{\frown}{CD})$

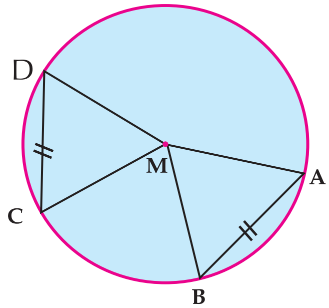
Corollary (2) *In the same circle (or in congruent circles), if the measures of arcs are equal, then their chords are equal in length, and conversely*



In circle M

If : $m(\overset{\frown}{AB}) = m(\overset{\frown}{CD})$

then : length of $\overline{AB}$ = length of $\overline{CD}$



And conversely

If : AB = CD

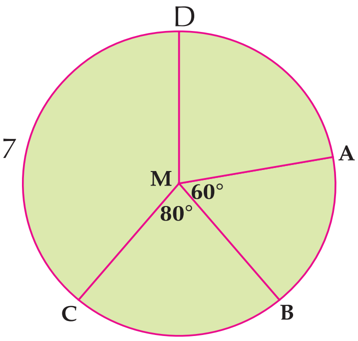
then : $m(\overset{\frown}{AB}) = m(\overset{\frown}{CD})$

Drill

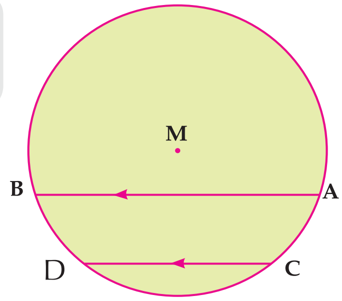
In the opposite figure :

$m(\overset{\frown}{AB}) = 60°$ et $m(\overset{\frown}{BC}) = 80°$, $m(\overset{\frown}{AD}) : m(\overset{\frown}{DC}) = 4 : 7$

1. Mention the arcs equal in measure.
2. Mention the arcs equal in length.
3. Draw the chords equal in length.



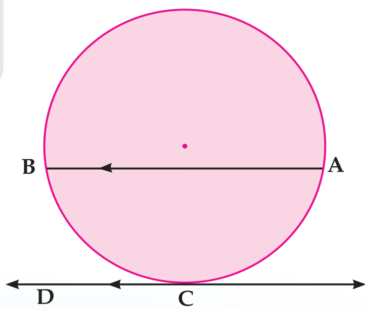
Corollary (3) *If two parallel chords are drawn in a circle, then the measures of the two arcs between them are equal.*



If $\overline{AB}$ and $\overline{CD}$ are two chords in circle M, $\overline{AB}$ // $\overline{CD}$

then : $m(\overset{\frown}{AC}) = m(\overset{\frown}{BD})$.

Corollary (4) *If a chord is parallel to a tangent of a circle, then the measures of the two arcs between them are equal.*



If $\overline{AB}$ is a chord of circle M, $\overleftrightarrow{CD}$ is a tangent at c,

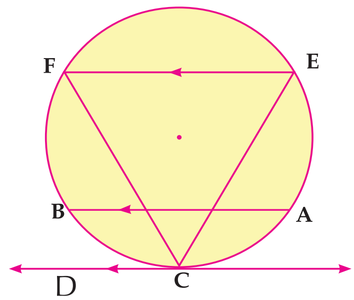
$\overline{AB}$ // $\overleftrightarrow{CD}$ **then** $m(\overset{\frown}{AC}) = m(\overset{\frown}{BC})$.

Drill

In the opposite figure :

M is a circle , $\overleftrightarrow{CD}$ is a tangent to the circle at C , $\overline{AB}$ and $\overline{EF}$ are two chords of the circle where : $\overline{AB} // \overline{EF} // \overleftrightarrow{CD}$

Complete the following to prove that CE = CF



Solution

∵ $\overline{AB} // \overline{EF}$ ∴ m $\overset{\frown}{(.......)}$ = m $\overset{\frown}{(.......)}$ (1)

∵ The tangent $\overleftrightarrow{CD} // \overline{AB}$ ∴ m $\overset{\frown}{(.......)}$ = m $\overset{\frown}{(.......)}$ (2)

By adding the two sides of (1) and (2)

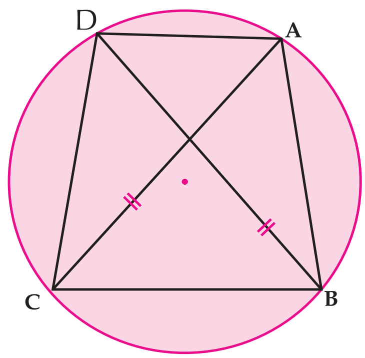
∴ m $(\overset{\frown}{EC})$ = m $\overset{\frown}{(.......)}$ ∴ CE = ...

Example 3

In the opposite figure :

ABCD is a quadrilateral inscribed in a circle in which AC = BD, AB = (3x - 5) cm , CD = (x + 3) cm.

Find with proof the length of $\overline{AB}$.



Solution

Given : ABCD is a quadrilateral inscribed in a circle, AC = BD , AB = (3x - 5) cm , CD= (X + 3) cm

R.T.P.: Find the length of $\overline{AB}$.

Proof : ∵ AC = BD given ∴ m $(\overset{\frown}{ABC})$ = m $(\overset{\frown}{BCD})$

∴ m $(\overset{\frown}{ABC})$ - m $(\overset{\frown}{BC})$ = m $(\overset{\frown}{BCD})$ - m $(\overset{\frown}{BC})$

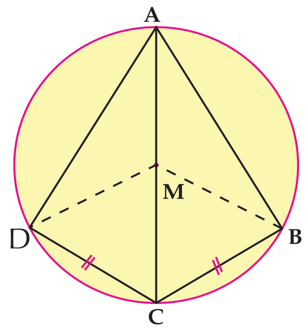
∴ m $(\overset{\frown}{AB})$ = m $(\overset{\frown}{DC})$ ∴ AB = CD

∵ AB = CD ∴ 3x - 5 = x + 3

2x = 8 ∴ x = 4

∵ AB = 3x - 5 ∴ AB = 3 × 4 - 5 = 7cm

Example 4



In the opposite figure :

ABCD is a quadrilateral inscribed in a circle M, $\overline{AC}$ is a diameter in the circle, CB = CD.

Prove that : m $(\overset{\frown}{AB})$ = m $(\overset{\frown}{AD})$

Solution

Given : $\overline{AC}$ is a diameter in a circle, CB = CD

R.T.P.: m $(\overset{\frown}{AB})$ = m $(\overset{\frown}{AD})$

Proof: ∵ CB = CD ∴ m $(\overset{\frown}{CB})$ = m $(\overset{\frown}{CD})$ (1)

∵ $\overline{AC}$ is a diameter in the circle

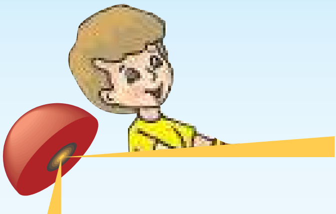
∴ m $(\overset{\frown}{AB})$ = 180°- m $(\overset{\frown}{CB})$, m $(\overset{\frown}{AD})$ = 180°- m $(\overset{\frown}{CD})$ (2)

from (1) **and** (2) **we get :** m $(\overset{\frown}{AB})$) = m $(\overset{\frown}{AD})$

For More Exercises, go to MOE website

5 - 2

The relation between the inscribed and central angles subtended by the same arc



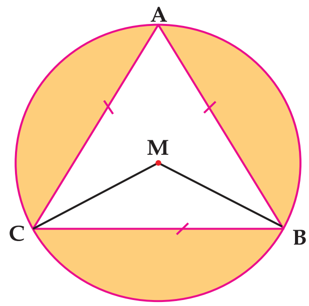
What you'll learn

☆ How to infer the relation between the measures of the inscribed and central angles subtended by the same arc

Key terms

☆ Inscribed angle.
☆ Central angle.

Think and Discuss



In the opposite figure :

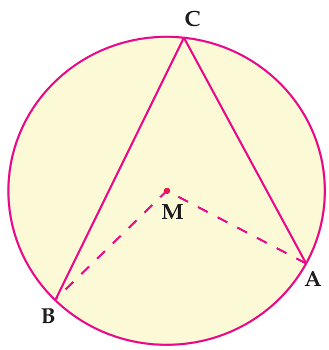
The circle M passes through the vertices of the equilateral triangle ABC.

- What is the measure of central ∠ BMC? **Explain your answer.**
- What is the vertices of ∠ BAC?
  Does the vertices of the angle belong to the set of points of circle M ?
- What are the two sides of ∠ BAC ?
- If ∠ BMC is central with arc $\overset{\frown}{BC}$. How do you describe ∠ BAC ?
- Compare between m(∠ BAC) and m (∠ BMC).

What do you notice ?

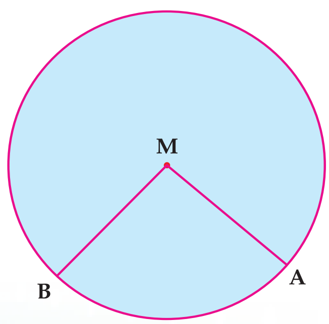
Inscribed angle	An angle the vertex of it lies on the circle and its sides contain two chords of the circle

In the opposite figure : Notice that :



1. ∠ ACB is an inscribed angle and $\overset{\frown}{AB}$ is the arc opposite to it.
2. For each inscribed angle, there is one central angle subtended by the same arc.

Think 

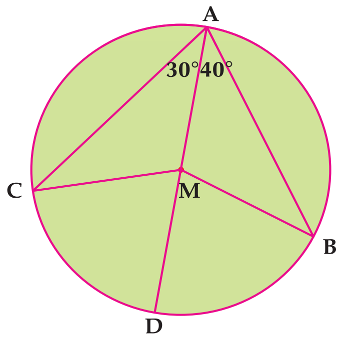


In the opposite figure :

What is the number of inscribed angles subtended with the central ∠ AMB at $\overset{\frown}{AB}$?

(Clarify your answer with a drawing)

Activity **In the opposite figure :**



$\overline{AD}$ **is a diameter in circle M. Study the figure, then answer the following questions :**

1. Mention two pairs of equal angles in measure.
2. If m (∠BAD) = 40° , find m (∠BMD) .
3. If m (∠CAD) = 30° , find m (∠CMD) .
4. Compare between m (∠BAC) , m (∠BMC). **What do you conclude?**

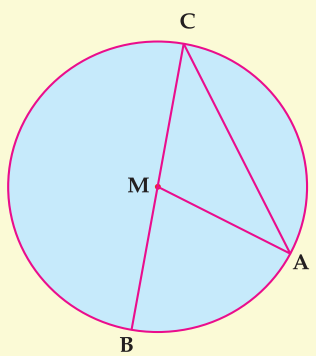
Theorem ***The measure of the inscribed angle is half the measure of the central angle, subtended by the same arc.***

Given : ∠ ACB is an inscribed angle, ∠ AMB is a central angle.

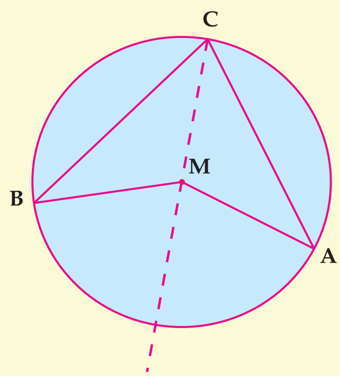
R.T.P. : Prove that m (∠ ACB) = $\frac{1}{2}$ m (∠ AMB).

Proof : **There are three cases to prove this theorem.**

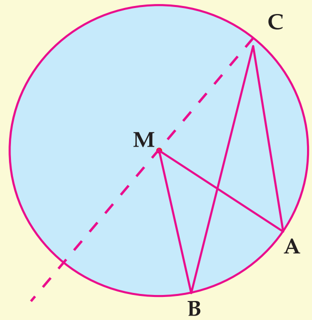
1. If M belongs to one of the two sides of the inscribed angle.



2. If M is a point inside the inscribed angle.



3. If M is a point outside the inscribed angle.



First case: If M belongs to one of the two sides of the inscribed angle.

∵ ∠AMB is outside Δ AMC

∴ m (∠AMB) = m (∠ A) + m (∠C) (1)

∵ AM = CM **(radii lengths)** ∴ m (∠A) = m (∠C) (2)

From (1) **and** (2) **we get :** m (∠AMB = 2 m (∠C)

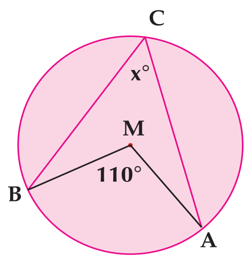
∴ m (∠ACB) = $\frac{1}{2}$ m (∠AMB) **(Q.E.D)**

Activity

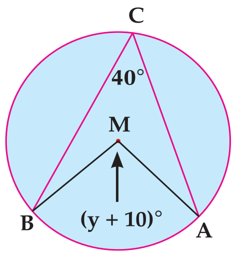
Prove that the theorem in the other two cases are correct and save your work in the portfolio.

Drill

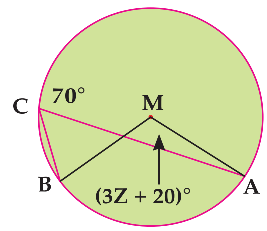
M is a circle. In each of the following figures, find the value of the symbol used in measuring:

1


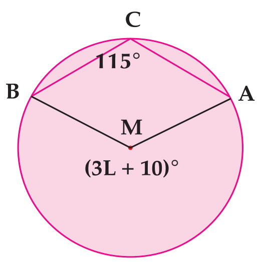
x = ...

2


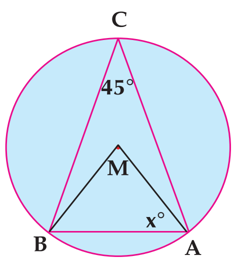
y = ...

3


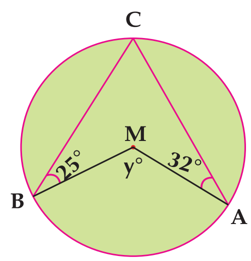
Z = ...

4


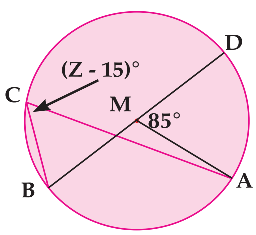
L = ...

5


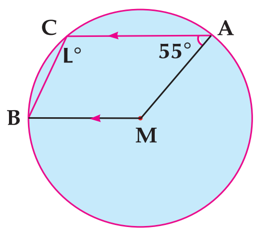
x = ...

6


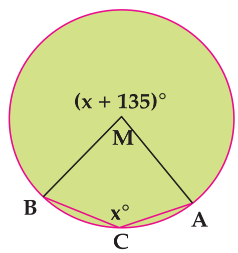
y = ...

7


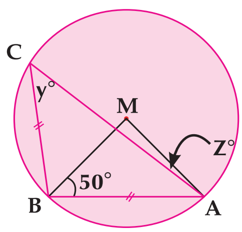
Z = ...

8


L = ...

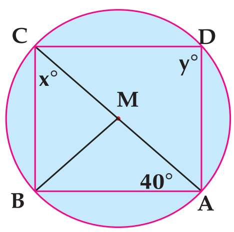
9


x = ...

10


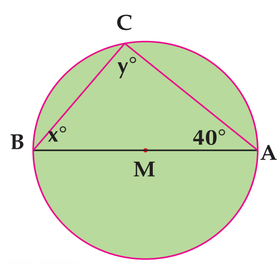
y = ...

Z = ...

11


x = ...

y = ...

12


x = ...

y = ...

Example 1

A is a point outside the circle M, $\overrightarrow{AB}$ is a tangent to the circle at B, $\overrightarrow{AM}$ intersects the circle M at C and D respectively m (∠ A) = 40° ***Find***. with proof m(∠ BDC).

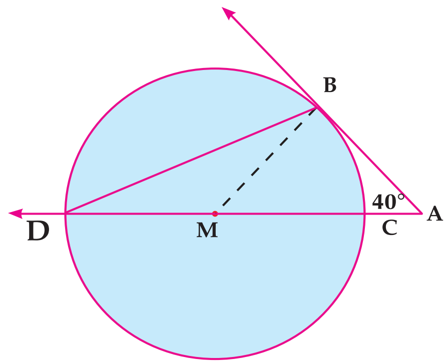
Solution

Given : $\overrightarrow{AB}$ is a tangent to the circle at B, m(∠ A) = 40° ,$\overrightarrow{AM}$ intersects the circle M at C and D.

R.T.P. : m (∠ BDC)

Construction: Draw the radius $\overline{BM}$.

Proof : ∵ $\overrightarrow{AB}$ is tangent to the circle at B, $\overline{BM}$ is a radius.



∴ m (∠ ABM) = 90°

In Δ ABM :

∵ m (∠ A) = 40° , m(∠ ABM) = 90°

∴ m (∠ BMC) = 180° - (40° + 90°) = 50°

∵ Inscribed ∠ BDC and central ∠ BMC are both subtended at $\overset{\frown}{BC}$.

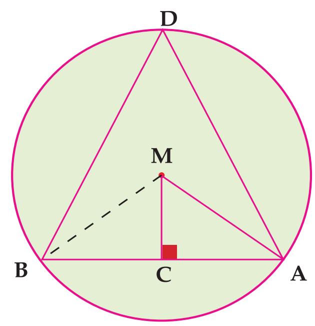
∴ m (∠ BDC) = $\frac{1}{2}$ m (∠ BMC)

∴ m (∠ BDC) = $\frac{1}{2}$ × 50 = 25 ° ***(Q.E.D.)***

Drill

In the opposite figure : $\overline{AB}$ is a chord of circle M, $\overline{MC}$ ⊥ $\overline{AB}$.

Prove that : m (∠ AMC) = m (∠ ADB)



Solution

Draw $\overline{BM}$, ***Complete :*** In Δ MAB :

∵ MA= MB , $\overline{MC}$ ⊥ $\overline{AB}$

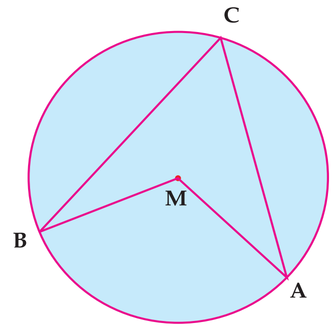
∴ m (∠ AMC) = m (∠ ....) = $\frac{1}{2}$ m (∠ ....) (1)

∵ inscribed ∠ ADB and central ∠ .... are subtended at $\overset{\frown}{.......}$

∴ m (∠ ....) = $\frac{1}{2}$ m (∠ ....) (2)

From (1) ***and*** (2) ***we get :*** m (∠ AMC) = m (∠ ....) .

Corollary (1) *The measure of an inscribed angle is half the measure of the subtended arc.*

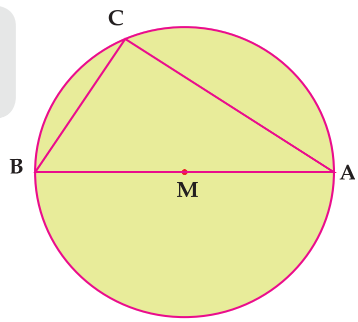


In the opposite figure :

$m(\angle C) = \frac{1}{2} m(\angle AMB)$, $m(\angle AMB) = m(\overset{\frown}{AB})$

$\therefore m(\angle C) = \frac{1}{2} m(\overset{\frown}{AB})$

Corollary (2) *The inscribed angle drawn in a semicircle is a right angle.*



i.e. : If the arc opposite to the inscribed angle equals the semicircle **then:** $m(\angle C) = \frac{1}{2} m(\overset{\frown}{AB})$

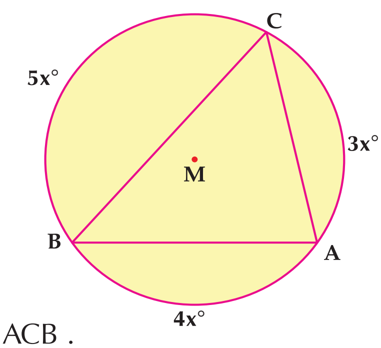
$\because m(\overset{\frown}{AB}) = 180°$ $\therefore m(\angle C) = 90°$

Think

- **What** is the type of the inscribed angle opposite to an arc less than a semicircle ? Why ?
- **What** is the type of the inscribed angle opposite to an arc greater than the semicircle ? Why ?
- **Is** the inscribed right angle inscribed in a semicircle ? **Explain your answer?**

Example 2

In the opposite figure : ABC is an inscribed triangle in circle M, $m(\overset{\frown}{AB}) : m(\overset{\frown}{BC}) : m(\overset{\frown}{AC}) = 4 : 5 : 3$. *find* $m(\angle ACB)$:



Solution

Suppose that : $m(\overset{\frown}{AB}) = 4x°$, $m(\overset{\frown}{BC}) = 5x°$, $m(\overset{\frown}{AC}) = 3x°$

$\therefore 4x + 5x + 3x = 360°$

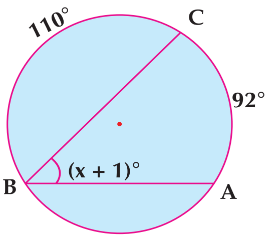
$12x = 360°$ $\therefore x = 30°$

$\therefore m(\overset{\frown}{AB}) = 4 \times 30 = 120°$ and opposite to the inscribed $\angle ACB$.

$\because m(\angle ACB) = \frac{1}{2} m(\overset{\frown}{AB})$ $\therefore m(\angle ACB) = \frac{1}{2} \times 120° = 60°$ *Q.E.D.*

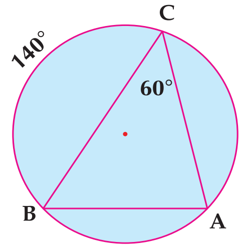
Drill

Study each of the following figures, then complete :

1


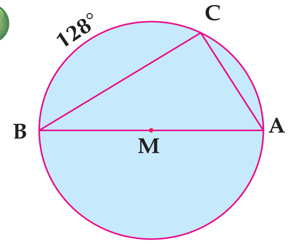
x = ...

m ($\overset{\frown}{AB}$) = ...

2


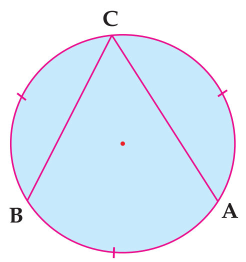
m (∠ A) = ...

m ($\overset{\frown}{AC}$) = ...

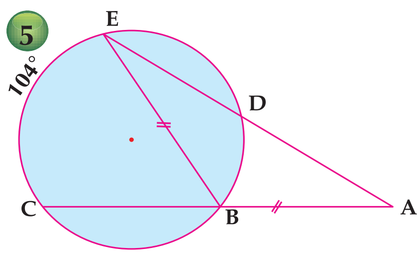
3


m (∠ C) = ...

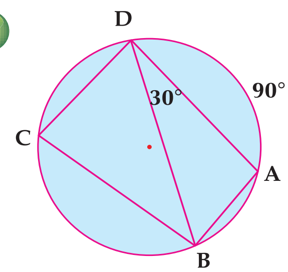
m (∠ B) = ...

4


m (∠ C) = ...

5


m (∠ EAC) = ...

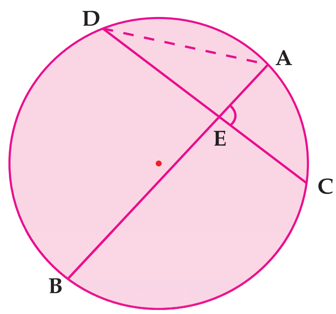
6


m (∠ DCB) = ...

Example 3

Well known problem (1)

If two chords intersect at a point inside a circle, then the measure of the included angle equals half of the sum of the measure of the two opposite arcs.



Solution

Given : $\overline{AB} \cap \overline{CD} = \{E\}$

R.T.P : m (∠AEC) = $\frac{1}{2}$ [m ($\overset{\frown}{AC}$) + m ($\overset{\frown}{BD}$)]

Construction: Draw $\overline{AD}$

Proof : ∵ ∠AEC is outside the Δ AED .

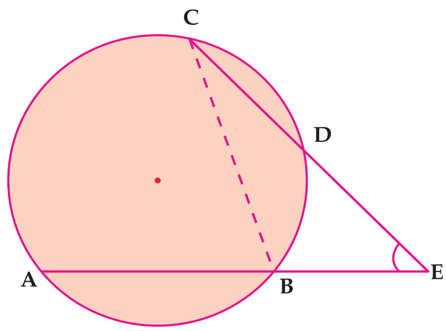
∵ m (∠AEC) = m (∠D) + m (∠A) = $\frac{1}{2}$ m ($\overset{\frown}{AC}$) + $\frac{1}{2}$ m($\overset{\frown}{BD}$)

= $\frac{1}{2}$ [m ($\overset{\frown}{AC}$) + m ($\overset{\frown}{BD}$)] . *Q.E.D.*

Example 4

Well known problem (2)

If two rays carrying two chords in a circle are intersecting outside it, then the measure of their intersecting angle equals half of the measure of the major arc subtracted from it half of the measure of the minor arc in which both are included by the two sides of this angle.



Solution

Given : $\overrightarrow{AB} \cap \overrightarrow{CD} = \{E\}$

R.T.P : $m(\angle E) = \frac{1}{2}[m(\overset{\frown}{AC}) - m(\overset{\frown}{BD})]$

Construction : Draw $\overline{BC}$.

Proof : $\because$ $\angle$ ABC is exterior to Δ BEC.

$\therefore m(\angle ABC) = m(\angle E) + m(\angle BCD)$

$\therefore m(\angle E) = m(\angle ABC) - m(\angle BCD)$

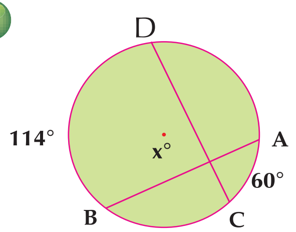
$= \frac{1}{2} m(\overset{\frown}{AC}) - \frac{1}{2} m(\overset{\frown}{BD})$

$= \frac{1}{2}[m(\overset{\frown}{AC}) - m(\overset{\frown}{BD})]$ *Q.E.D*

Drill

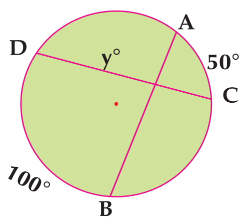
In each of the following figures, find the value of the symbol used in measuring:

1



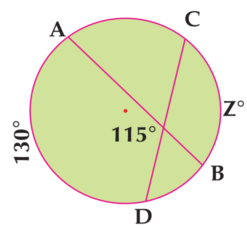
X = ...

2



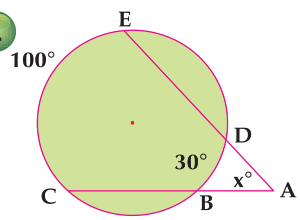
Y = ...

3



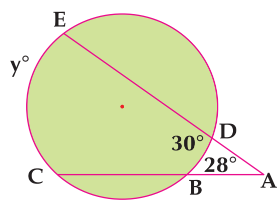
Z = ...

4



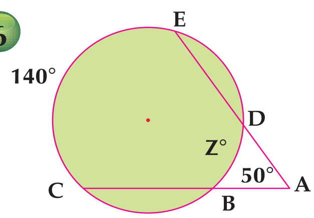
X = ...

5



Y = ...

6



Z = ...

Example 5 **In the opposite figure :**

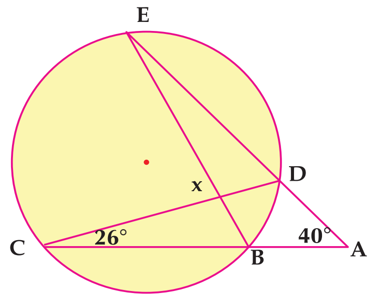
$\overrightarrow{CB} \cap \overrightarrow{ED} = \{A\}$, m ($\angle$A) = 40° , $\overline{DC} \cap \overline{BE} = \{X\}$, m ($\angle$BCD) = 26°

Find : **A** m ($\overset{\frown}{CE}$) **B** m ($\angle$EXC) .

Solution

Given : $\overrightarrow{CB} \cap \overrightarrow{ED} = \{A\}$, m ($\angle$A) = 40° , $\overline{DC} \cap \overline{BE} = \{X\}$, m ($\angle$BCD) = 26°

R.T.P. : **A** m ($\overset{\frown}{CE}$) **B** m ($\angle$EXC) .



Proof : $\because$ m ($\angle$BCD) = 26°

$\therefore$ m ($\overset{\frown}{BD}$) = 2m ($\angle$BCD) = 52°

$\because$ $\overrightarrow{CB} \cap \overrightarrow{ED} = \{A\}$

$\therefore$ m ($\angle$A) = $\frac{1}{2}$ [m($\overset{\frown}{CE}$) - m ($\overset{\frown}{BD}$)]

$\therefore$ 40 = $\frac{1}{2}$ [m($\overset{\frown}{CE}$) - 52]

m($\overset{\frown}{CE}$) = 80 + 52 = 132° *(Q.E.D. 1)*

$\because$ $\overline{DC} \cap \overline{BE} = \{X\}$ $\therefore$ m ($\angle$EXC) = $\frac{1}{2}$ [m($\overset{\frown}{CE}$) + m ($\overset{\frown}{BD}$)]

m ($\angle$EXC) = $\frac{1}{2}$ [132 + 52] = $\frac{1}{2}$ × 184 = 92°

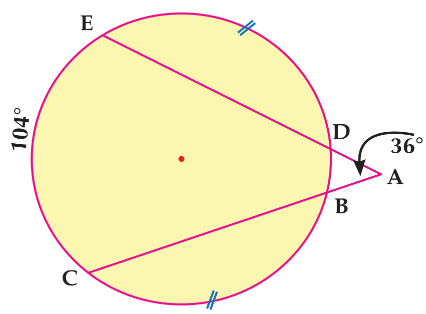
(Q.E.D. 2)

Drill

In the opposite figure :

m ($\angle$A) = 36° , m($\overset{\frown}{EC}$)= 104° , m ($\overset{\frown}{BC}$) = m ($\overset{\frown}{DE}$)

Find : **A** m ($\overset{\frown}{BD}$) **B** m ($\overset{\frown}{DE}$) .



Solution

Complete: $\because$ $\overrightarrow{CB} \cap \overrightarrow{ED} = \{A\}$

$\therefore$ m ($\angle$A) = $\frac{1}{2}$ [......................................]

$\therefore$ 36 = $\frac{1}{2}$ [......................................]

$\therefore$ m ($\overset{\frown}{BD}$) = ... *(Q.E.D. 1)*

$\because$ m ($\overset{\frown}{DE}$) +m ($\overset{\frown}{BC}$) = 360° - (......... + ...) = ...

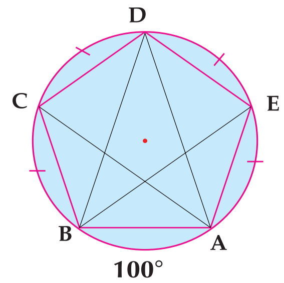
$\because$ m ($\overset{\frown}{DE}$) = m ($\overset{\frown}{BC}$) $\therefore$ 2 m ($\overset{\frown}{DE}$) =

$\therefore$ m ($\overset{\frown}{DE}$) = ... *(Q.E.D. 2)*

For More Exercises, go to MOE website

5 - 3 Inscribed Angles Subtended by the Same Arc

Think and Discuss



In the opposite figure: m $(\overset{\frown}{AB}) = 100°$

- **Do the inscribed angles** $\angle AEB$, $\angle ADB$ et $\angle ACB$ include the same arc?
- *Find* m ($\angle$AEB), m ($\angle$ADB) , m ($\angle$ACB).

  What do you notice ?

- Do the inscribed angles that include equal arcs in measure are equal in measue? **Explain your answer.**

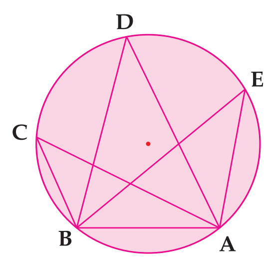
What you'll learn

☆ How to infer the relation between the inscribed angles that include equal arcs in measure

> ***Theorem 2*** *In the same circle, the measures of all inscribed angles subtended by the same arc are equal in measure.*

Given : $\angle C$, $\angle D$ and $\angle E$ are common inscribed angles at $\overset{\frown}{AB}$

R.T.P : m ($\angle$C) = m ($\angle$D) = m ($\angle$E)



Proof: $\because$ m ($\angle$C) = $\frac{1}{2}$ m $(\overset{\frown}{AB})$

, m($\angle$D) = $\frac{1}{2}$ m $(\overset{\frown}{AB})$

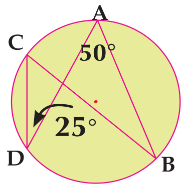
, m ($\angle$E) = $\frac{1}{2}$ m $(\overset{\frown}{AB})$

$\therefore$ m ($\angle$C) = m ($\angle$D) = m ($\angle$E) *(Q.E.D.)*

Drill

Study each of the following figures, then complete :

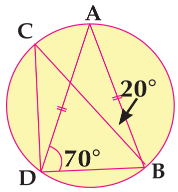
1



m (∠C) = ... °

m (∠B) = ... °

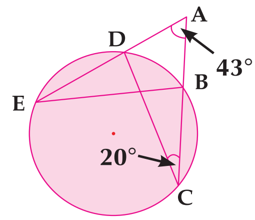
2



m (∠C) = ... °

m (∠BDC) = ... °

3

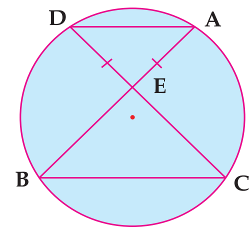


m (∠BED) = ...°

m (∠ABE) = ...°

Example 1

In the opposite figure :



$\overline{AB} \cap \overline{CD} = \{E\}$, EA = ED

Prove that : EB = EC .

Solution

In Δ AED ∵ EA = ED ∴ m (∠D) = m (∠A) (1)

∵ ∠ABC , ∠ADC are both inscribed and include $\overset{\frown}{AC}$ ∴ m (∠B) = m (∠D) (2)

∵ ∠DCB , ∠ DAB are both inscribed and include $\overset{\frown}{BD}$ ∴ m (∠C) = m (∠A) (3)

From (1), (2) **and** (3) **we deduce that :** m (∠B) = m (∠C)

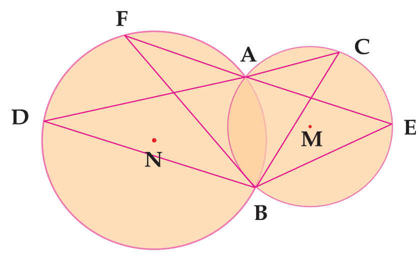
In Δ EBC: ∵ m (∠B) = m (∠C) ∴ EB = EC *(Q.E.D.)*

Drill

In the opposite figure :

M and N are two intersecting circles at A and B. $\overleftrightarrow{AC}$ intersects the circle M at C and intersects the circle N at D, $\overleftrightarrow{AE}$ intersects the circle M at E, and the circle N at F .

Prove that : m (∠EBC) = m (∠FBD)

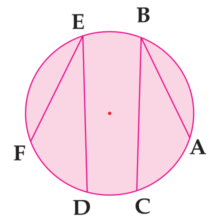


Corollary *In the same circle or in congruent circles, the measures of the inscribed angles subtended by arcs of equal measures are equal*

Notice that :

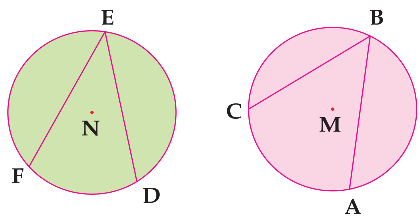
1. In the circle M if : m $(\overset{\frown}{AC})$ = m $(\overset{\frown}{DF})$

   then : m (∠B) = m (∠E)



2. For any two circles M and N, if : m $(\overset{\frown}{AC})$ = m $(\overset{\frown}{DF})$

   then : m(∠B) = m (∠E)



3 The converse of the previous corollary is true:

i.e. : In the same circle or in congruent circles, the inscribed angles of equal measures subtend arcs of equal measures.

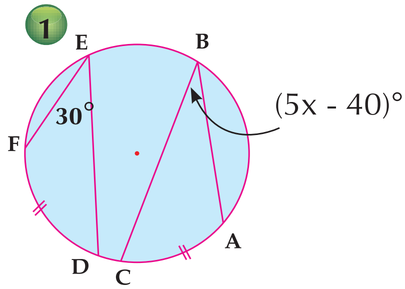
Think :

Are each two chords not intersecting inside a circle and subtended by two congruent arcs are parallel? Explain your answer.

Drill

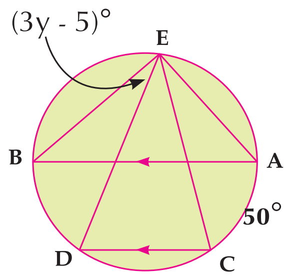
In each of the following figures, find the value of the symbol used in measuring :

1



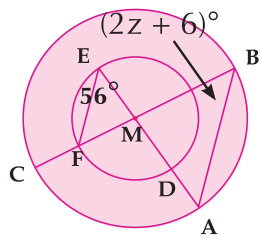
X = ...

2



Y = ...

3



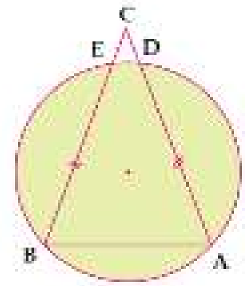
Z = ...

Example 3

In the opposite figure :

$\overline{AD}$ and $\overline{BE}$ are two equal chords in length in the circle ,

$\overrightarrow{AD} \cap \overrightarrow{BE} = \{C\}$. **Prove that :** $CD = CE$.



Solution

$AD = BE$

Prove that : $CD = CE$

$\because AD = BE$ $\therefore m(\overset{\frown}{AD}) = m(\overset{\frown}{BE})$

by adding $m(\overset{\frown}{DE})$ to each of the two sides, we get : $m(\overset{\frown}{ADE}) = m(\overset{\frown}{BED})$

$\therefore m(\angle B) = m(\angle A)$

in $\triangle ABC$ $\because m(\angle A) = m(\angle B)$ $\therefore AC = BC$ (1)

$\because AD = BE$ (2)

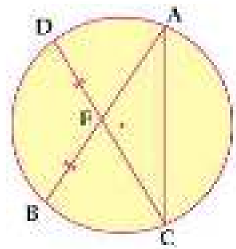
By subtracting the two sides of (2) **from** (1) **we get :** $CD = CE$

Drill

In the opposite figure :

$\overline{AB}$ and $\overline{CD}$ are two equal chords in length in the circle, $\overline{AB} \cap \overline{CD} = \{E\}$.

Prove that : the triangle ACE is an isosceles triangle.

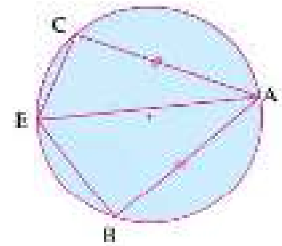


Drill

In the opposite figure :

$AB = AC$, $E \in \overset{\frown}{BC}$

Prove that : $m(\angle AEB) = m(\angle AEC)$



Think

What is the number of bisectors of $\angle BEC$? Explain your answer.

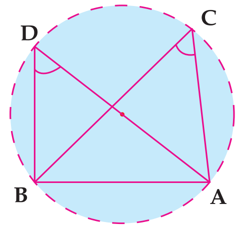
The converse of theorm 2

If two angles subtended to the same base and on the same side of it, have the same measure, then their vertices are on an arc of a circle and the base is a chord in it.

In the opposite figure, notice that :

$\angle C$, $\angle D$ are both drawn on the base $\overline{AB}$, and on one side of it , m ($\angle$ C) = m ($\angle$ D)

Then : The points A , B , C and D lie on one circle where $\overline{AB}$ is a chord in it.



Example 4

In the opposite figure : AB = AD , m ($\angle$ A) = 80° , m ($\angle$ C) = 50°

Prove that : The points A, B, C and D have one circle passing through them.

Solution

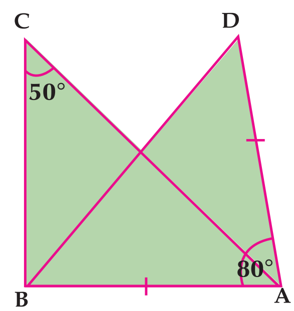
In Δ ABD ∵ A B = AD , m ($\angle$ A) = 80°

∴ $m(\angle D) = m(\angle ABD) = \frac{180° - 80°}{2} = 50°$

∵ m ($\angle$ D) = m ($\angle$ C) = 50°

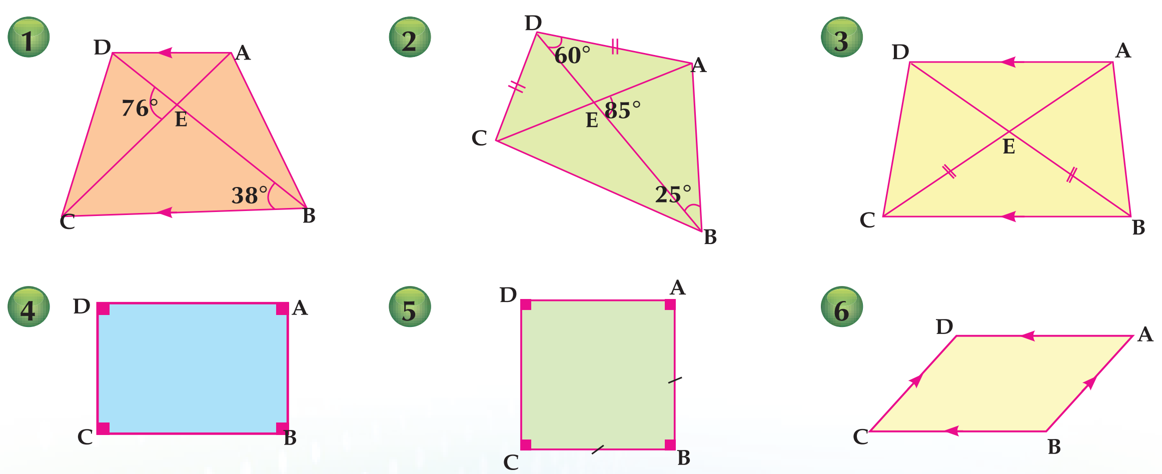
They are both drawn angles on one base $\overline{AB}$ and on one side of it .

∴ The points A , B , C and D have one circle passing through them **(Q.E.D.)**



Drill

Which of the following figures can have a circle passing through the points A, B, C and D ? Mention the reason.



For More Exercises, go to MOE website

5 - 4

Cyclic Quadrilaterals

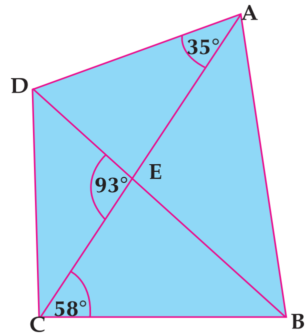
What you'll learn

- ☆ The concept of the cyclic quadrilateral.
- ☆ Identifying when the shape is cyclic quadrilateral.

Key terms

- ☆ Cyclic quadrilateral.

Think and Discuss



In the opposite figure :

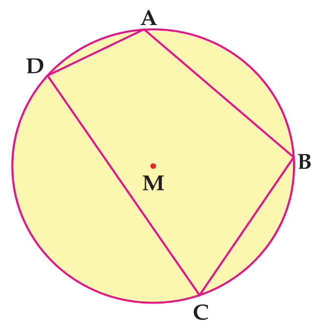
ABCD is a quadrilateral, its diagonals intersect at E,
m (∠ ACB) = 58 ° , m(∠ CAD) = 35°,
m (∠ CED) = 93°.

Can a circle be drawn passing through the vertices of the quadrilateral ABCD ? Explain your answer?

Cyclic quadrilateral	is a quadrilateral figure whose four vertices belong to one circle.

Notice :

1 **The figure ABCD** is a cyclic quadrilateral because its vertices A, B, C and D belong to the circle M.

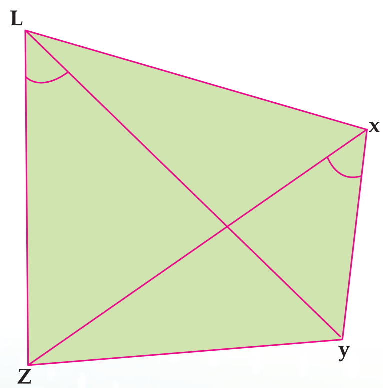


2 **The figure XYZL is a cyclic quadrilateral because:**

m (∠ YXZ) = m (∠ YLZ)

They are two drawn angles on the base $\overline{YZ}$ and in one direction of it, A circle can be drawn passing through the points X, Y, Z and L.

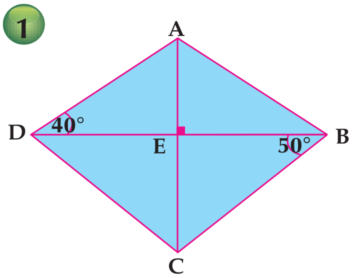
i.e. The vertices of figure XYZL belong to one circle.



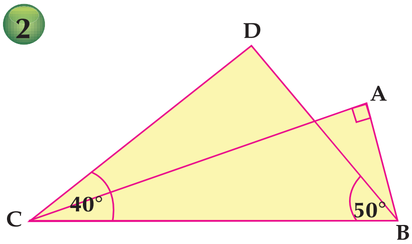
Drill

Which of the following figures is a cyclic quadrilateral ? Explain your answer .

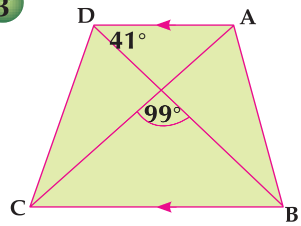
1



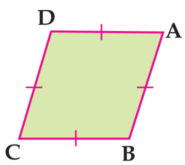
2



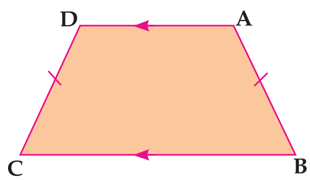
3



4



5

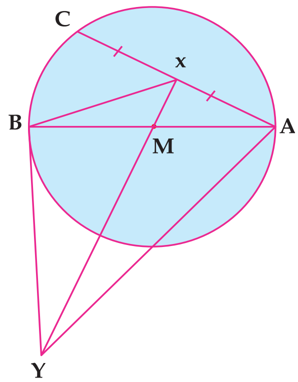


Example 1

In the opposite figure :

$\overline{AB}$ is a diameter in circle M, X is the midpoint of $\overline{AC}$ and $\overrightarrow{XM}$ intersecting the tangent of the circle at B in Y .

Prove that : the figure AXBY is a cyclic quadrilateral.



Solution

Given : $\overline{AB}$ is a diameter in the circle M where AX = CX , $\overrightarrow{BY}$ is a tangent to the circle at B.

R.T.P. : AXBY is a cyclic quadrilateral.

Proof : $\because$ X is the midpoint of $\overline{AC}$ $\therefore$ $\overline{MX} \perp \overline{AC}$, m (∠ AXY) = 90°

$\because$ $\overline{AB}$ is a diameter and, $\overrightarrow{BY}$ is a tangent at B $\therefore$ $\overrightarrow{BY} \perp \overline{AB}$, m (∠ ABY) = 90°

$\because$ m (∠ AXY) = m (∠ ABY) = 90°

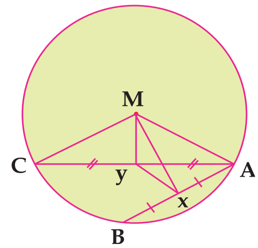
They are two drawn angles on the base $\overline{AY}$ and in one direction of it.

$\therefore$ Figure AXBY is a cyclic quadrilateral.

Think *In the previous example, where is the center of the circle passing through the vertices of the figure AXBY ? located? Explain your answer.*

Drill

In the opposite figure :



A circle with center M. X and Y are the two midpoints of $\overline{AB}$ and $\overline{AC}$ respectively.

Prove that : First: AXYM is a cyclic quadrilateral.

Second: m (∠MXY) = m (∠MCY)

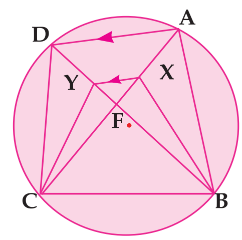
Third : $\overline{AM}$ is a diameter in the circle passing through the points A, X, Y and M

Example 2

ABCD is a cyclic quadrilateral with diagouals intersecting at F, X ∈ $\overline{AF}$ and Y ∈ $\overline{DF}$ where $\overline{XY}$ // $\overline{AD}$.

Prove that : First: BXYC is cyclic quadrilateral. **Second:** m (∠XBY) = m (∠XCY)

Solution



Given: ABCD is a quadrilateral inscribed inside a circle, $\overline{XY}$ // $\overline{AD}$

R.T.P.: **Prove that : First:** BXYC is cyclic quadrilateral.

Second: m (∠XBY) = m (∠XCY)

Proof: ∵ $\overline{XY}$//$\overline{AD}$ ∴ m (∠CAD) = m (∠CXY) *Corresponding*

∵ m (∠CAD) = m (∠CBD) *both are inscribed and common in* $\overset{\frown}{CD}$

∴ m (∠CXY) = m (∠CBY)

and they are two inscribed angles on the base $\overline{CY}$ and in one direction of it.

∴ BXYC is a cyclic quadrilateral *(Q.E.D 1)*

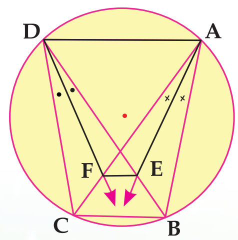
∵ BXYC is a cyclic quadrilateral *(Proof)*

∴ m (∠XBY) = m (∠XCY)

because they are both inscribed angles common at $\overset{\frown}{CD}$. *(Q.E.D 2)*

Drill

In the opposite figure



In the opposite figure : ABCD is a cyclic quadrilateral which has :

$\overrightarrow{AE}$ bisects ∠BAC and $\overrightarrow{DF}$ bisects ∠ BDC ,

Prove that : First: AEFD is a cyclic quadrilateral

Second: $\overline{EF}$ // $\overline{BC}$.

For More Exercises, go to MOE website

5 - 5 Properties of Cyclic Quadrilaterals

What you'll learn

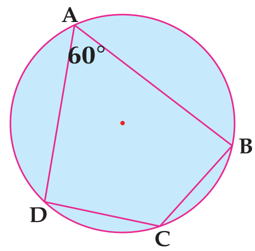
- ☆ Properties of the cyclic quadrilateral shape.
- ☆ How to solve problems on the Properties of the cyclic quadrilateral shape.

Key terms

- ☆ Cyclic quadrilateral.

Think and Discuss

In the opposite figure :



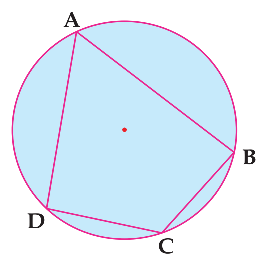
m (∠A) = 60° , then m ($\overparen{BCD}$) = ………°

- ◆ If m ($\overparen{BAD}$) = ………. °
- ◆ If m (∠BCD) = ………°
- ◆ If m (∠B) = 80° then m (∠D) = ……….
- ◆ What do you notice on the sum of the two opposite angles in the cyclic quadrilateral ?

Theorem 3 *In a cyclic quadrilateral, each two opposite angles are supplementary.*

Given: ABCD is a cyclic quadrilateral.

R.T.P: **Prove that :** (1) m (∠A) + m (∠C) = 180°
(2) m (∠B) + m (∠D) = 180°



Proof: ∵ m (∠A) = $\frac{1}{2}$ m ($\overparen{BCD}$)

, m (∠C) = $\frac{1}{2}$ m ($\overparen{BAD}$)

∴ m (∠A) + m (∠C)

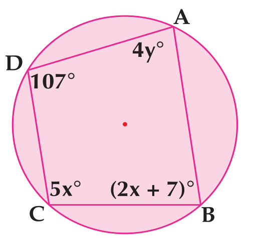
= $\frac{1}{2}$ [m ($\overparen{BCD}$) + m ($\overparen{BAD}$)]

= $\frac{1}{2}$ × 360° = 180°

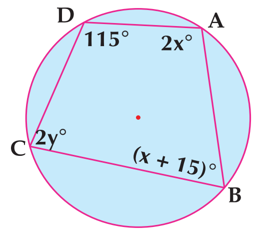
Similarly : m (∠B) + m (∠D) = 180°

(Q.E.D.)

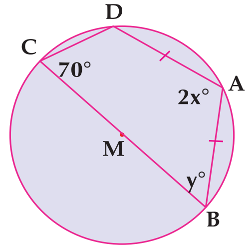
Drill **In each of the following figures, find the value of the symbol used in measuring :**

1


X = ... , Y = ...

2


X = ... , Y = ...

3


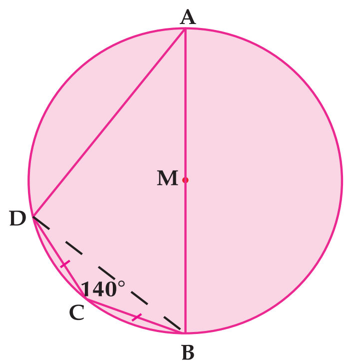
X = ... , Y = ...

Example 1

ABCD is a quadrilateral inscribed in circle M where M ∈ $\overline{AB}$, CB = CD , m ∠(BCD) = 140°

Find : First : m (∠A) Second : m (∠D)

Solution



∵ ABCD is a cyclic quadrilateral

∴ m (∠A) + m (∠C) = 180° ***(theorem)***

∴ m (∠A) = 180° - 140° = 40° ***(Q.E.D first)***

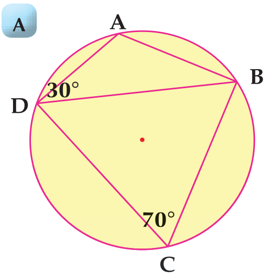
Draw $\overline{BD}$, in Δ BCD ∵ CB = CD

∴ m (∠CDB) = m (∠CBD) = $\frac{180 - 140}{2}$ = 20°

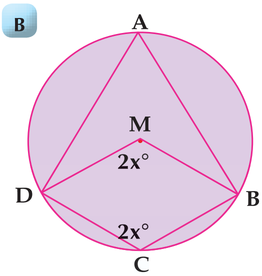
∵ $\overline{AB}$ is a diameter in circle M ∴ m (∠ADB) = 90°

∴ m (∠ADC) = 90° + 20° = 110° ***(Q.E.D second)***

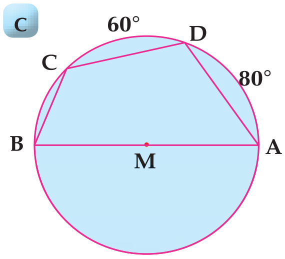
Drill **With the assistance of the given figures, find with proof :**

A


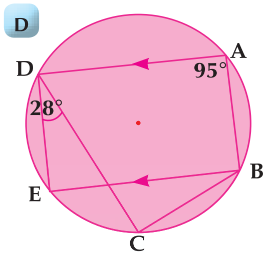
m (∠ABD)

B


m (∠A)

C


measures of figure's angles ABCD

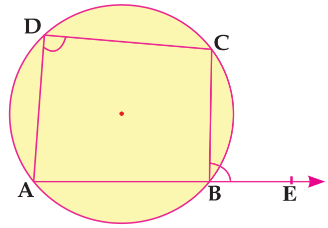
D


measures of figure's angles ABCD

Corollary

The measure of the exterior angle at a vertex of a cyclic quadrilateral is equal to the measure of the interior angle at the opposite vertex.

In the opposite figure :



ABCD is a cyclic quadrilateral, $E \in \overrightarrow{AB}$, $E \notin \overline{AB}$

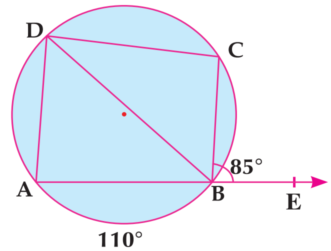
∴ ∠EBC is an angle outside the cyclic quadrilateral ABCD,

∠ D is the inner angle opposite to it.

Thus : m (∠EBC) = m (∠D) *(The supplements of one angle is equal in measure)*

Example 2

In the opposite figure :



$E \in \overrightarrow{AB}$, $E \notin \overline{AB}$, m $(\overset{\frown}{AB})$ = 110°, m (∠CBE) = 85°

Find m (∠BDC) .

Solution

∵ m $(\overset{\frown}{AB})$ = 110° , ∠ ADB is an inscribed angle with arc $\overset{\frown}{AB}$

∴ m (∠ ADB) = $\frac{1}{2}$ m $(\overset{\frown}{AB})$ = 55° .

∵ ∠CBE is exterior angle at a vertex of the cyclic quadrilateral ABCD

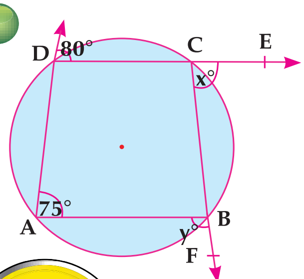
∴ m (∠CBE) = m (∠CDA) = 85° *(Corollary)*

∴ m (∠BDC) = 85° - 55° = 30° *(Q.E.E.)*

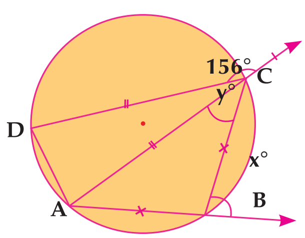
Drill

In each of the following figures, find the value of the symbol used in measuring.

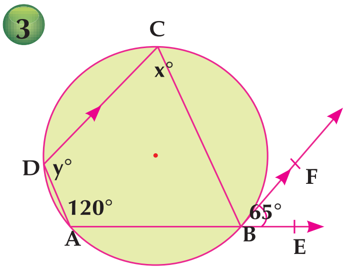
1



2



3



The converse of theorem 3

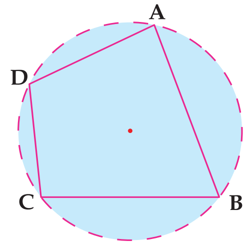
If two opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

In the opposite figure :

If m (∠A) + m (∠C) = 180°

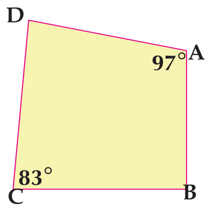
or : m (∠B+ m (∠D) = 180 °

So, ABCD is a cyclic quadrilateral.



Drill

In each of the following figures, prove that ABCD is a cyclic quadrilateral :

1


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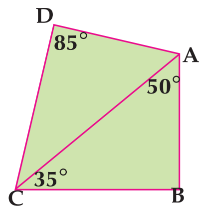

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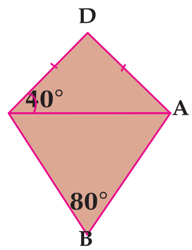


Corollary

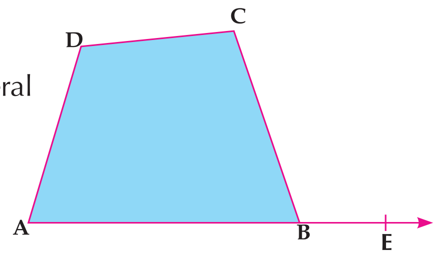
If the measure of the exterior angle at a vertex of a quadrilateral figure is equal to the measure of the interior angle at the opposite vertex, then the figure is a cyclic quadrilateral.

In the opposite figure :

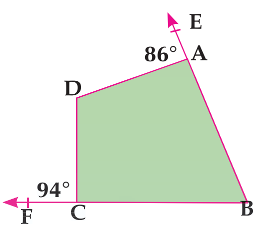
ABCD is a quadrilateral , $E \in \overrightarrow{AB}$, $E \notin \overline{AB}$

∴ ∠EBC is an exterior angle at a vertex of the quadrilateral ABCD and, ∠ D is the inner angle opposite to it.

If m (∠EBC) = m (∠D) then ABCD is a cyclic quadrilateral.



Drill **Prove that each of the following figures is a cyclic quadrilateral:**

1


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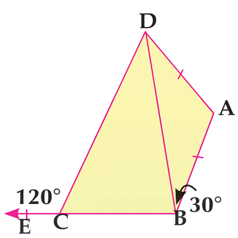

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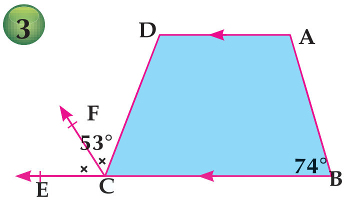
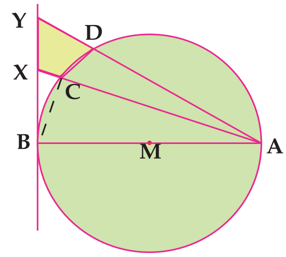


Example 3

In the opposite figure :



$\overline{AB}$ is a diameter in circle M , $\overline{AC}$ and $\overline{AD}$ are two chords in it and in one side from $\overline{AB}$.

A tangent to the circle was drawn from B and intersected $\overrightarrow{AC}$ at X and $\overrightarrow{AD}$ at Y .

Prove that : XYDC is a cyclic quadrilateral.

Solution

Draw $\overline{BC}$ ∵ $\overline{AB}$ is a diameter

∴ m (∠ACB) = 90° and ∠ ABC is complement to ∠BAX (1)

∵ $\overline{AB}$ is a diameter and $\overrightarrow{BY}$ is tangent to the circle at B.

∴ m (∠ABX) = 90° and ∠AXB is complement to ∠BAX (2)

From (1) **and** (2)

∴ m (∠ABC) = m (∠AXB)

∵ ∠ YDC is an exterior angle of the cyclic quadrilateral ABCD

∴ m (∠YDC) = m (∠ ABC) = m (∠ AXB)

∵ ∠ AXB is an exterior angle at the vertex of the quadrilateral XYDC and ∠ YDC is opposite to it.

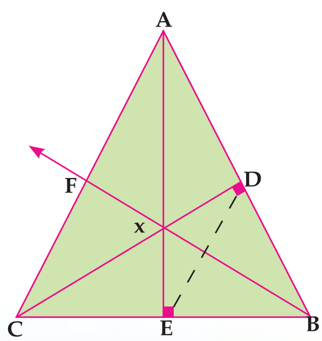
∴ XYDC is a cyclic quadrilateral.

Think

State the cases of the quadrilateral to be cyclic. Mention all the possible cases.

Drill

In the opposite figure, prove that :



The perpindicular line segments on the sides of the triangle from the opposite vertices intersect at one point.

What is the number of cyclic quadrilaterals in the opposite figure? and what are they?

For More Exercises, go to MOE website

5 - 6 The relation between the tangents of a circle

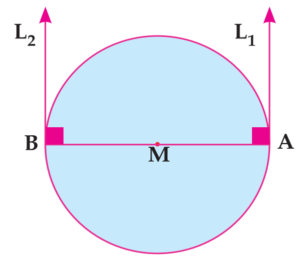
What you'll learn

- How to infer the relation between the two tangent segments drawn from a point outside the circle.
- The concept of a circle inscribed in a polygon.
- How to infer the relation lon the relation between the tangents of a circle.

Key terms

- Chord of tangency.
- A circle inscribed in a polygon.
- Common tangents.

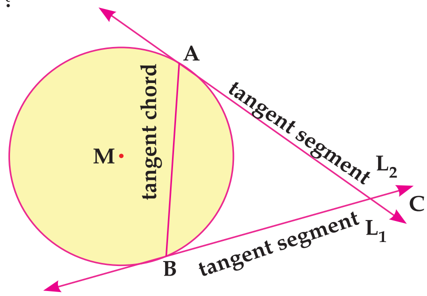
Think and Discuss



You know that the two tangents drawn at the two ends of a diameter in a circle are parallel.

What is the relation between the two tangents drawn at the two ends of a chord of a circle that does not pass through its center?

In the opposite figure :



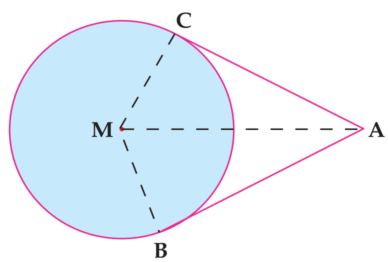
Notice that :

If $\overline{AB}$ is a chord in circle M, then the two tangents L_1 and L_2 intersect at the point C.

Both $\overline{CA}$ and $\overline{CB}$ are called a tangent line segment and $\overline{AB}$ is called a chord of tangency.

Theorem 4 *The two tangent - segments drawn to a circle from a point outside it are equal in length.*

Given : A is a point outside the circle M, $\overline{AB}$ and $\overline{AC}$ are two tangent segments of the circle at B and C.



R.T.P: Prove that : AB = AC

onstruction :

Draw $\overline{MB}$, $\overline{MC}$ and $\overline{MA}$

Proof : $\because$ $\overline{AB}$ is a tangent segment to circle M

$\therefore$ m ($\angle$ABM) = 90°

$\because$ $\overline{AC}$ is a tangent segment to circle M

$\therefore$ m ($\angle$ACM) = 90°

∵ The two triangles ABM and ACM have :

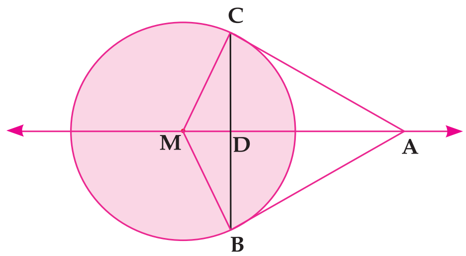
m(∠B) = m (∠C) = 90° *(Proof)*

MB = MC *(Lengths of radii)*

$\overline{AM}$ is a common side. ∴ Δ ABM ≡ Δ ACM

We get : $\overline{AB} \equiv \overline{AC}$ ∴ AB = AC *(Q.E.D.)*

Think **In the opposite figure :**



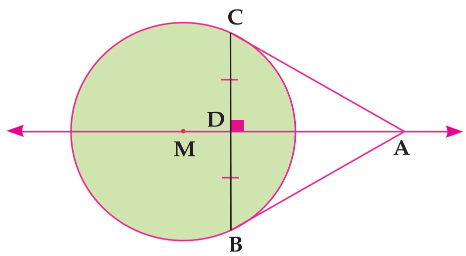
- Why is $\overleftrightarrow{MA}$ the axis of $\overline{BC}$?
- Why does $\overrightarrow{AM}$ bisect ∠BAC?
- Why does $\overrightarrow{MA}$ bisect ∠BMC?

Theorem corollaries:

Corollary 1

The straight line passing through the center of the circle and the intersection point of the two tangent is an axis of symmetry to the chord of tangency of those two tangents.

In the opposite figure :



$\overline{AB}$ and $\overline{AC}$ are two tangents to circle M at B and C.

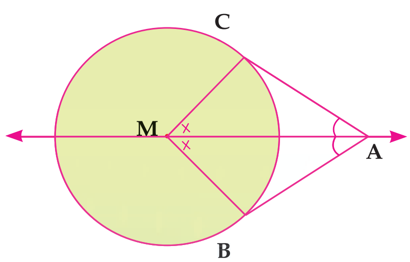
Then : $\overleftrightarrow{AM}$ is the axis of $\overline{BC}$

Thus : $\overleftrightarrow{AM} \perp \overline{BC}$, and BD = CD

Corollary 2

The straight line passing through the center of the circle and the intersection point of its two tangents bisects the angle between these two tangents. It also bisects the angle between the two radii passing through the two points of tangenvy.

In the opposite figure :



$\overline{AB}$ and $\overline{AC}$ are two tangents to the circle M at B and C.

Then : $\overrightarrow{AM}$ bisects ∠ A

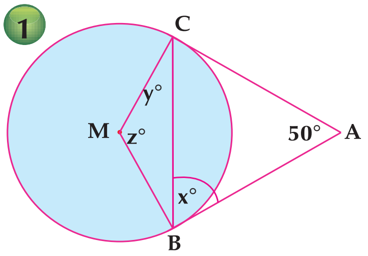
∴ m (∠BAM) = m (∠CAM)

, $\overrightarrow{MA}$ bisects ∠ BMC

∴ m (∠ AMB) = m (∠ AMC)

Drill

In each of the following figures, $\overline{AB}$ and $\overline{AC}$ are two tangent segments to the circle M.
Find the value of the symbol used in measuring :

1


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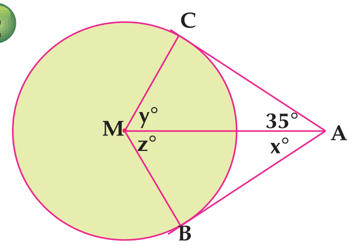

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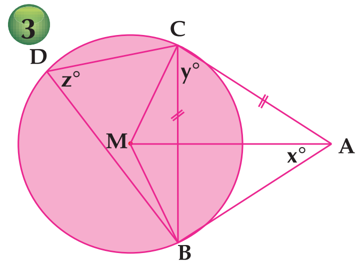
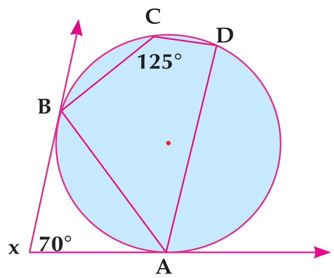


Example 1

In the opposite figure :



$\overrightarrow{XA}$ and $\overrightarrow{XB}$ are two tangents to the circle at A and B.

m ($\angle$ AXB) = 70°, m ($\angle$ DCB) = 125°

Prove that : First : $\overrightarrow{AB}$ bisects $\angle$ DAX . **Second:** $\overrightarrow{AD}$ // $\overrightarrow{XB}$.

Solution

Given: $\overrightarrow{XA}$ and $\overrightarrow{XB}$ are two tangents to the circle,
m ($\angle$AXB) = 70° and m ($\angle$ DCB) = 125° .

R.T.P.: **First :** $\overrightarrow{AB}$ bisects $\angle$DAX

Second: $\overrightarrow{AD}$ // $\overrightarrow{XB}$.

Proof : $\therefore$ $\overline{XA}$ and $\overline{XB}$ are two tangent segments. $\therefore$ XA = XB

in Δ XAB

$\because$ m ($\angle$XAB) = m ($\angle$XBA), m ($\angle$X) = 70°

$\therefore$ m ($\angle$XAB) = $\frac{180° - 70°}{2}$ = 55° (1)

$\because$ ABCD is a cyclic quadrilateral, m ($\angle$C) = 125°

$\therefore$ m ($\angle$ DAB) = 180° - 125° = 55° *(theorem)* (2)

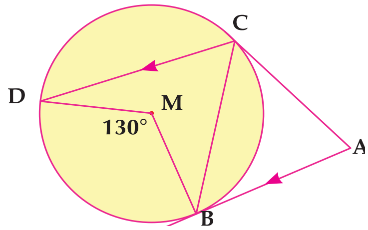
From (1) **and** (2) **we get :** m ($\angle$XAB) = m ($\angle$ DAB) = 55°

$\therefore$ $\overrightarrow{AB}$ bisects $\angle$DAX *(Q.E.D First)*

$\because$ m ($\angle$XBA) = m ($\angle$DAB) = 55° *alternating angle*

$\therefore$ $\overrightarrow{AD}$ // $\overrightarrow{XB}$ *(Q.E.D Second)*

Drill



In the opposite figure :

$\overline{AB}$ and $\overline{AC}$ are two tangent segments to the circle M,

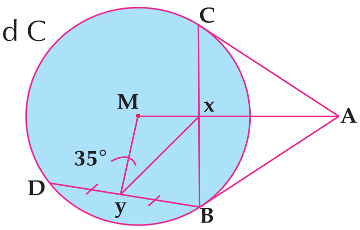
$\overline{AB} \parallel \overline{CD}$, m ($\angle$BMD) = 130° .

1 **Prove that :** $\overrightarrow{CB}$ bisects $\angle$ ACD

2 ***Find*** m ($\angle$A) .

Example 2

In the opposite figure :



$\overline{AB}$ and $\overline{AC}$ are two tangent segments to the circle M at B and C

$\overline{AM} \cap \overline{BC}$ = {X}, Y is the midpoint of the chord $\overline{BD}$

m ($\angle$XYM) = 35° .

A **Prove that :** XBYM is a cyclic quadrilateral.

B ***Find*** m ($\angle$A) .

Solution

$\because$ $\overline{AB}$, and $\overline{AC}$ are two tangent segments to the circle M at B and C

$\therefore$ $\overleftrightarrow{AM}$ is the axis of $\overline{BC}$, m($\angle$BXM) = 90° (1)

$\because$ Y is the midpoint of the chord $\overline{BD}$

$\therefore$ m ($\angle$BYM) = 90° (2)

From (1) **and** (2)

$\therefore$ XBYM is a cyclic quadrilateral. ***(Q.E.D 1)***

Draw $\overline{BM}$

$\because$ XBYM is a cyclic quadrilateral, m ($\angle$XYM) = 35° .

$\therefore$ m ($\angle$XBM) = m ($\angle$XYM) = 35°

$\because$ $\overline{AB}$ is a tangent segment and $\overline{BM}$ is a radius.

$\therefore$ m ($\angle$ ABM) = 90°

$\therefore$ m ($\angle$ ABC) = 90° - 35° = 55°

$\because$ AB = AC

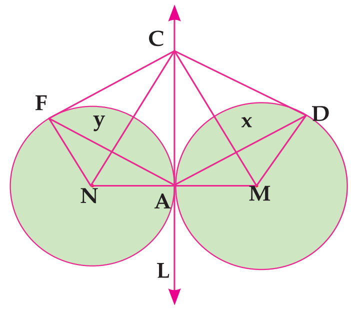
$\therefore$ m ($\angle$ABC) = m ($\angle$ACB) = 55°

$\therefore$ m ($\angle$A) = 180° - (55° + 55°) = 70° ***(Q.E.D 2)***

Drill

In the opposite figure :

M and N are two circles touching externally at A. The line L is a common tangent for both of them at A, C ∈ L, Two other tangents were drawn from C to the two circles M and N touching them at D and E respectively $\overline{CM} \cap \overline{DA} = \{X\}$ and $\overline{CN} \cap \overline{AE} = \{Y\}$



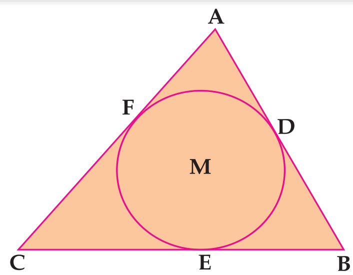
1. **What is the number of cyclic quadrilaterals in the opposite figure ? and what are they ?**
2. **Prove that : CD = CA = CE, and explain this geometrically.**

Definition **The inscribed circle of a polygon is the circle which touches all of its sides internally**

In the opposite figure :

M is the inscribed circle of the triangle ABC because it touches all of its sides internally at D, E and F .

i.e. : The triangle ABC is drawn outside the circle M.



Think ***Is the center of the inscribed circle for any triangle the intersection point of the bisectors of its interior angles ? Explain your answer.***

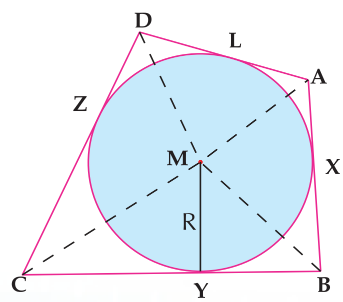
Example 3

In the opposite figure :

M is an inscribed circle to the quadrilateral ABCD with radius length of 5 cm.

AB = 9cm and CD = 12 cm.

Find the perimeter of ABCD, then calculate its area.



Solution

∵ The circle M is an inscribed circle to the quadrilateral ABCD

∴ The circle M touches the sides of ABCD at X , Y , Z and L

∵ $\overline{AX}$ and $\overline{AL}$ are two tangent segments to the circle M

∴ AX = AL

$\because$ $\overline{BX}$ and $\overline{BY}$ are two tangent segments to circle M $\therefore$ BX = BY

Similiarly, CZ= CY $\therefore$ DZ = DL

By addition, we get : (AX + BX) + (CZ + DZ) = AL + BY + CY + DL

$\therefore$ AB + CD = AD+ BC = $\frac{1}{2}$ the perimeter of ABCD

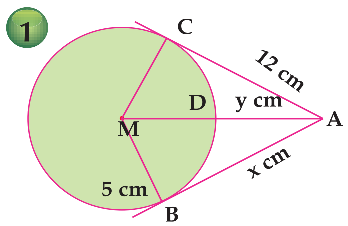
Perimeter of ABCD = 2(9 + 12) = 42cm ,

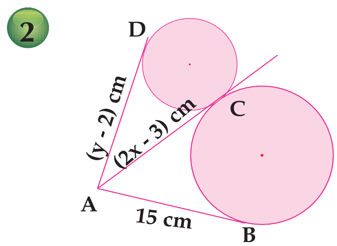
Area of ABCD $= \frac{1}{2} AB \times r + \frac{1}{2} BC \times r + \frac{1}{2} CD \times r + \frac{1}{2} AD \times r$

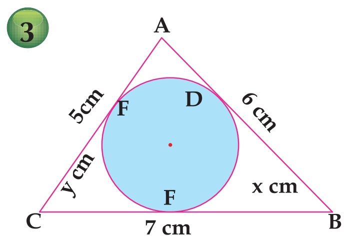
$= \frac{1}{2}$ perimeter $\times r = \frac{1}{2} \times 42 \times 5 = 105 cm^2$

Drill

Find the value of the symbol used in measuring :

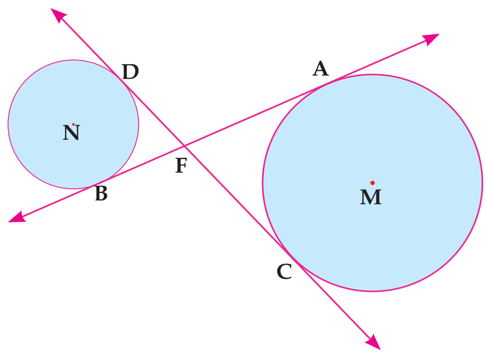






Common tangents of two distant circles :

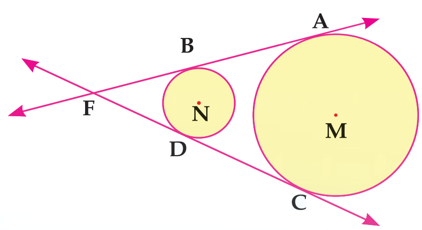
A $\overleftrightarrow{AB}$ is called a common internal tangent to the two circles M and N because the two circles M and N are located at two different sides of $\overleftrightarrow{AB}$, **Also** $\overleftrightarrow{CD}$ is an internal tangent to the two circles.



Notice that : $\overline{AB} \cap \overline{CD}$ = {E}

In the opposite figure : Prove that : AB = CD

B $\overleftrightarrow{AB}$ is called a common external tangent to the two circles M and N because the two circles M and N are are located in the same side of , $\overleftrightarrow{AB}$, also $\overleftrightarrow{CD}$ is an external tangent to the two circles.

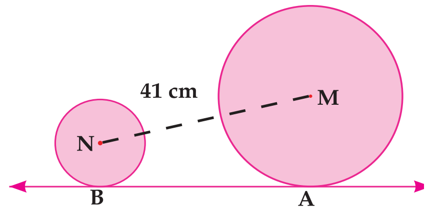


Notice that : $\overrightarrow{AB} \cap \overrightarrow{CD}$ = {F}

In the opposite figure : Prove that : AB = CD

Drill

In the opposite figure : $\overleftrightarrow{AB}$ is a common tangent to the two circles M and N externally at A and B respectively. Their two radii lengths are 17 cm and 8 cm respecticely. If MN = 41 cm, Find the length of $\overline{AB}$



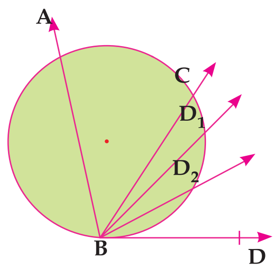
5 - 7

Angles of Tangency

Think and Discuss

In the opposite figure :

$\angle ABC$ is an inscribed angle with the two sides $\overrightarrow{BA}$ and $\overrightarrow{BC}$ and arc $\overset{\frown}{AC}$, $\overrightarrow{BD}$ is a tangent to the circle at B. If we imagine the revoltion of one of the sides of the inscribed angle, let it be $\overrightarrow{BC}$ moving away from $\overrightarrow{BA}$ so, it takes one of the positions $\overrightarrow{BC_1}$, $\overrightarrow{BC_2}$, ...



- **Does** the measure of the resulted inscribed angles increase such as $\angle ABC_1$ and $\angle ABC_2$, ...
- **Do the mesures of** m ($\overset{\frown}{AC_1}$) and m ($\overset{\frown}{AC_2}$) increase, ... ?
- If $\overrightarrow{BC}$ and $\overrightarrow{BD}$ are congruent, what do you notice ?

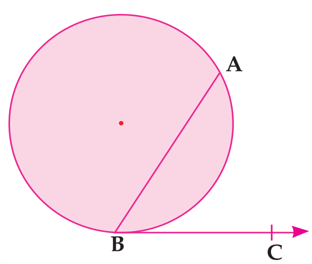
Notice that We get a larger inscribed angle in measure when $\overrightarrow{BC}$ and $\overrightarrow{BD}$ are about to be congruent $\angle$ ABD is called the angle of tangency it is a special case of the tangent angle :

$$m(\angle ABD) = \frac{1}{2} m(\overset{\frown}{ACD})$$

Angle of Tangency	The angle which is composed of the union of two rays, one is a tangent to the circle and the other contains a chord of the circle passing through the point of tangency.

Thus :
The measure of the angle of tangency is half the measure of the arc between the two sides.

i.e. : $m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AB})$



What you'll learn

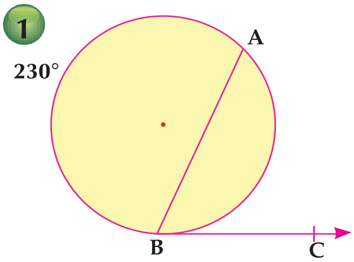
- The concept of the angle of tangency
- How to infer the relation between the angle of tangency and the inscribed angle subtended by the same arc.
- The relation between the angle of tangency and the centeral angle subtended by the same arc.
- How to solve problems on angels of tangency.

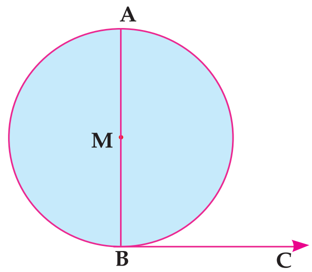
Key terms

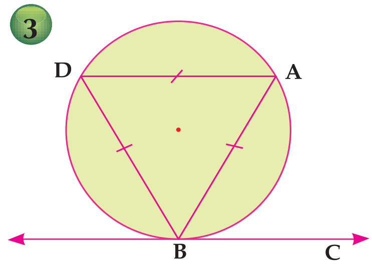
- Angle of tangency.
- Inscribed angle.
- Central angle.

Drill

In each of the following figures, calculate m $(\angle ABC)$.







Theorem 5 *The measure of the angle of tangency is equal to the measure of the inscribed angle subtended by the same arc.*

Given : $\angle ABC$ is an angle of tangency and, $\angle D$ is an inscribed angle.

R.T.P : **Prove that :** $m(\angle ABC) = m(\angle D)$

Proof : $\because$ $\angle ABC$ is an angle of tangency

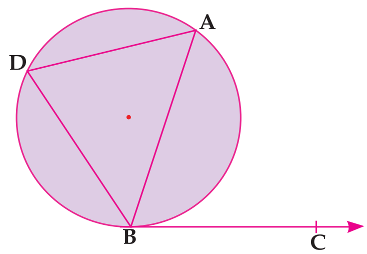
$\therefore$ $m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AB})$ (1)

$\because$ $\angle D$ is an inscribed angle

$\therefore$ $m(\angle D) = \frac{1}{2} m(\overset{\frown}{AB})$ (2)

From (1) **and** (2) **we get :**

$m(\angle ABC) = m(\angle\ \ C)$



Q.E.D.

Corollary *The measure of the angle of tangency is half the measure of the central angle subtended by the same arc.*

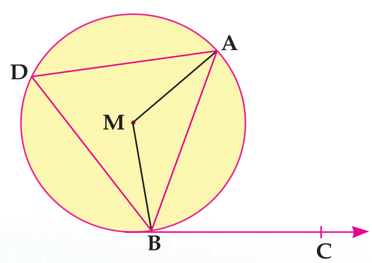
In the opposite figure :

$\overrightarrow{BC}$ is tangent to circle M, $\overline{AB}$ is a chord of tangency

$\therefore$ $m(\angle\ ABC) = m(\angle D\ \ )$ *(theorem)*

$\because$ $m(\angle D) = \frac{1}{2} m(\angle AMB)$ *(theorem)*

$\therefore$ $m(\angle\ ABC) = \frac{1}{2} m(\angle\ AMB)$

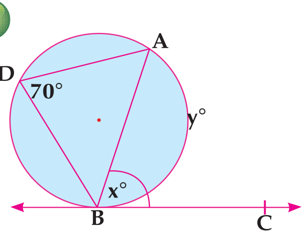


Drill

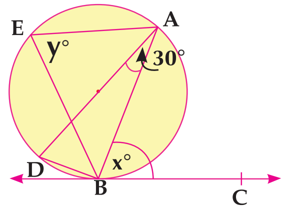
In each of the following figures : $\overleftrightarrow{BC}$ is tangent to the circle.

Find the value of the symbol used in measuring.

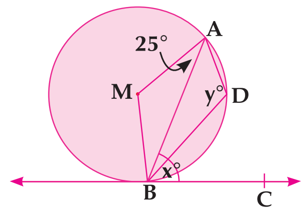
1



2



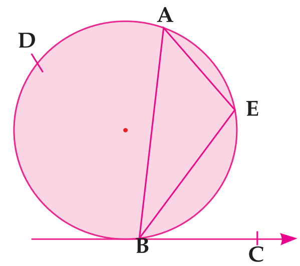
3



Important notice :

The angle of tangency is supplementary to the drawn inscribed angle on the chord of the angle of tangency and in one side of it.

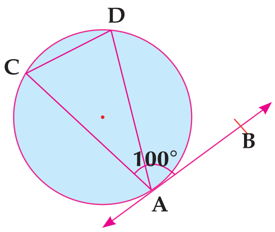
i.e. : ∠ ABC is supplementary to ∠ AEB .



Drill

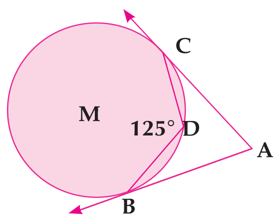
With the assistance of the given figures, complete :

1



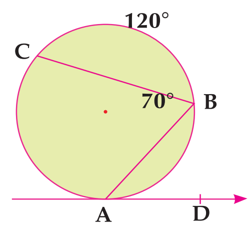
m (∠ADC) = ... °

2



m (∠ BAC) = ... °

3

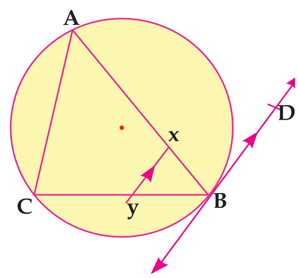


m (∠ BAD) = ... °

Example 1

ABC is a triangle inscribed in a circle, $\overleftrightarrow{BD}$ is a tangent to the circle at B, X ∈ $\overline{AB}$, Y ∈ $\overline{BC}$ Where $\overline{XY}$ // $\overleftrightarrow{BD}$.

Prove that : AXYC is a cyclic quadrilateral.



Proof :

∵ $\overleftrightarrow{BD}$ is tangent to the circle at B, $\overline{AB}$ is a chord of tangency.

∴ m (∠DBA) = m (∠C)

∵ $\overline{XY}$ // $\overleftrightarrow{DB}$, $\overleftrightarrow{AB}$ *intersecting both of them*

∴ m (∠ DBA) = m (∠BXY) ∴ m (∠BXY) = m (∠C)

∵ ∠BXY is exterior from the quadrilateral XYCA.

∴ XYCA is a cyclic quadrilateral. *(Q.E.D.)*

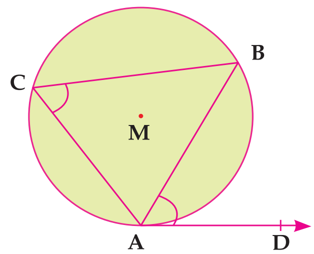
The converse of theorm 5

If a ray is drawn from one end of a chord of a circle so that the angle between this ray and the chord is equal in measure to the inscribed angle subtended by the chord in the alternate side, then this ray is a tangent to this circle.

i.e. :

If we draw $\overrightarrow{AD}$ from one end of the chord $\overline{AB}$ in circle M and :

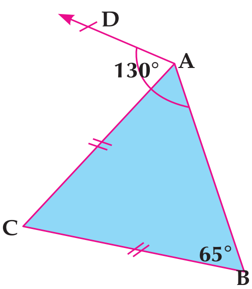
m ($\angle$DAB) = m ($\angle$C) **then :** $\overrightarrow{AD}$ is a tangent to circle M.



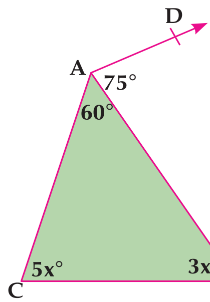
Drill

In each of the following shapes show that $\overleftrightarrow{AD}$ is a tangent to the circle passing through the vertices of the triangle ABC.

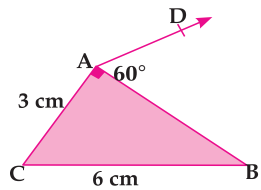
1



2



3

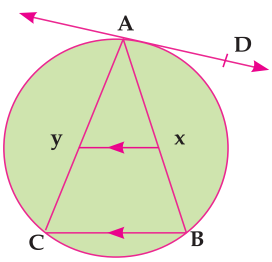


Example 4

ABC is a triangle inscribed in a circle, $\overleftrightarrow{AD}$ is a tangent to the circle at A, X ∈ $\overline{AB}$, Y ∈ $\overline{AC}$ where $\overline{XY}$ // $\overline{BC}$

Prove that : $\overleftrightarrow{AD}$ is a tangent to the circle passing through the points A, X and Y .

Solution



Given: $\overleftrightarrow{AD}$ is a tangent to the circle and, $\overline{XY}$ // $\overline{BC}$

R.T.P.: **Prove that :** $\overleftrightarrow{AD}$ is a tangent to the circle passing through the points A , X and Y.

Proof: ∵ $\overleftrightarrow{AD}$ is a tangent and, $\overline{AB}$ is the chord of tangeny

∴ m ($\angle$D A B) = m ($\angle$C) (1)

∵ $\overline{XY}$ // $\overline{BC}$, $\overleftrightarrow{AC}$ *intersector* ∴ m ($\angle$AYX) = m ($\angle$C) (2)

From (1) **and** (2) **we get :** m ($\angle$ DAB) = m ($\angle$ AYX)

i.e.: m ($\angle$ DAX) = m ($\angle$ AYX)

∴ $\overleftrightarrow{AD}$ is a tangent to the circle passing through the points A, X and Y .

For More Exercises, go to MOE website

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